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△-convergence for mixed-type total asymptotically nonexpansive mappings in hyperbolic spaces

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Abstract
In this paper, we prove some △-convergence theorems in a hyperbolic space. A mixed Agarwal-O'Regan-Sahu type iterative scheme for approximating a common fixed point of total asymptotically nonexpansive mappings is constructed. Our results extend some results in the literature.
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1 Introduction and preliminaries

In this paper, we work in the setting of hyperbolic spaces introduced by Kohlenbach [1]. Concretely, (X,d,W) is called a hyperbolic space if (X,d) is a metric space and $W: X \times X \times [0,1] \to X$ is a function satisfying

(I) $\forall x,y,z \in X$, $\forall \lambda \in [0,1]$, $d(z, W(x,y,\lambda)) \leq (1-\lambda)d(z,x) + \lambda d(z,y)$;
(II) $\forall x,y \in X$, $\forall \lambda_1,\lambda_2 \in [0,1]$, $d(W(x,y,\lambda_1), W(x,y,\lambda_2)) = |\lambda_1 - \lambda_2| \cdot d(x,y)$;
(III) $\forall x,y \in X$, $\forall \lambda \in [0,1]$, $W(x,y,\lambda) = W(y,x,(1-\lambda))$;
(IV) $\forall x,y,z,w \in X$, $\forall \lambda \in [0,1]$, $d(W(x,z,\lambda), W(y,w,\lambda)) \leq (1-\lambda)d(x,y) + \lambda d(z,w)$.

If a space satisfies only (I), it coincides with the convex metric space introduced by Takahashi [2]. The concept of hyperbolic spaces in [1] is more restrictive than the hyperbolic type introduced by Goebel [3] since (I)-(III) together are equivalent to (X,d,W) being a space of hyperbolic type in [3]. But it is slightly more general than the hyperbolic space defined in Reich [4] (see [1]). This class of metric spaces in [1] covers all normed linear spaces, $\mathbb{R}$-trees in the sense of Tits, the Hilbert ball with the hyperbolic metric (see [5]), Cartesian products of Hilbert balls, Hadamard manifolds (see [4, 6]) and CAT(0) spaces in the sense of Gromov (see [7]). A thorough discussion of hyperbolic spaces and a detailed treatment of examples can be found in [1] (see also [3–5]).

A hyperbolic space is *uniformly convex* [8] if for $u,x,y \in X$, $r > 0$ and $\epsilon \in (0,2]$, there exists $\delta \in (0,1]$ such that

$$d\left(W\left(x,y,\frac{1}{2}\right),u\right) \leq (1-\delta)r,$$

provided that $d(x,u) \leq r$, $d(y,u) \leq r$ and $d(x,y) \geq \epsilon r$.

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A map $\eta : (0,\infty) \times (0,2] \to (0,1]$ is called *modulus of uniform convexity* if $\delta = \eta(r,\epsilon)$ for given $r > 0$. Besides, η is *monotone* if it decreases with r (for a fixed ϵ), that is,

$$\eta(r_2,\epsilon) \le \eta(r_1,\epsilon), \quad \forall r_2 \ge r_1 > 0.$$

A subset C of a hyperbolic space X is *convex* if $W(x,y,\lambda) \in C$ for all $x,y \in C$ and $\lambda \in [0,1]$.

Let (X,d) be a metric space, and let C be a nonempty subset of X. Recall that $T : C \to C$ is said to be a $(\{\nu_n\},\{\mu_n\},\zeta)$*-total asymptotically nonexpansive mapping* if there exist nonnegative sequences $\{\nu_n\}$, $\{\mu_n\}$ with $\nu_n \to 0$, $\mu_n \to 0$ and a strictly increasing continuous function $\zeta : [0,\infty) \to [0,\infty)$ with $\zeta(0) = 0$ such that

$$d\bigl(T^n x, T^n y\bigr) \le d(x,y) + \nu_n \zeta\bigl(d(x,y)\bigr) + \mu_n, \quad \forall n \ge 1, x,y \in C. \tag{1}$$

It is well known that each nonexpansive mapping is an asymptotically nonexpansive mapping and each asymptotically nonexpansive mapping is a $(\{\nu_n\},\{\mu_n\},\zeta)$-total asymptotically nonexpansive mapping.

