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# An extension of Jensen's discrete inequality to partially convex functions

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**Abstract**

This paper deals with a new extension of Jensen's discrete inequality to a partially convex function $f$, which is defined on a real interval $\mathbb{I}$, convex on a subinterval $[a,b] \subset \mathbb{I}$, decreasing for $u \le c$ and increasing for $u \ge c$, where $c \in [a,b]$. Several relevant applications are given to show the effectiveness of the proposed partially convex function theorem.

**MSC:** 26D07; 26D10; 41A44

**Keywords:** Jensen's discrete inequality; partially convex function; increasing/decreasing function

## 1 Introduction

Let $\mathbf{x} = \{x_1, x_2, \ldots, x_n\}$ be a sequence of real numbers belonging to a given real interval $\mathbb{I}$, and let $\mathbf{p} = \{p_1, p_2, \ldots, p_n\}$ be a sequence of given positive weights associated to $\mathbf{x}$ and satisfying $p_1 + p_2 + \cdots + p_n = 1$. If $f$ is a convex function on $\mathbb{I}$, then

$$\sum_{i=1}^{n} p_i f(x_i) \ge f\left(\sum_{i=1}^{n} p_i x_i\right)$$

is the classical Jensen discrete inequality (see [1, 2]).

In [3], we extended the weighted Jensen discrete inequality to a half convex function $f$, defined on a real interval $\mathbb{I}$ and convex for $u \le s$ or $u \ge s$, where $s \in \mathbb{I}$.

**WHCF-Theorem** *Let $f$ be a function defined on a real interval $\mathbb{I}$ and convex for $u \le s$ or $u \ge s$, where $s \in \mathbb{I}$, and let $p_1, p_2, \ldots, p_n$ be positive real numbers such that*

$$p = \min\{p_1, p_2, \ldots, p_n\}, \qquad p_1 + p_2 + \cdots + p_n = 1.$$

*The inequality*

$$p_1 f(x_1) + p_2 f(x_2) + \cdots + p_n f(x_n) \ge f(s)$$

*holds for all $x_1, x_2, \ldots, x_n \in \mathbb{I}$ satisfying $p_1x_1 + p_2x_2 + \cdots + p_nx_n = s$ if and only if*

$$pf(x) + (1-p)f(y) \ge f(s)$$

*for all $x, y \in \mathbb{I}$ such that $px + (1-p)y = s$.*

For the particular case $p_1 = p_2 = \cdots = p_n = 1/n$, from the weighted half convex function theorem, we get the half convex function theorem (see [4, 5]).

**HCF-Theorem** *Let $f$ be a function defined on a real interval $\mathbb{I}$ and convex for $u \le s$ or $u \ge s$, where $s \in \mathbb{I}$. The inequality*

$$f(x_1) + f(x_2) + \cdots + f(x_n) \ge nf(s)$$

*holds for all $x_1, x_2, \ldots, x_n \in \mathbb{I}$ satisfying $x_1 + x_2 + \cdots + x_n = ns$ if and only if*

$$f(x) + (n-1)f(y) \ge nf(s)$$

*for all $x, y \in \mathbb{I}$ which satisfy $x + (n-1)y = ns$.*

Applying HCF-Theorem and WHCF-Theorem to the function $f$ defined by $f(u) = g(e^u)$ and replacing $s$ by $\ln r$, $x$ by $\ln x$, $y$ by $\ln y$, and each $x_i$ by $\ln a_i$ for $i = 1, 2, \ldots, n$, we get the following corollaries, respectively.

**HCF-Corollary** *Let $g$ be a function defined on a positive interval $\mathbb{I}$ such that the function $f$ defined by $f(u) = g(e^u)$ is convex for $e^u \le r$ or $e^u \ge r$, where $r \in \mathbb{I}$. The inequality*

$$g(a_1) + g(a_2) + \cdots + g(a_n) \ge ng(r)$$

*holds for all $a_1, a_2, \ldots, a_n \in \mathbb{I}$ satisfying $a_1 a_2 \cdots a_n = r^n$ if and only if*

