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On asymptotically double lacunary statistical equivalent sequences in ideal context

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Abstract

An ideal $\mathcal{I}$ is a family of subsets of positive integers $\mathbb{N} \times \mathbb{N}$ which is closed under taking finite unions and subsets of its elements. In this paper, we present some definitions which are a natural combination of the definition of asymptotic equivalence, statistical convergence, lacunary statistical convergence, double sequences and an ideal. In addition, we also present asymptotically $\mathcal{I}$-equivalent double sequences and study some properties of this concept.

MSC: 40A35; 40G15

Keywords: asymptotic equivalence; statistical convergence; lacunary statistical convergence; ideal; ideal convergence; double sequence

1 Introduction

Pobyvancts [1] introduced the concept of asymptotically regular matrices which preserve the asymptotic equivalence of two nonnegative numbers sequences. Marouf [2] presented definitions for asymptotically equivalent and asymptotic regular matrices. Patterson [3] extended these concepts by presenting an asymptotically statistical equivalent analog of these definitions and natural regularity conditions for nonnegative summability matrices. Patterson and Savaş [4] introduced the concept of an asymptotically lacunary statistical equivalent sequences of real numbers.

The idea of statistical convergence was formerly given under the name 'almost convergence' by Zygmund in the first edition of his celebrated monograph published in Warsaw in 1935 [5]. The concept was formally introduced by Steinhaus [6] and Fast [7], and later, it was introduced by Schoenberg [8] and also independently by Buck [9]. A lot of developments have been made in this area after the works of Šalát [10] and Fridy [11]. Over the years and under different names, statistical convergence has been discussed in the theory of Fourier analysis, ergodic theory and number theory. Fridy and Orhan [12] introduced the concept of lacunary statistical convergence. Mursaleen and Mohiuddine [13] introduced the concept of lacunary statistical convergence with respect to the intuitionistic fuzzy normed space. Savaş and Patterson [14, 15] introduced the concept of lacunary statistical convergence for double sequences. Recently Mohiuddine *et al.* [16] introduced statistical convergence of double sequences in locally solid Riesz spaces. For details related to lacunary statistical convergence, we refer to [12, 17–24].

Kostyrko *et al.* [25] introduced the notion of *I*-convergence with the help of an admissible ideal, *I* denotes the ideal of subsets of $\mathbb{N}$, which is a generalization of statistical convergence. Quite recently, Das *et al.* [26] unified these two approaches to introduce new

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concepts of *I*-statistical convergence, *I*-lacunary statistical convergence and investigated some of their consequences. The notion of lacunary ideal convergence of real sequences was introduced in [27, 28]. Hazarika [29, 30] introduced the lacunary ideal convergent sequences of fuzzy real numbers and studied some basic properties of this notion. Kumar and Sharma [31] studied asymptotically generalized statistical equivalent sequences using ideals. Recently Savas [32] and Savas and Gumus [33] studied ideal asymptotically lacunary statistical equivalent single sequences. For more applications of ideals, we refer to [34–47].

In this paper, we define asymptotically lacunary statistical equivalent double sequences using an ideal and establish some basic results regarding this notion.

2 Definitions and preliminaries

In this section, we recall some definitions and notations, which form the base for the present study.

A family of sets $I \subset P(\mathbb{N})$ (power sets of $\mathbb{N}$) is called an *ideal* if and only if for each $A, B \in I$, we have $A \cup B \in I$, and for each $A \in I$ and each $B \subset A$, we have $B \in I$. A non-empty family of sets $\mathcal{F} \subset P(\mathbb{N})$ is a *filter* on $\mathbb{N}$ if and only if $\phi \notin \mathcal{F}$, for each $A, B \in \mathcal{F}$, we have $A \cap B \in \mathcal{F}$ and each $A \in \mathcal{F}$ and each $B \supset A$, we have $B \in \mathcal{F}$. An ideal I is called non-trivial ideal if $I \neq \phi$ and $\mathbb{N} \notin I$. Clearly, $I \subset P(\mathbb{N})$ is a non-trivial ideal if and only if $\mathcal{F} = \mathcal{F}(I) = \{\mathbb{N} - A : A \in I\}$ is a filter on $\mathbb{N}$. A non-trivial ideal $I \subset P(\mathbb{N})$ is called *admissible* if and only if $\{\{x\} : x \in \mathbb{N}\} \subset I$. A non-trivial ideal I is *maximal* if there cannot exists any non-trivial ideal $J \neq I$ containing I as a subset. Further details on ideals of $P(\mathbb{N})$ can be found in Kostyrko *et al.* [25]. Recall that a sequence $x = (x_k)$ of points in $\mathbb{R}$ is said to be *I*-convergent to a real number ℓ if $\{k \in \mathbb{N} : |x_k - \ell| \geq \varepsilon\} \in I$ for every $\varepsilon > 0$ [25]. In this case, we write $I - \lim x_k = \ell$.

By a lacunary sequence $\theta = (k_r)$, where $k_0 = 0$, we mean an increasing sequence of non-negative integers with $h_r := k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ will be denoted by $J_r := (k_{r-1}, k_r]$, and the ratio $\frac{k_r}{k_{r-1}}$ will be defined by q_r (see [48]).

The notion of statistical convergence depends on the density (asymptotic or natural) of subsets of $\mathbb{N}$. A subset of $\mathbb{N}$ is said to have natural density $\delta(E)$ if

$$\delta(E) = \lim_{n \to \infty} \frac{1}{n} \big| \{k \leq n : k \in E\} \big| \text{ exists.}$$

Definition 2.1 A real or complex number sequence $x = (x_k)$ is said to be *statistically convergent* to L if for every $\varepsilon > 0$,

$$\lim_n \frac{1}{n} \big| \big\{ k \leq n : |x_k - L| \geq \varepsilon \big\} \big| = 0.$$

In this case, we write $S - \lim x = L$ or $x_k \to L(S)$, and S denotes the set of all statistically convergent sequences.

Definition 2.2 [12] A sequence $x = (x_k)$ is said to be *lacunary statistically convergent* to the number L if for every $\varepsilon > 0$,

$$\lim_{r \to \infty} \frac{1}{h_r} \big| \big\{ k \in J_r : |x_k - L| \geq \varepsilon \big\} \big| = 0.$$

Let S_θ denote the set of all lacunary statistically convergent sequences. If $\theta = (2^r)$, then S_θ is the same as S.