$T : C \to C$ is said to be *uniformly L-Lipschitzian* if there exists a constant $L > 0$ such that

$$d\bigl(T^n x, T^n y\bigr) \le L d(x,y), \quad \forall n \ge 1, x,y \in C.$$

The following iteration process is a translation of the mixed Agarwal-O'Regan-Sahu type iterative scheme introduced in [9] from Banach spaces to hyperbolic spaces. The iteration rate of convergence is similar to the Picard iteration process and faster than other fixed point iteration processes. Besides, it is independent of Mann and Ishikawa iteration processes.

$$\begin{cases} x_1 \in C, \\ x_{n+1} = W(S_1^n x_n, T_1^n y_n, \alpha_n), \quad n \ge 1, \\ y_n = W(S_2^n x_n, T_2^n x_n, \beta_n), \end{cases} \tag{2}$$

where C is a nonempty closed and convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η. $T_i : C \to C$, $i = 1,2$, is a uniformly L_i-Lipschitzian and $(\{\nu_n^{(i)}\},\{\mu_n^{(i)}\},\zeta^{(i)})$-total asymptotically nonexpansive mapping, and $S_i : C \to C$, $i = 1,2$, is a uniformly $\widetilde{L}_i$-Lipschitzian and $(\{\widetilde{\nu}_n^{(i)}\},\{\widetilde{\mu}_n^{(i)}\},\widetilde{\zeta}^{(i)})$-total asymptotically nonexpansive mapping such that the following conditions are satisfied:

(1) $\sum_{n=1}^\infty \nu_n^{(i)} < \infty$, $\sum_{n=1}^\infty \mu_n^{(i)} < \infty$, $\sum_{n=1}^\infty \widetilde{\nu}_n^{(i)} < \infty$, $\sum_{n=1}^\infty \widetilde{\mu}_n^{(i)} < \infty$, $i = 1,2$;

(2) There exists a constant $M > 0$ such that $\zeta^{(i)}(r) \le Mr$, $\widetilde{\zeta}^{(i)}(r) \le Mr$, $\forall r \ge 0$, $i = 1,2$.

Remark 1 Without loss of generality, we can assume that $T_i : C \to C$ and $S_i : C \to C$, $i = 1,2$, both are uniformly L-Lipschitzian and $(\{\nu_n\},\{\mu_n\},\zeta)$-total asymptotically nonexpansive mappings satisfying conditions (1) and (2). In fact, letting $\nu_n = \max\{\nu_n^{(i)},\widetilde{\nu}_n^{(i)}, i = 1,2\}$, $\mu_n = \max\{\mu_n^{(i)},\widetilde{\mu}_n^{(i)}, i = 1,2\}$, $L = \max\{L_i,\widetilde{L}_i, i = 1,2\}$ and $\zeta = \max\{\zeta^{(i)},\widetilde{\zeta}^{(i)}, i = 1,2\}$, then S_i and T_i, $i = 1,2$, are the required mappings.

Chang [10] proved some strong convergence theorems and $\triangle$-convergence theorems for approximating a common fixed point of total asymptotically nonexpansive mappings in a CAT(0) space using the mixed Agarwal-O'Regan-Sahu type iterative scheme. More precisely, one of the results is as follows.

Theorem 1 [10] *Let C be a bounded closed and convex subset of a complete* CAT(0) *space X. Let $T_i : C \to C$ and $S_i : C \to C$, $i = 1, 2$, be uniformly L-Lipschitzian and $(\{\nu_n\}, \{\mu_n\}, \zeta)$-total asymptotically nonexpansive mappings. If $\mathcal{F} := \bigcap_{i=1}^{2} F(T_i) \cap F(S_i) \neq \emptyset$ and the following conditions are satisfied*:

(i) $\sum_{n=1}^{\infty} \nu_n < \infty$ *and* $\sum_{n=1}^{\infty} \mu_n < \infty$;