$$g(a) + (n-1)g(b) \ge ng(r)$$

*for all $a, b \in \mathbb{I}$ which satisfy $ab^{n-1} = r^n$.*

**WHCF-Corollary** *Let $g$ be a function defined on a positive interval $\mathbb{I}$ such that the function $f$ defined by $f(u) = g(e^u)$ is convex for $e^u \le r$ or $e^u \ge r$, where $r \in \mathbb{I}$, and let $p_1, p_2, \ldots, p_n$ be positive real numbers such that*

$$p = \min\{p_1, p_2, \ldots, p_n\}, \qquad p_1 + p_2 + \cdots + p_n = 1.$$

*The inequality*

$$p_1 g(a_1) + p_2 g(a_2) + \cdots + p_n g(a_n) \ge g(r)$$

*holds for all $a_1, a_2, \ldots, a_n \in \mathbb{I}$ satisfying $a_1^{p_1} a_2^{p_2} \cdots a_n^{p_n} = r$ if and only if*

$$pg(a) + (1-p)g(b) \ge g(r)$$

*for all $a, b \in \mathbb{I}$ such that $a^p b^{1-p} = r$.*

In this paper, we will use HCF-Theorem and WHCF-Theorem to extend Jensen's inequality to partially convex functions, which are defined on a real interval $\mathbb{I}$ and convex only on a subinterval $[a, b] \subset \mathbb{I}$.

**Remark 1.1** Clearly, HCF-Theorem is a particular case of WHCF-Theorem. However, we posted here the both theorems because HCF-Theorem is much more useful to prove many inequalities of extended Jensen type.

**Remark 1.2** Actually, in HCF-Theorem and WHCF-Theorem, it suffices to consider that

$$x \geq s \geq y$$

when $f$ is convex for $u \leq s$, and

$$x \leq s \leq y$$

when $f$ is convex for $u \geq s$ (see [3]). Also, in HCF-Corollary and WHCF-Corollary, it suffices to consider that

$$a \geq r \geq b$$

when $f$ is convex for $e^u \leq r$, and

$$a \leq r \leq b$$

when $f$ is convex for $e^u \geq r$.

## 2 Main results

The main results of the paper are given by the following two theorems: partially convex function theorem (PCF-Theorem) and weighted partially convex function theorem (WPCF-Theorem).

**PCF-Theorem** *Let $f$ be a function defined on a real interval $\mathbb{I}$, decreasing for $u \leq s_0$ and increasing for $u \geq s_0$, where $s_0 \in \mathbb{I}$. In addition, assume that $f$ is convex on $[s_0, s]$ or $[s, s_0]$, where $s \in \mathbb{I}$. The inequality*

$$f(x_1) + f(x_2) + \cdots + f(x_n) \geq nf(s)$$

*holds for all $x_1, x_2, \ldots, x_n \in \mathbb{I}$ satisfying $x_1 + x_2 + \cdots + x_n = ns$ if and only if*

$$f(x) + (n-1)f(y) \geq nf(s)$$

*for all $x, y \in \mathbb{I}$ which satisfy $x + (n-1)y = ns$.*

**WPCF-Theorem** *Let $f$ be a function defined on a real interval $\mathbb{I}$, decreasing for $u \leq s_0$ and increasing for $u \geq s_0$, where $s_0 \in \mathbb{I}$, and let $p_1, p_2, \ldots, p_n$ be positive real numbers such that*

$$p = \min\{p_1, p_2, \ldots, p_n\}, \quad p_1 + p_2 + \cdots + p_n = 1.$$

*In addition, assume that $f$ is convex on $[s_0, s]$ or $[s, s_0]$, where $s \in \mathbb{I}$. The inequality*

$$p_1 f(x_1) + p_2 f(x_2) + \cdots + p_n f(x_n) \geq f(s)$$

*holds for all* $x_1, x_2, \dots, x_n \in \mathbb{I}$ *satisfying* $p_1x_1 + p_2x_2 + \cdots + p_nx_n = s$ *if and only if*

$$pf(x) + (1-p)f(y) \ge f(s)$$

*for all* $x, y \in \mathbb{I}$ *such that* $px + (1-p)y = s$.

Applying PCF-Theorem and WPCF-Theorem to the function $f$ defined by $f(u) = g(e^u)$ and replacing $s_0$ by $\ln r_0$, $s$ by $\ln r$, $x$ by $\ln x$, $y$ by $\ln y$, and each $x_i$ by $\ln a_i$ for $i = 1, 2, \dots, n$, we get the following corollaries, respectively.