Definition 2.3 [26] Let $I \subset P(\mathbb{N})$ be a non-trivial ideal. A sequence (x_k) is *I-statistically convergent* to L if for each $\varepsilon > 0$ and $\delta > 0$,

$$\left\{ n \in \mathbb{N} : \frac{1}{n} \left| \left\{ k \leq n : |x_k - L| \geq \varepsilon \right\} \right| \geq \delta \right\} \in I.$$

In this case, we write $I(S) - \lim x_k = L$.

Definition 2.4 [26] Let $I \subset P(\mathbb{N})$ be a non-trivial ideal. A sequence (x_k) is said to be *I-lacunary statistically convergent* to L if for each $\varepsilon > 0$ and $\delta > 0$,

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \left| \left\{ k \in J_r : |x_k - L| \geq \varepsilon \right\} \right| \geq \delta \right\} \in I.$$

In this case, we write $I(S_\theta) - \lim x_k = L$. If $\theta = (2^r)$, then $I(S_\theta)$ is the same as $I(S)$.

Definition 2.5 [2] Two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be *asymptotically equivalent* if

$$\lim_k \frac{x_k}{y_k} = 1,$$

denoted by $x \sim y$.

Definition 2.6 [3] Two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be *asymptotically statistical equivalent* of multiple L provided that for every $\varepsilon > 0$,

$$\lim_n \frac{1}{n} \left| \left\{ k \leq n : \left| \frac{x_k}{y_k} - L \right| \geq \varepsilon \right\} \right| = 0,$$

denoted by $x \sim^{S^L} y$ and simply asymptotically statistical equivalent if $L = 1$.

Patterson and Savas [4] defined the asymptotically lacunary statistical equivalent sequences as follows.

Definition 2.7 Two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be *asymptotically lacunary statistical equivalent* of multiple L provided that for every $\varepsilon > 0$,

$$\lim_r \frac{1}{h_r} \left| \left\{ k \in J_r : \left| \frac{x_k}{y_k} - L \right| \geq \varepsilon \right\} \right| = 0$$

denoted by $x \sim^{S_\theta^L} y$ and simply asymptotically lacunary statistical equivalent if $L = 1$. If we take $\theta = (2^r)$, then we get Definition 2.6.

By the convergence of a double sequence, we mean the convergence in Pringsheim's sense [49]. A double sequence $x = (x_{k,l})$ has a *Pringsheim* limit L (denoted by $P - \lim x = L$) provided that given an $\varepsilon > 0$, there exists an $n \in \mathbb{N}$ such that $|x_{k,l} - L| < \varepsilon$, whenever $k, l > n$. We describe such an $x = (x_{k,l})$ more briefly as '*P-convergent*'. We denote the space of all *P*-convergent sequences by c^2. The double sequence $x = (x_{k,l})$ is bounded if there exists

a positive number M such that $|x_{k,l}| < M$ for all k and l. We denote all bounded double sequences by l_∞^2.

Let $K \subset \mathbb{N} \times \mathbb{N}$ and $K(m,n)$ denote the number of (i,j) in K such that $i \le m$ and $j \le n$ (see [50]). Then the lower natural density of K is defined by $\underline{\delta}_2(K) = \liminf_{m,n\to\infty} \frac{K(m,n)}{mn}$. In case the sequence $(\frac{K(m,n)}{mn})$ has a limit in Pringsheim's sense, then we say that K has a *double natural density* and is defined by $P - \lim_{m,n\to\infty} \frac{K(m,n)}{mn} = \delta_2(K)$.

For example, let $K = \{(i^2, j^2) : i,j \in \mathbb{N}\}$. Then

$$\delta_2(K) = P - \lim_{m,n\to\infty} \frac{K(m,n)}{mn} \le P - \lim_{m,n\to\infty} \frac{\sqrt{m}\sqrt{n}}{mn} = 0,$$

i.e., the set K has double natural density zero, while the set $\{(i,3j) : i,j \in \mathbb{N}\}$ has double natural density $\frac{1}{3}$.

Definition 2.8 [50] A real double sequence $x = (x_{k,l})$ is said to be *P-statistically convergent* to ℓ provided that for each $\varepsilon > 0$,

$$P - \lim_{mn} \frac{1}{mn} \left|\left\{(k,l) : k < m \text{ and } l < n, |x_{k,l} - \ell| \ge \varepsilon\right\}\right| = 0.$$

We denote the set of all statistical convergent double sequences by $\mathcal{S}$.

The double sequence $\overline{\theta} = \theta_{r,s} = \{(k_r, l_s)\}$ is called *double lacunary sequence* if there exist two increasing sequences of integers such that (see [15])

$$k_o = 0, \qquad h_r = k_r - k_{r-1} \to \infty \quad \text{as } r \to \infty$$

and

$$l_o = 0, \qquad \overline{h}_s = l_s - l_{s-1} \to \infty \quad \text{as } s \to \infty.$$

Notations: $k_{r,s} = k_r l_s$, $h_{r,s} = h_r \overline{h}_s$ and $\theta_{r,s}$ is determined by

$$J_{r,s} = \left\{(k,l) : k_{r-1} < k \le k_r \text{ and } l_{s-1} < l \le l_s\right\},$$

$$q_r = \frac{k_r}{k_{r-1}}, \qquad \overline{q}_s = \frac{l_s}{l_{s-1}} \quad \text{and} \quad q_{r,s} = q_r \overline{q}_s.$$

Definition 2.9 A double sequence $(x_{k,l})$ is said to be *double lacunary convergent* to L if

$$P - \lim_{r,s} \frac{1}{h_{r,s}} \sum_{(k,l)\in J_{r,s}} x_{k,l} = L.$$

In this case, we write $\overline{\theta} - \lim_{k,l} x_{k,l} = L$. We denote $N_{\overline{\theta}}$ the set of all double lacunary convergent sequences.

Definition 2.10 [51] Let $\mathcal{I} \subset P(\mathbb{N} \times \mathbb{N})$ be a non-trivial ideal. A double sequence $(x_{k,l})$ is said to be *$\mathcal{I}$-convergent* to L if for each $\varepsilon > 0$,

$$\left\{(k,l) \in \mathbb{N} \times \mathbb{N} : |x_{k,l} - L| \ge \varepsilon\right\} \in \mathcal{I}.$$

In this case, we write $\mathcal{I} - \lim x_{k,l} = L$.

Throughout the paper, we denote $\mathcal{I}$ as admissible ideal of subsets of $\mathbb{N} \times \mathbb{N}$, unless otherwise stated.