(ii) *there exist constants $a, b \in (0, 1)$ with $0 < b(1 - a) \leq \frac{1}{2}$ such that $\{\alpha_n\} \subset [a, b]$*;

(iii) *there exists a constant $M > 0$ such that $\zeta(r) \leq Mr$, $r \geq 0$*;

(iv) *$d(x, T_i y) \leq d(S_i x, T_i y)$ for all $x, y \in C$ and $i = 1, 2$,*

then the sequence $\{x_n\}$ defined by (2) *$\triangle$-converges to a common fixed point of T_i and S_i, $i = 1, 2$.*

Theorem 1 can be viewed as a improvement and extension of several well-known results in Banach spaces and CAT(0) spaces, such as [9] and [11]. Our purpose of this paper is to extend Theorem 1 from the CAT(0) spaces setting to the general setup of uniformly convex hyperbolic spaces.

Let $\{x_n\}$ be a bounded sequence in a hyperbolic space X. For $x \in X$, we define

$$r(x, \{x_n\}) = \limsup_{n \to \infty} d(x, x_n).$$

The *asymptotic radius $r(\{x_n\})$ of $\{x_n\}$* is given by

$$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\}.$$

The *asymptotic radius $r_C(\{x_n\})$ of $\{x_n\}$ with respect to $C \subset X$* is given by

$$r_C(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in C\}.$$

The *asymptotic center $A(\{x_n\})$ of $\{x_n\}$* is the set

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}.$$

The *asymptotic center $A_C(\{x_n\})$ of $\{x_n\}$ with respect to $C \subset X$* is the set

$$A_C(\{x_n\}) = \{x \in C : r(x, \{x_n\}) = r_C(\{x_n\})\}.$$

Recall that a sequence $\{x_n\}$ in X is said to *$\triangle$-converge to $x \in X$* if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we call x the *$\triangle$-limit of $\{x_n\}$*. The following lemmas are important in our paper.

Lemma 1 [12, 13] *Let (X, d, W) be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity, and let C be a nonempty closed convex subset of X. Then every bounded sequence $\{x_n\}$ in X has a unique asymptotic center with respect to C.*

Lemma 2 [12] *Let (X, d, W) be a uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Let $x \in X$ and $\{\alpha_n\}$ be a sequence in $[a, b]$ for some $a, b \in (0, 1)$. If $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\limsup_{n\to\infty} d(x_n, x) \leq c$, $\limsup_{n\to\infty} d(y_n, x) \leq c$ and $\lim_{n\to\infty} d(W(x_n, y_n, \alpha_n), x) = c$ for some $c \geq 0$, then*

$$\lim_{n\to\infty} d(x_n, y_n) = 0.$$

Lemma 3 [12] *Let C be a nonempty closed convex subset of a uniformly convex hyperbolic space, and let $\{x_n\}$ be a bounded sequence in C such that $A(\{x_n\}) = \{y\}$ and $r(\{x_n\}) = \rho$. If $\{y_m\}$ is another sequence in C such that $\lim_{m\to\infty} r(y_m, \{x_n\}) = \rho$, then $\lim_{m\to\infty} y_m = y$.*

Lemma 4 [10] *Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be sequences of nonnegative numbers such that*

$$a_{n+1} \leq (1 + b_n)a_n + c_n, \quad \forall n \geq 1.$$

If $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.

2 Main results

In this section, we prove our main theorems.

Theorem 2 *Let C be a nonempty closed and convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η. Let $T_i : C \to C$ and $S_i : C \to C$, $i = 1, 2$, be uniformly L-Lipschitzian and $(\{\nu_n\}, \{\mu_n\}, \zeta)$-total asymptotically nonexpansive mappings. If $\mathcal{F} := \bigcap_{i=1}^{2} F(T_i) \cap F(S_i) \neq \emptyset$ and the following conditions are satisfied*:

(i) $\sum_{n=1}^{\infty} \nu_n < \infty$ *and* $\sum_{n=1}^{\infty} \mu_n < \infty$;

(ii) *there exist constants* $a, b \in (0, 1)$ *such that* $\{\alpha_n\} \subset [a, b]$;

(iii) *there exists a constant* $M > 0$ *such that* $\zeta(r) \leq Mr$, $r \geq 0$;

(iv) $d(x, T_i y) \leq d(S_i x, T_i y)$ *for all* $x, y \in C$ *and* $i = 1, 2$,

then the sequence $\{x_n\}$ defined by (2) $\triangle$*-converges to a common fixed point of T_i and S_i, $i = 1, 2$.*

Proof We divide our proof into three steps.

Step 1. In the sequel, we shall show that for each $p \in \mathcal{F}$,

$$\lim_{n\to\infty} d(x_n, p) \quad \text{and} \quad \lim_{n\to\infty} d(x_n, \mathcal{F}) \text{ exist.} \tag{3}$$

In fact, by conditions (1), (2), (I) and (iii), one gets

$$\begin{aligned} d(y_n, p) &= d\big(W\big(S_2^n x_n, T_2^n x_n, \beta_n\big), p\big) \\ &\leq (1 - \beta_n) d\big(S_2^n x_n, p\big) + \beta_n d\big(T_2^n x_n, p\big) \\ &\leq (1 - \beta_n)\big\{d(x_n, p) + \nu_n \zeta\big(d(x_n, p)\big) + \mu_n\big\} + \beta_n\big\{d(x_n, p) + \nu_n \zeta\big(d(x_n, p)\big) + \mu_n\big\} \\ &= d(x_n, p) + \nu_n \zeta\big(d(x_n, p)\big) + \mu_n \\ &\leq (1 + \nu_n M) d(x_n, p) + \mu_n \end{aligned} \tag{4}$$

and

$$\begin{aligned} d(x_{n+1},p) &= d\big(W\big(S_1^n x_n, T_1^n y_n, \alpha_n\big),p\big) \\ &\le (1-\alpha_n)d\big(S_1^n x_n,p\big) + \alpha_n d\big(T_1^n y_n,p\big) \\ &\le (1-\alpha_n)\big\{d(x_n,p) + \nu_n\zeta\big(d(x_n,p)\big) + \mu_n\big\} + \alpha_n\big\{d(y_n,p) + \nu_n\zeta\big(d(y_n,p)\big) + \mu_n\big\} \\ &\le (1-\alpha_n)\big\{(1+\nu_n M)d(x_n,p) + \mu_n\big\} + \alpha_n\big\{(1+\nu_n M)d(y_n,p) + \mu_n\big\}. \end{aligned} \tag{5}$$

Combining (4) and (5), one has

$$d(x_{n+1},p) \le (1+\sigma_n)d(x_n,p) + \xi_n, \quad \forall n \ge 1 \text{ and } p \in \mathcal{F}, \tag{6}$$

and

$$d(x_{n+1},\mathcal{F}) \le (1+\sigma_n)d(x_n,\mathcal{F}) + \xi_n, \quad \forall n \ge 1, \tag{7}$$

where $\sigma_n = \nu_n M(1+\alpha_n(1+\nu_n M))$, $\xi_n = (1+\alpha_n(1+\nu_n M))\mu_n$. Furthermore, using condition (i), one has

$$\sum_{n=1}^{\infty} \sigma_n < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \xi_n < \infty. \tag{8}$$

Consequently, a combination of (6), (7), (8) and Lemma 4 shows that (3) is proved.

Step 2. We claim that

$$\lim_{n\to\infty} d(x_n, T_i x_n) = 0 \quad \text{and} \quad \lim_{n\to\infty} d(x_n, S_i x_n) = 0, \quad i = 1,2. \tag{9}$$

In fact, it follows from (3) that $\lim_{n\to\infty} d(x_n,p)$ exists for each given $p \in \mathcal{F}$. Without loss of generality, we assume that

$$\lim_{n\to\infty} d(x_n,p) = c \ge 0. \tag{10}$$

By (4) and (10), one has

$$\liminf_{n\to\infty} d(y_n,p) \le \limsup_{n\to\infty} d(y_n,p) \le \lim_{n\to\infty} \big\{(1+\nu_n M)d(x_n,p) + \mu_n\big\} = c. \tag{11}$$