3 Asymptotically lacunary statistical equivalent double sequences using ideals

In this section, we define asymptotically $\mathcal{I}$-equivalent, asymptotically $\mathcal{I}$-statistical equivalent, asymptotically $\mathcal{I}$-lacunary statistical equivalent and asymptotically lacunary $\mathcal{I}$-equivalent double sequences and obtain some analogous results from these new definitions point of views.

Definition 3.1 Let $\mathcal{I} \subset P(\mathbb{N} \times \mathbb{N})$ be a non-trivial ideal. A double sequence $(x_{k,l})$ is said to be $\mathcal{I}$-*statistically convergent* to L if for each $\varepsilon > 0$ and $\delta > 0$,

$$\left\{ (m, n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} \left| \left\{ k \leq m; l \leq n : |x_{k,l} - L| \geq \varepsilon \right\} \right| \geq \delta \right\} \in \mathcal{I}.$$

In this case, we write $\mathcal{I}(\mathcal{S}) - \lim x_{k,l} = L$.

Definition 3.2 Let $\mathcal{I} \subset P(\mathbb{N} \times \mathbb{N})$ be a non-trivial ideal. A double sequence $(x_{k,l})$ is said to be $\mathcal{I}$-*lacunary statistically convergent* to L if for each $\varepsilon > 0$ and $\delta > 0$,

$$\left\{ (r, s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{r,s}} \left| \left\{ (k, l) \in J_{r,s} : |x_{k,l} - L| \geq \varepsilon \right\} \right| \geq \delta \right\} \in \mathcal{I}.$$

In this case, we write $\mathcal{I}(\mathcal{S}_{\bar{\theta}}) - \lim x_{k,l} = L$.

Definition 3.3 Two nonnegative double sequences $x = (x_{k,l})$ and $y = (y_{k,l})$ are said to be *P-asymptotically equivalent* if

$$P - \lim_{k,l} \frac{x_{k,l}}{y_{k,l}} = 1,$$

denoted by $x \sim^{P} y$.

Definition 3.4 Two nonnegative double sequences $x = (x_{k,l})$ and $y = (y_{k,l})$ are said to be *asymptotically statistical equivalent* of multiple L provided that for every $\varepsilon > 0$

$$P - \lim_{m,n} \frac{1}{mn} \left| \left\{ k \leq m, l \leq n : \left| \frac{x_{k,l}}{y_{k,l}} - L \right| \geq \varepsilon \right\} \right| = 0,$$

denoted by $x \sim^{\mathcal{S}^{L}} y$ and simply asymptotically statistical equivalent if $L = 1$.

Definition 3.5 Two nonnegative double sequences $x = (x_{k,l})$ and $y = (y_{k,l})$ are said to be *asymptotically lacunary statistical equivalent* of multiple L provided that for every $\varepsilon > 0$,

$$P - \lim_{r,s} \frac{1}{h_{r,s}} \left| \left\{ (k, l) \in J_{r,s} : \left| \frac{x_{k,l}}{y_{k,l}} - L \right| \geq \varepsilon \right\} \right| = 0$$

denoted by $x \sim^{\mathcal{S}^{L}_{\bar{\theta}}} y$ and simply asymptotically lacunary statistical equivalent if $L = 1$. If we take $\bar{\theta} = (2^{r}, 2^{s})$, then we get Definition 3.4.

Definition 3.6 Two non-negative double sequences $(x_{k,l})$ and $(y_{k,l})$ are said to be *asymptotically $\mathcal{I}$-equivalent* of multiple L provided that for every $\varepsilon > 0$,

$$\left\{(k,l) \in \mathbb{N} \times \mathbb{N} : \left|\frac{x_{k,l}}{y_{k,l}} - L\right| \geq \varepsilon\right\} \in \mathcal{I},$$

denoted by $(x_{k,l}) \sim^{\mathcal{I}^L} (y_{k,l})$ and simply asymptotically $\mathcal{I}$-equivalent if $L = 1$.

Lemma 3.1 *Let $\mathcal{I} \subset P(\mathbb{N} \times \mathbb{N})$ be an admissible ideal. Let $(x_{k,l})$, $(y_{k,l})$ be two double sequences and $(x_{k,l}), (y_{k,l}) \in \ell_\infty^2$ with $\mathcal{I} - \lim_{k,l} x_{k,l} = 0 = \mathcal{I} - \lim_{k,l} y_{k,l}$ such that $(x_{k,l}) \sim^{\mathcal{I}^L} (y_{k,l})$. Then there exists a sequence $(z_{k,l}) \in \ell_\infty^2$ with $\mathcal{I} - \lim_{k,l} z_{k,l} = 0$ such that $(x_{k,l}) \sim^{\mathcal{I}^L} (z_{k,l}) \sim^{\mathcal{I}^L} (y_{k,l})$.*

Definition 3.7 Two non-negative double sequences $(x_{k,l})$ and $(y_{k,l})$ are said to be *asymptotically $\mathcal{I}$-statistically equivalent* of multiple L provided that for every $\varepsilon > 0$ and for every $\delta > 0$,

$$\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn}\left|\left\{k \leq m, l \leq n : \left|\frac{x_{k,l}}{y_{k,l}} - L\right| \geq \varepsilon\right\}\right| \geq \delta\right\} \in \mathcal{I},$$

denoted by $(x_{k,l}) \sim^{\mathcal{I}(\mathcal{S})^L} (y_{k,l})$ and simply asymptotically $\mathcal{I}$-statistical equivalent if $L = 1$.

Definition 3.8 Two non-negative double sequences $(x_{k,l})$ and $(y_{k,l})$ are said to be *Cesaro asymptotically $\mathcal{I}$-equivalent* (or $\mathcal{I}(\sigma_1)$-equivalent) of multiple L provided that for every $\delta > 0$,

$$\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \left|\frac{1}{mn}\sum_{k,l=1}^{m,n}\frac{x_{k,l}}{y_{k,l}} - L\right| \geq \delta\right\} \in \mathcal{I}$$

denoted by $(x_{k,l}) \sim^{\mathcal{I}(\sigma_1)^L} (y_{k,l})$ and simply asymptotically $\mathcal{I}(\sigma_1)$-equivalent if $L = 1$.

Definition 3.9 Two non-negative double sequences $(x_{k,l})$ and $(y_{k,l})$ are said to be *strongly Cesaro asymptotically $\mathcal{I}$-equivalent* (or $\mathcal{I}(|\sigma_1|)$-equivalent) of multiple L provided that for every $\delta > 0$,

$$\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn}\sum_{k,l=1}^{m,n}\left|\frac{x_{k,l}}{y_{k,l}} - L\right| \geq \delta\right\} \in \mathcal{I}$$

denoted by $(x_{k,l}) \sim^{\mathcal{I}(|\sigma_1|)^L} (y_{k,l})$ and simply strongly Cesaro asymptotically $\mathcal{I}(|\sigma_1|)$-equivalent if $L = 1$.

Definition 3.10 Two non-negative double sequences $(x_{k,l})$ and $(y_{k,l})$ are said to be *strongly asymptotically lacunary equivalent* of multiple L provided that

$$P - \lim_{r,s} \frac{1}{h_{r,s}} \sum_{(k,l) \in J_{r,s}} \left|\frac{x_{k,l}}{y_{k,l}} - L\right| = 0$$

denoted by $(x_{k,l}) \sim^{[N_{\bar{\theta}}]^L} (y_{k,l})$ and simply strongly asymptotically lacunary equivalent if $L = 1$.

Definition 3.11 Two non-negative double sequences $(x_{k,l})$ and $(y_{k,l})$ are said to be *asymptotically $\mathcal{I}$-lacunary equivalent* (or $\mathcal{I}(N_{\bar{\theta}})$-equivalent) of multiple L provided that for every $\delta > 0$,

$$\left\{ (r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{r,s}} \sum_{(k,l) \in J_{r,s}} \left| \frac{x_{k,l}}{y_{k,l}} - L \right| \geq \delta \right\} \in \mathcal{I}$$

denoted by $(x_{k,l}) \sim^{\mathcal{I}(N_{\bar{\theta}})^L} (y_{k,l})$ and simply asymptotically $\mathcal{I}(N_{\bar{\theta}})$-equivalent if $L = 1$.

Definition 3.12 Two non-negative double sequences $(x_{k,l})$ and $(y_{k,l})$ are said to be *asymptotically $\mathcal{I}$-lacunary statistically equivalent* (or $\mathcal{I}(\mathcal{S}_{\bar{\theta}})$-equivalent) of multiple L provided that for every $\varepsilon > 0$, for every $\delta > 0$,

$$\left\{ (r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{r,s}} \left| \left\{ (k,l) \in J_{r,s} : \left| \frac{x_{k,l}}{y_{k,l}} - L \right| \geq \varepsilon \right\} \right| \geq \delta \right\} \in \mathcal{I}$$

denoted by $(x_{k,l}) \sim^{\mathcal{I}(\mathcal{S}_{\bar{\theta}})^L} (y_{k,l})$ and simply asymptotically $\mathcal{I}(\mathcal{S}_{\bar{\theta}})$-equivalent if $L = 1$.

Theorem 3.1 *Let $\mathcal{I} \subset P(\mathbb{N} \times \mathbb{N})$ be a non-trivial ideal. Let $\bar{\theta} = \{(k_r, l_s)\}$ be a double lacunary sequence. If $(x_{k,l}), (y_{k,l}) \in \ell_\infty^2$ and $(x_{k,l}) \sim^{\mathcal{I}(\mathcal{S})^L} (y_{k,l})$. Then $(x_{k,l}) \sim^{\mathcal{I}(\sigma_1)^L} (y_{k,l})$.*

Proof (a) Suppose that $(x_{k,l}), (y_{k,l}) \in \ell_\infty^2$ and $(x_{k,l}) \sim^{\mathcal{I}(\mathcal{S})^L} (y_{k,l})$. Then we can assume that

$$\left| \frac{x_{k,l}}{y_{k,l}} - L \right| \leq M \quad \text{for almost all } k, l.$$

Let $\varepsilon > 0$. Then we have

$$\left| \frac{1}{mn} \sum_{k,l=1}^{m,n} \left(\frac{x_{k,l}}{y_{k,l}} - L \right) \right| \leq \frac{1}{mn} \sum_{k,l=1}^{m,n} \left| \frac{x_{k,l}}{y_{k,l}} - L \right|$$

$$\leq \frac{1}{mn} \sum_{\substack{k,l; \\ |\frac{x_{k,l}}{y_{k,l}} - L| \geq \varepsilon}} \left| \frac{x_{k,l}}{y_{k,l}} - L \right| + \frac{1}{mn} \sum_{\substack{k,l; \\ |\frac{x_{k,l}}{y_{k,l}} - L| < \varepsilon}} \left| \frac{x_{k,l}}{y_{k,l}} - L \right|$$

$$\leq M \cdot \frac{1}{mn} \left| \left\{ k \leq m; l \leq n : \left| \frac{x_{k,l}}{y_{k,l}} - L \right| \geq \varepsilon \right\} \right| + \frac{1}{mn} \cdot mn \cdot \varepsilon.$$

Consequently, if $\delta > \varepsilon > 0$, δ and ε are independent, put $\delta_1 = \delta - \varepsilon > 0$, we have

$$\left\{ (m,n) \in \mathbb{N} \times \mathbb{N} : \left| \frac{1}{mn} \sum_{k,l=1}^{m,n} \left(\frac{x_{k,l}}{y_{k,l}} - L \right) \right| \geq \delta \right\}$$

$$\subseteq \left\{ (m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} \left| \left\{ k \leq m; l \leq n : \left| \frac{x_{k,l}}{y_{k,l}} - L \right| \geq \varepsilon \right\} \right| \geq \frac{\delta_1}{M} \right\} \in \mathcal{I}.$$

This shows that $(x_{k,l}) \sim^{\mathcal{I}(\sigma_1)^L} (y_{k,l})$. □

Corollary 3.1 *Let $\mathcal{I} \subset P(\mathbb{N} \times \mathbb{N})$ be a non-trivial ideal. If $(x_{k,l}), (y_{k,l}) \in \ell_\infty^2$ and $(x_{k,l}) \sim^{\mathcal{I}(\mathcal{S})^L} (y_{k,l})$. Then $(x_{k,l}) \sim^{\mathcal{I}(|\sigma_1|)^L} (y_{k,l})$.*

Theorem 3.2 *Let $\mathcal{I} \subset P(\mathbb{N} \times \mathbb{N})$ be a non-trivial ideal. Let $\bar{\theta} = \{(k_r, l_s)\}$ be a double lacunary sequence. Then*

(a) *$(x_{k,l}) \sim^{\mathcal{I}(N_{\bar{\theta}})^L} (y_{k,l}) \Rightarrow (x_{k,l}) \sim^{\mathcal{I}(\mathcal{S}_{\bar{\theta}})^L} (y_{k,l})$.*