Noting

$$\begin{aligned} d\big(T_1^n y_n,p\big) &= d\big(T_1^n y_n, T_1^n p\big) \\ &\le d(y_n,p) + \nu_n\zeta\big(d(y_n,p)\big) + \mu_n \\ &\le (1+\nu_n M)d(y_n,p) + \mu_n, \quad \forall n \ge 1, \end{aligned}$$

and

$$\begin{aligned} d\big(S_1^n x_n,p\big) &= d\big(S_1^n x_n, S_1^n p\big) \\ &\le d(x_n,p) + \nu_n\zeta\big(d(x_n,p)\big) + \mu_n \\ &\le (1+\nu_n M)d(x_n,p) + \mu_n, \quad \forall n \ge 1, \end{aligned}$$

by (10) and (11), one has

$$\limsup_{n\to\infty} d\bigl(T_1^n y_n, p\bigr) \le c \quad \text{and} \quad \limsup_{n\to\infty} d\bigl(S_1^n x_n, p\bigr) \le c. \tag{12}$$

Besides, by (6) one gets

$$d(x_{n+1}, p) = d\bigl(W\bigl(S_1^n x_n, T_1^n y_n, \alpha_n\bigr), p\bigr) \le (1+\sigma_n) d(x_n, p) + \xi_n,$$

which yields that

$$\lim_{n\to\infty} d\bigl(W\bigl(S_1^n x_n, T_1^n y_n, \alpha_n\bigr), p\bigr) = c. \tag{13}$$

Now, by (12), (13) and Lemma 2, we have

$$\lim_{n\to\infty} d\bigl(S_1^n x_n, T_1^n y_n\bigr) = 0. \tag{14}$$

Using the same method, we can also have that

$$\lim_{n\to\infty} d\bigl(S_2^n x_n, T_2^n x_n\bigr) = 0. \tag{15}$$

It follows from (14), (15) and condition (iv) that

$$\lim_{n\to\infty} d\bigl(x_n, T_1^n y_n\bigr) \le \lim_{n\to\infty} d\bigl(S_1^n x_n, T_1^n y_n\bigr) = 0 \tag{16}$$

and

$$\lim_{n\to\infty} d\bigl(x_n, T_2^n x_n\bigr) \le \lim_{n\to\infty} d\bigl(S_2^n x_n, T_2^n x_n\bigr) = 0. \tag{17}$$

By virtue of (15), one has

$$\begin{aligned} d\bigl(y_n, S_2^n x_n\bigr) &= d\bigl(W\bigl(S_2^n x_n, T_2^n x_n, \beta_n\bigr), S_2^n x_n\bigr) \\ &\le \beta_n d\bigl(T_2^n x_n, S_2^n x_n\bigr) \to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{18}$$

Because we have

$$d(x_n, y_n) \le d\bigl(x_n, T_2^n x_n\bigr) + d\bigl(T_2^n x_n, S_2^n x_n\bigr) + d\bigl(S_2^n x_n, y_n\bigr),$$

it follows from (15), (17) and (18) that

$$\lim_{n\to\infty} d(x_n, y_n) = 0. \tag{19}$$

Combining (16) and (19), one obtains

$$\begin{aligned} d\bigl(x_n, T_1^n x_n\bigr) &\le d\bigl(x_n, T_1^n y_n\bigr) + d\bigl(T_1^n y_n, T_1^n x_n\bigr) \\ &\le d\bigl(x_n, T_1^n y_n\bigr) + d(x_n, y_n) + \nu_n \zeta\bigl(d(x_n, y_n)\bigr) + \mu_n \\ &\le d\bigl(x_n, T_1^n y_n\bigr) + (1 + \nu_n M) d(x_n, y_n) + \mu_n \to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{20}$$

Moreover, it follows from (16) and (19) that

$$\begin{aligned} d\big(S_1^n x_n, T_1^n x_n\big) &\le d\big(S_1^n x_n, T_1^n y_n\big) + d\big(T_1^n y_n, T_1^n x_n\big) \\ &\le d\big(S_1^n x_n, T_1^n y_n\big) + Ld(y_n, x_n) \to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{21}$$