(b) *$\mathcal{I}(N_{\bar{\theta}})^L$ is a proper subset of $\mathcal{I}(\mathcal{S}_{\bar{\theta}})^L$.*

(c) *Let $(x_{k,l}), (y_{k,l}) \in \ell_\infty^2$ and $(x_{k,l}) \sim^{\mathcal{I}(\mathcal{S}_{\bar{\theta}})^L} (y_{k,l})$, then $(x_{k,l}) \sim^{\mathcal{I}(N_{\bar{\theta}})^L} (y_{k,l})$.*

(d) *$\mathcal{I}(\mathcal{S}_{\bar{\theta}})^L \cap \ell_\infty^2 = \mathcal{I}(N_{\bar{\theta}})^L \cap \ell_\infty^2$.*

Proof (a) Let $\varepsilon > 0$ and $(x_{k,l}) \sim^{\mathcal{I}(N_{\bar{\theta}})^L} (y_{k,l})$. Then we can write

$$
\sum_{(k,l) \in J_{r,s}} \left| \frac{x_{k,l}}{y_{k,l}} - L \right| \geq \sum_{\substack{(k,l) \in J_{r,s} \\ |\frac{x_{k,l}}{y_{k,l}} - L| \geq \varepsilon}} \left| \frac{x_{k,l}}{y_{k,l}} - L \right|
$$

$$
\geq \varepsilon \left| \left\{ (k,l) \in J_{r,s} : \left| \frac{x_{k,l}}{y_{k,l}} - L \right| \geq \varepsilon \right\} \right|
$$

$$
\Rightarrow \quad \frac{1}{\varepsilon \cdot h_{r,s}} \sum_{(k,l) \in J_{r,s}} \left| \frac{x_{k,l}}{y_{k,l}} - L \right| \geq \frac{1}{h_{r,s}} \left| \left\{ (k,l) \in J_{r,s} : \left| \frac{x_{k,l}}{y_{k,l}} - L \right| \geq \varepsilon \right\} \right|.
$$

Thus, for any $\delta > 0$,

$$
\frac{1}{h_{r,s}} \left| \left\{ (k,l) \in J_{r,s} : \left| \frac{x_{k,l}}{y_{k,l}} - L \right| \geq \varepsilon \right\} \right| \geq \delta
$$

implies that

$$
\frac{1}{h_{r,s}} \sum_{(k,l) \in J_{r,s}} \left| \frac{x_{k,l}}{y_{k,l}} - L \right| \geq \varepsilon \delta.
$$

Therefore, we have

$$
\left\{ (r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{r,s}} \left| \left\{ (k,l) \in J_{r,s} : \left| \frac{x_{k,l}}{y_{k,l}} - L \right| \geq \varepsilon \right\} \right| \geq \delta \right\}
$$

$$
\subset \left\{ (r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{r,s}} \sum_{(k,l) \in J_{r,s}} \left| \frac{x_{k,l}}{y_{k,l}} - L \right| \geq \varepsilon \delta \right\}.
$$

Since $(x_{k,l}) \sim^{\mathcal{I}(N_{\bar{\theta}})^L} (y_{k,l})$, so that

$$
\left\{ (r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{r,s}} \sum_{(k,l) \in J_{r,s}} \left| \frac{x_{k,l}}{y_{k,l}} - L \right| \geq \varepsilon \delta \right\} \in \mathcal{I},
$$

which implies that

$$
\left\{ (r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{r,s}} \left| \left\{ (k,l) \in J_{r,s} : \left| \frac{x_{k,l}}{y_{k,l}} - L \right| \geq \varepsilon \right\} \right| \geq \delta \right\} \in \mathcal{I}.
$$

This shows that $(x_{k,l}) \sim^{\mathcal{I}(\mathcal{S}_{\bar{\theta}})^L} (y_{k,l})$.

(b) Suppose that $\mathcal{I}(N_{\bar{\theta}})^L \subset \mathcal{I}(\mathcal{S}_{\bar{\theta}})^L$. Let $(x_{k,l})$ and $(y_{k,l})$ be two sequences defined as follows:

$$x_{k,l} = \begin{cases} kl, & \text{if } k_{r-1} < k \le k_{r-1} + [\sqrt{h_r}], l_{s-1} < l \le l_{s-1} + [\sqrt{h_s}], r,s = 1,2,3,\ldots; \\ 0, & \text{otherwise}, \end{cases}$$

and

$$y_{k,l} = 1 \quad \text{for all } k,l \in \mathbb{N}.$$

It is clear that $(x_{k,l}) \notin \ell_\infty^2$, and for $\varepsilon > 0$,

$$\frac{1}{h_{r,s}} \left| \left\{ (k,l) \in J_{r,s} : \left| \frac{x_{k,l}}{y_{k,l}} - 1 \right| \ge \varepsilon \right\} \right| \le \frac{[\sqrt{h_{r,s}}]}{h_{r,s}} \quad \text{and} \quad \frac{[\sqrt{h_{r,s}}]}{h_{r,s}} \to 0 \quad \text{as } r,s \to \infty. \quad (3.1)$$

This implies that

$$\left\{ (r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{r,s}} \left| \left\{ (k,l) \in J_{r,s} : \left| \frac{x_{k,l}}{y_{k,l}} - 1 \right| \ge \varepsilon \right\} \right| \ge \delta \right\}$$

$$\subseteq \left\{ (r,s) \in \mathbb{N} \times \mathbb{N} : \frac{[\sqrt{h_{r,s}}]}{h_{r,s}} \ge \delta \right\}.$$

By virtue of last part of (3.1), the set on the right side is a finite set, and so it belongs to $\mathcal{I}$. Consequently, we have

$$\left\{ (r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{r,s}} \left| \left\{ (k,l) \in J_{r,s} : \left| \frac{x_{k,l}}{y_{k,l}} - 1 \right| \ge \varepsilon \right\} \right| \ge \delta \right\} \in \mathcal{I}.$$

Therefore, $(x_{k,l}) \sim^{\mathcal{I}(\mathcal{S}_{\bar{\theta}})^1} (y_{k,l})$.

On the other hand, we shall show that $(x_{k,l}) \sim^{\mathcal{I}(N_{\bar{\theta}})^1} (y_{k,l})$ is not satisfied. Suppose that $(x_{k,l}) \sim^{\mathcal{I}(N_{\bar{\theta}})^1} (y_{k,l})$. Then for every $\delta > 0$, we have

$$\left\{ (r,l) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{r,s}} \sum_{(k,l) \in J_{r,s}} \left| \frac{x_{k,l}}{y_{k,l}} - 1 \right| \ge \delta \right\} \in \mathcal{I}. \quad (3.2)$$

Now,

$$\lim_{r,s} \frac{1}{h_{r,s}} \sum_{(k,l) \in J_{r,s}} \left| \frac{x_{k,l}}{y_{k,l}} - 1 \right| = \frac{1}{h_{r,s}} \left(\frac{[\sqrt{h_{r,s}}]([\sqrt{h_{r,s}}] - 1)}{2} \right) \to \frac{1}{2} \quad \text{as } r,s \to \infty.$$

It follows for the particular choice $\delta = \frac{1}{4}$ that

$$\left\{ (r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{r,s}} \sum_{(k,l) \in J_{r,s}} \left| \frac{x_{k,l}}{y_{k,l}} - 1 \right| \ge \frac{1}{4} \right\}$$

$$= \left\{ (r,s) \in \mathbb{N} \times \mathbb{N} : \left(\frac{[\sqrt{h_{r,s}}]([\sqrt{h_{r,s}}] - 1)}{h_{r,s}} \right) \ge \frac{1}{2} \right\}$$

$$= \left\{ (m,n), (m+1,n+1), (m+2,n+2), \ldots \right\}$$

for some $m, n \in \mathbb{N}$ which belongs to $\mathcal{F}$ as $\mathcal{I}$ is admissible. This contradicts (3.2) for the choice $\delta = \frac{1}{4}$. Therefore, $(x_{k,l}) \not\sim^{\mathcal{I}(N_{\bar{\theta}})^1} (y_{k,l})$.

(c) Suppose that $(x_{k,l}) \sim^{\mathcal{I}(S_{\bar{\theta}})^L} (y_{k,l})$ and $(x_{k,l}), (y_{k,l}) \in \ell_\infty^2$. We assume that $|\frac{x_{k,l}}{y_{k,l}} - L| \leq M$ and for all $k, l \in \mathbb{N}$. Given $\varepsilon > 0$, we get

$$
\frac{1}{h_{r,s}} \sum_{(k,l) \in J_{r,s}} \left| \frac{x_{k,l}}{y_{k,l}} - L \right| = \frac{1}{h_{r,s}} \sum_{\substack{(k,l) \in J_{r,s} \\ |\frac{x_{k,l}}{y_{k,l}} - L| \geq \varepsilon}} \left| \frac{x_{k,l}}{y_{k,l}} - L \right| + \frac{1}{h_{r,s}} \sum_{\substack{(k,l) \in J_{r,s} \\ |\frac{x_{k,l}}{y_{k,l}} - L| < \varepsilon}} \left| \frac{x_{k,l}}{y_{k,l}} - L \right|
$$

$$
\leq \frac{M}{h_{r,s}} \left| \left\{ (k,l) \in J_{r,s} : \left| \frac{x_{k,l}}{y_{k,l}} - L \right| \geq \varepsilon \right\} \right| + \varepsilon.
$$

If we put

$$
A(\varepsilon) = \left\{ (r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{r,s}} \sum_{(k,l) \in J_{r,s}} \left| \frac{x_{k,l}}{y_{k,l}} - L \right| \geq \varepsilon \right\}
$$

and

$$
B(\varepsilon_1) = \left\{ (r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{r,s}} \left| \left\{ (k,l) \in J_{r,s} : \left| \frac{x_{k,l}}{y_{k,l}} - L \right| \geq \varepsilon \right\} \right| \geq \frac{\varepsilon_1}{M} \right\},
$$

where $\varepsilon_1 = \delta - \varepsilon > 0$, ($\delta$ and ε are independent), then we have $A(\varepsilon) \subset B(\varepsilon_1)$, and so $A(\varepsilon) \in \mathcal{I}$. This shows that $(x_{k,l}) \sim^{\mathcal{I}(N_{\bar{\theta}})^L} (y_{k,l})$.

(d) It follows from (a), (b) and (c). $\qquad \square$

Theorem 3.3 *Let $\mathcal{I} \subset P(\mathbb{N} \times \mathbb{N})$ be an admissible ideal. Suppose that for given $\delta > 0$ and every $\varepsilon > 0$ such that*

$$
\left\{ (m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} \left| \left\{ 0 \leq k \leq m-1; 0 \leq l \leq n-1 : \left| \frac{x_{k,l}}{y_{k,l}} - L \right| \geq \varepsilon \right\} \right| < \delta \right\} \in \mathcal{F},
$$

then $(x_{k,l}) \sim^{\mathcal{I}(S)^L} (y_{k,l})$.

Proof Let $\delta > 0$ be given. For every $\varepsilon > 0$, choose m_1, n_1 such that

$$
\frac{1}{mn} \left| \left\{ 0 \leq k \leq m-1; 0 \leq l \leq n-1 : \left| \frac{x_{k,l}}{y_{k,l}} - L \right| \geq \varepsilon \right\} \right|
$$
$$
< \frac{\delta}{2}, \quad \text{for all } m \geq m_1, n \geq n_1. \tag{3.3}
$$

It is sufficient to show that there exists m_2, n_2 such that for $m \geq m_2, n \geq n_2$,

$$
\frac{1}{mn} \left| \left\{ 0 \leq k \leq m-1; 0 \leq l \leq n-1 : \left| \frac{x_{k,l}}{y_{k,l}} - L \right| \geq \varepsilon \right\} \right| < \frac{\delta}{2}. \tag{3.4}
$$

Let $m_0 = \max\{m_1, m_2\}$; $n_0 = \max\{n_1, n_2\}$. The relation (3.3) will be true for $m > m_0, n > n_0$. If s_0, t_0 chosen fixed, then we get

$$
\left| \left\{ 0 \leq k \leq s_0-1; 0 \leq l \leq t_0-1 : \left| \frac{x_{k,l}}{y_{k,l}} - L \right| \geq \varepsilon \right\} \right| = M.
$$

Now, for $m > s_0$, $n > t_0$, we have

$$\frac{1}{mn}\left|\left\{0 \le k \le m-1; 0 \le l \le n-1 : \left|\frac{x_{k,l}}{y_{k,l}} - L\right| \ge \varepsilon\right\}\right|$$

$$\le \frac{1}{mn}\left|\left\{0 \le k \le s_0 - 1; 0 \le l \le t_0 - 1 : \left|\frac{x_{k,l}}{y_{k,l}} - L\right| \ge \varepsilon\right\}\right|$$

$$+ \frac{1}{mn}\left|\left\{s_0 \le k \le m-1; t_0 \le l \le n-1 : \left|\frac{x_{k,l}}{y_{k,l}} - L\right| \ge \varepsilon\right\}\right|$$

$$\le \frac{M}{mn} + \frac{1}{mn}\left|\left\{s_0 \le k \le m-1; t_0 \le l \le n-1 : \left|\frac{x_{k,l}}{y_{k,l}} - L\right| \ge \varepsilon\right\}\right| \le \frac{M}{mn} + \frac{\delta}{2}.$$