This jointly with (16) and (20) yields that

$$d\big(S_1^n x_n, x_n\big) \le d\big(S_1^n x_n, T_1^n x_n\big) + d\big(T_1^n x_n, x_n\big) \to 0 \quad \text{as } n \to \infty,$$

and

$$\begin{aligned} d(x_{n+1}, x_n) &\le d\big(W\big(S_1^n x_n, T_1^n y_n, \alpha_n\big), x_n\big) \\ &\le (1-\alpha_n) d\big(S_1^n x_n, x_n\big) + \alpha_n d\big(T_1^n y_n, x_n\big) \to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{22}$$

Now by (17), (20) and (22), for each $i = 1, 2$, one gets

$$\begin{aligned} d(x_n, T_i x_n) &\le d(x_n, x_{n+1}) + d\big(x_{n+1}, T_i^{n+1} x_{n+1}\big) + d\big(T_i^{n+1} x_{n+1}, T_i^{n+1} x_n\big) \\ &\quad + d\big(T_i^{n+1} x_n, T_i x_n\big) \\ &\le (1+L) d(x_n, x_{n+1}) + d\big(x_{n+1}, T_i^{n+1} x_{n+1}\big) \\ &\quad + Ld\big(T_i^n x_n, x_n\big) \to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{23}$$

For each $i = 1, 2$, the combination of (15), (17), (20), (21) and (iv) yields that

$$\begin{aligned} d(x_n, S_i x_n) &\le d\big(x_n, T_i^n x_n\big) + d\big(S_i x_n, T_i^n x_n\big) \\ &\le d\big(x_n, T_i^n x_n\big) + d\big(S_i^n x_n, T_i^n x_n\big) \to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{24}$$

Therefore, (9) is proved.

Step 3. Now we are in a position to prove the $\triangle$-convergence of $\{x_n\}$. Since $\{x_n\}$ is bounded, by Lemma 1, it has a unique asymptotic center $A_C(\{x_n\}) = \{x^*\}$. Let $\{u_n\}$ be any subsequence of $\{x_n\}$ with $A_C(\{u_n\}) = \{u\}$, then by (23) and (24), for each $i = 1, 2$, we have

$$\lim_{n\to\infty} d(u_n, T_i u_n) = 0 \quad \text{and} \quad d(u_n, S_i u_n) = 0. \tag{25}$$

We claim that $u \in \mathcal{F}$. In fact, we define a sequence $\{z_m\}$ in C by $z_m = T_1^m u$. Then one has

$$\begin{aligned} d(z_m, u_n) &\le d\big(T_1^m u, T_1^m u_n\big) + d\big(T_1^m u_n, T_1^{m-1} u_n\big) + \cdots + d(T_1 u_n, u_n) \\ &\le d(u, u_n) + \nu_n \zeta\big(d(u, u_n)\big) + \mu_n + Ld(T_1 u_n, u_n) + \cdots + d(T_1 u_n, u_n). \end{aligned}$$

By (25), one gets

$$\limsup_{n\to\infty} d(z_m, u_n) \le \limsup_{n\to\infty} d(u, u_n) = r\big(u, \{u_n\}\big),$$

which yields that

$$\big|r\big(z_m, \{u_n\}\big) - r\big(u, \{u_n\}\big)\big| \to 0 \quad \text{as } m \to \infty.$$

Lemma 3 shows that $\lim_{m\to\infty} T_1^m u = u$. Because T_1 is uniformly continuous, we have

$$T_1 u = T_1\Big(\lim_{m\to\infty} T_1^m u\Big) = \lim_{m\to\infty} T_1^{m+1} u = u.$$

Hence, $u \in F(T_1)$. Using the same method, we can prove that $u \in \mathcal{F}$. By the uniqueness of asymptotic centers, we get that $x^* = u$. It implies that x^* is the unique asymptotic center of $\{u_n\}$ for each subsequence $\{u_n\}$ of $\{x_n\}$, that is, $\{x_n\}$ $\triangle$-converges to $x^* \in \mathcal{F}$. The proof is completed. □

Competing interests

The author declares that they have no competing interests.

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