Thus, for sufficiently large n,

$$\frac{1}{mn}\left|\left\{s_0 \le k \le m-1; t_0 \le l \le n-1 : \left|\frac{x_{k,l}}{y_{k,l}} - L\right| \ge \varepsilon\right\}\right| \le \frac{M}{mn} + \frac{\delta}{2} < \delta.$$

This established the result. □

Theorem 3.4 *Let $\mathcal{I} \subset P(\mathbb{N} \times \mathbb{N})$ be a non-trivial ideal. Let $\bar{\theta} = (k_r, l_s)$ be a double lacunary sequence with $\liminf_{r,s} q_{r,s} > 1$. Then $(x_{k,l}) \sim^{\mathcal{I}(S)^L} (y_{k,l}) \Rightarrow (x_{k,l}) \sim^{\mathcal{I}(S_{\bar\theta})^L} (y_{k,l})$.*

Proof Suppose that $\liminf_{r,s} q_{r,s} > 1$, then there exists an $\alpha > 0$ such that $q_{r,s} \ge 1 + \alpha$ for sufficiently large r, s. Then we have

$$\frac{h_{r,s}}{k_r l_s} \ge \frac{\alpha}{1+\alpha}.$$

If $(x_{k,l}) \sim^{\mathcal{I}(S^L)} (y_{k,l})$, then for every $\varepsilon > 0$ and for sufficiently large r, s, we have

$$\frac{1}{k_r l_s}\left|\left\{k \le k_r; l \le l_s : \left|\frac{x_{k,l}}{y_{k,l}} - L\right| \ge \varepsilon\right\}\right|$$

$$\ge \frac{1}{k_r l_s}\left|\left\{(k,l) \in J_{r,s} : \left|\frac{x_{k,l}}{y_{k,l}} - L\right| \ge \varepsilon\right\}\right|$$

$$\ge \frac{\alpha}{1+\alpha} \cdot \frac{1}{h_{r,s}}\left|\left\{(k,l) \in J_{r,s} : \left|\frac{x_{k,l}}{y_{k,l}} - L\right| \ge \varepsilon\right\}\right|.$$

Therefore, for any $\delta > 0$, we have

$$\left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{r,s}}\left|\left\{(k,l) \in J_{r,s} : \left|\frac{x_{k,l}}{y_{k,l}} - L\right| \ge \varepsilon\right\}\right| \ge \delta\right\}$$

$$\subseteq \left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{k_r l_s}\left|\left\{k \le k_r; l \le l_s : \left|\frac{x_{k,l}}{y_{k,l}} - L\right| \ge \varepsilon\right\}\right| \ge \frac{\alpha\delta}{1+\alpha}\right\} \in \mathcal{I}.$$

This completes the proof. □

Theorem 3.5 *Let $\mathcal{I} = \mathcal{I}_{\text{fin}} = \{A \subset \mathbb{N} \times \mathbb{N} : A \text{ is a finite set}\}$ be a non-trivial ideal, and let $\bar{\theta} = (k_r, l_s)$ be a double lacunary sequence with $\limsup_{r,s} q_{r,s} < \infty$. Then $(x_{k,l}) \sim^{\mathcal{I}(S_{\bar\theta})^L} (y_{k,l}) \Rightarrow (x_{k,l}) \sim^{\mathcal{I}(S)^L} (y_{k,l})$.*

Proof If $\limsup_{r,s} q_{r,s} < \infty$. Then there exists a $K > 0$ such that $q_{r,s} < K$ for all $r, s \geq 1$. Let $(x_{k,l}) \sim^{\mathcal{I}(\mathcal{S}_{\bar{\theta}})^L} (y_{k,l})$. Then there exists $B > 0$ and $\varepsilon > 0$, we put

$$M_{r,s} = \frac{1}{h_{r,s}} \left| \left\{ (k,l) \in J_{r,s} : \left| \frac{x_{k,l}}{y_{k,l}} - L \right| \geq \varepsilon \right\} \right|.$$

Since $(x_{k,l}) \sim^{\mathcal{I}(\mathcal{S}_{\bar{\theta}})^L} (y_{k,l})$. Then for every $\varepsilon > 0$ and $\delta > 0$, we have

$$\left\{ (r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{r,s}} \left| \left\{ (k,l) \in J_{r,s} : \left| \frac{x_{k,l}}{y_{k,l}} - L \right| \geq \varepsilon \right\} \right| \geq \delta \right\}$$

$$= \left\{ (r,s) \in \mathbb{N} \times \mathbb{N} : \frac{M_{r,s}}{h_{r,s}} \geq \delta \right\} \in \mathcal{I},$$

and, therefore, it is a finite set. We choose integers $r_0, s_0 \in \mathbb{N}$ such that

$$\frac{M_{r,s}}{h_{r,s}} < \delta \quad \text{for all } r > r_0, s > s_0. \tag{3.5}$$

Let $M = \max\{M_{r,s} : 1 \leq r \leq r_0, 1 \leq s \leq s_0\}$ and m, n be two integers with satisfying $k_{r-1} < m \leq k_r$, $l_{s-1} < n \leq l_s$, then we have

$$\frac{1}{mn} \left| \left\{ k \leq m; l \leq n : \left| \frac{x_{k,l}}{y_{k,l}} - L \right| \geq \varepsilon \right\} \right|$$

$$\leq \frac{1}{k_{r-1}l_{s-1}} \left| \left\{ k \leq k_r; l \leq l_s : \left| \frac{x_{k,l}}{y_{k,l}} - L \right| \geq \varepsilon \right\} \right|$$

$$= \frac{1}{k_{r-1}l_{s-1}} \{M_{1,1} + M_{2,2} + \cdots + M_{r_0,s_0} + M_{r_{r_0+1,s_0+1}} + \cdots + M_{r,s}\}$$

$$\leq \frac{M}{k_{r-1}l_{s-1}} \cdot r_0 s_0 + \frac{1}{k_{r-1}l_{s-1}} \left\{ h_{r_0+1,s_0+1} \left(\frac{M_{r_0+1,s_0+1}}{h_{r_0+1,s_0+1}} \right) + \cdots + h_{r,s} \frac{M_{r,s}}{k_{r,s}} \right\}$$

$$\leq \frac{M}{k_{r-1}l_{s-1}} \cdot r_0 s_0 + \frac{1}{k_{r-1}l_{s-1}} \left(\sup_{r > r_0, s > s_0} \frac{M_{r,s}}{h_{r,s}} \right) \{h_{r_0+1,s_0+1} + \cdots + h_{r,s}\}$$

$$\leq \frac{M}{k_{r-1}l_{s-1}} \cdot r_0 s_0 + \delta \left(\frac{k_r l_s - k_{r_0} l_{s_0}}{k_{r-1}l_{s-1}} \right)$$

$$\leq \frac{M}{k_{r-1}l_{s-1}} \cdot r_0 s_0 + \delta \cdot q_{r,s} \leq \frac{M}{k_{r-1}l_{s-1}} \cdot r_0 s_0 + \delta K.$$

This completes the proof of the theorem. $\qquad \square$

Definition 3.13 Let $p \in (0, \infty)$. Two non-negative double sequences $(x_{k,l})$ and $(y_{k,l})$ are said to be *strongly asymptotically lacunary p-equivalent* of multiple L if

$$P - \lim_{r,s} \frac{1}{h_{r,s}} \sum_{(k,l) \in J_{r,s}} \left| \frac{x_{k,l}}{y_{k,l}} - L \right|^p = 0$$

denoted by $(x_{k,l}) \sim^{[N_{\bar{\theta}}]_p^L} (y_{k,l})$ and simply strongly asymptotically lacunary p-equivalent if $L = 1$.

Definition 3.14 Let $p \in (0, \infty)$. Two non-negative double sequences $(x_{k,l})$ and $(y_{k,l})$ are said to be *asymptotically lacunary p-statistically equivalent* of multiple L if for every $\varepsilon > 0$,

$$P - \lim_{r,s} \frac{1}{h_{r,s}} \left| \left\{ (k,l) \in J_{r,s} : \left| \frac{x_{k,l}}{y_{k,l}} - L \right|^p \geq \varepsilon \right\} \right| = 0$$

denoted by $(x_{k,l}) \sim^{S^L_{\bar{\theta}_p}} (y_{k,l})$ and simply asymptotically lacunary p-statistical equivalent if $L = 1$.

Theorem 3.6 Let $\bar{\theta} = (k_r, l_s)$ be a double lacunary sequence. Then

(a) $(x_{k,l}) \sim^{[N_{\bar{\theta}}]^L_p} (y_{k,l}) \Rightarrow (x_{k,l}) \sim^{S^L_{\bar{\theta}_p}} (y_{k,l})$.

(b) $[N_{\bar{\theta}}]^L_p$ is a proper subset of $S^L_{\bar{\theta}_p}$.

(c) Let $(x_{k,l}), (y_{k,l}) \in \ell^2_\infty$ and $(x_{kml}) \sim^{S^L_{\bar{\theta}_p}} (y_{k,l})$, then $(x_{k,l}) \sim^{[N_{\bar{\theta}}]^L_p} (y_{k,l})$.

(d) $S^L_{\bar{\theta}_p} \cap \ell^2_\infty = [N_{\bar{\theta}}]^L_p \cap \ell^2_\infty$.

The proof of the above theorem is similar to Theorem 3.2 for $I = I_{\text{fin}}$.

Definition 3.15 Let $p \in (0, \infty)$. We say that two non-negative double sequences $(x_{k,l})$ and $(y_{k,l})$ are *strongly asymptotically $\mathcal{I}$-lacunary p-equivalent* of multiple L if for every $\varepsilon > 0$,

$$\left\{ (r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{r,s}} \sum_{(k,l) \in J_{r,s}} \left| \frac{x_{k,l}}{y_{k,l}} - L \right|^p \geq \varepsilon \right\} \in \mathcal{I}$$

denoted by $(x_{k,l}) \sim^{\mathcal{I}(N_{\bar{\theta}})^L_p} (y_{k,l})$ and simply strongly asymptotically $\mathcal{I}$-lacunary p-equivalent if $L = 1$.

Definition 3.16 Let $p \in (0, \infty)$. We say that two non-negative double sequences $(x_{k,l})$ and $(y_{k,l})$ are *asymptotically $\mathcal{I}$-lacunary p-statistically equivalent* of multiple L if for every $\varepsilon > 0$, for every $\delta > 0$

$$\left\{ (r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{r,s}} \left| \left\{ (k,l) \in J_{r,s} : \left| \frac{x_{k,l}}{y_{k,l}} - L \right|^p \geq \varepsilon \right\} \right| \geq \delta \right\} \in \mathcal{I}$$

denoted by $(x_{k,l}) \sim^{\mathcal{I}(S_{\bar{\theta}_p})^L} (y_{k,l})$ and simply asymptotically $\mathcal{I}$-lacunary p-statistical equivalent if $L = 1$.

Theorem 3.7 Let $\mathcal{I} \subset P(\mathbb{N} \times \mathbb{N})$ be a non-trivial ideal, and let $\bar{\theta} = (k_r, l_s)$ be a double lacunary sequence. Then

(a) $(x_{k,l}) \sim^{\mathcal{I}(N_{\bar{\theta}_p})^L} (y_{k,l}) \Rightarrow (x_{k,l}) \sim^{\mathcal{I}(S_{\bar{\theta}_p})^L} (y_{k,l})$.

(b) $\mathcal{I}(N_{\bar{\theta}_p})^L$ is a proper subset of $\mathcal{I}(S_{\bar{\theta}_p})^L$.

(c) Let $(x_{k,l}), (y_{k,l}) \in \ell^2_\infty$ and $(x_{k,l}) \sim^{\mathcal{I}(S_{\bar{\theta}_p})^L} (y_{k,l})$, then $(x_k) \sim^{I(N_{\theta_p})^L} (y_k)$.

(d) $\mathcal{I}(S_{\bar{\theta}_p})^L \cap \ell^2_\infty = \mathcal{I}(N_{\bar{\theta}_p})^L \cap \ell^2_\infty$.

Proof The proof of the theorem follows from the proofs of the Theorems 3.2 and 3.6. □

For $\mathcal{I} = \mathcal{I}_{\text{fin}} = \{ A \subseteq \mathbb{N} \times \mathbb{N} : A \text{ is finite} \}$, this theorem reduces to Theorem 3.6.

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