**PCF-Corollary** *Let $g$ be a function defined on a positive interval $\mathbb{I}$, decreasing for $t \le r_0$ and increasing for $t \ge r_0$, where $r_0 \in \mathbb{I}$. In addition, assume that the function $f$ defined by $f(u) = g(e^u)$ is convex for $r_0 \le e^u \le r$ or $r \le e^u \le r_0$, where $r \in \mathbb{I}$. The inequality*

$$g(a_1) + g(a_2) + \cdots + g(a_n) \ge ng(r)$$

*holds for all* $a_1, a_2, \dots, a_n \in \mathbb{I}$ *satisfying* $a_1a_2\cdots a_n = r^n$ *if and only if*

$$g(a) + (n-1)g(b) \ge ng(r)$$

*for all* $a, b \in \mathbb{I}$ *which satisfy* $ab^{n-1} = r^n$.

**WPCF-Corollary** *Let $g$ be a function defined on a positive interval $\mathbb{I}$, decreasing for $t \le r_0$ and increasing for $t \ge r_0$, where $r_0 \in \mathbb{I}$, and let $p_1, p_2, \dots, p_n$ be positive real numbers such that*

$$p = \min\{p_1, p_2, \dots, p_n\}, \quad p_1 + p_2 + \cdots + p_n = 1.$$

*In addition, assume that the function $f$ defined by $f(u) = g(e^u)$ is convex for $r_0 \le e^u \le r$ or $r \le e^u \le r_0$, where $r \in \mathbb{I}$. The inequality*

$$p_1g(a_1) + p_2g(a_2) + \cdots + p_ng(a_n) \ge g(r)$$

*holds for all* $a_1, a_2, \dots, a_n \in \mathbb{I}$ *satisfying* $a_1^{p_1}a_2^{p_2}\cdots a_n^{p_n} = r$ *if and only if*

$$pg(a) + (1-p)g(b) \ge g(r)$$

*for all* $a, b \in \mathbb{I}$ *such that* $a^pb^{1-p} = r$.

In order to prove WPCF-Theorem, we need the following lemmas.

**Lemma 2.1** *Let $f$ be a function defined on a real interval $\mathbb{I}$, decreasing for $u \le s_0$ and increasing for $u \ge s_0$, where $s_0 \in \mathbb{I}$, and let $p_1, p_2, \dots, p_n$ be positive real numbers such that*

$$p_1 + p_2 + \cdots + p_n = 1.$$

*For $s \in \mathbb{I}$, $s \geq s_0$, if the inequality*

$$p_1 f(x_1) + p_2 f(x_2) + \cdots + p_n f(x_n) \geq f(s)$$

*holds for all $x_1, x_2, \ldots, x_n \in \mathbb{I}$ such that*

$$x_1, x_2, \ldots, x_n \geq s_0, \quad p_1 x_1 + p_2 x_2 + \cdots + p_n x_n = s,$$

*then it holds for all $x_1, x_2, \ldots, x_n \in \mathbb{I}$ such that*

$$p_1 x_1 + p_2 x_2 + \cdots + p_n x_n = s.$$

**Lemma 2.2** *Let $f$ be a function defined on a real interval $\mathbb{I}$, decreasing for $u \leq s_0$ and increasing for $u \geq s_0$, where $s_0 \in \mathbb{I}$, and let $p_1, p_2, \ldots, p_n$ be positive real numbers such that*

$$p_1 + p_2 + \cdots + p_n = 1.$$

*For $s \in \mathbb{I}$, $s \leq s_0$, if the inequality*

$$p_1 f(x_1) + p_2 f(x_2) + \cdots + p_n f(x_n) \geq f(s)$$

*holds for all $x_1, x_2, \ldots, x_n \in \mathbb{I}$ such that*

$$x_1, x_2, \ldots, x_n \leq s_0, \quad p_1 x_1 + p_2 x_2 + \cdots + p_n x_n = s,$$

*then it holds for all $x_1, x_2, \ldots, x_n \in \mathbb{I}$ such that*

$$p_1 x_1 + p_2 x_2 + \cdots + p_n x_n = s.$$

Notice that in the case $s \geq s_0$, WPCF-Theorem follows immediately from Lemma 2.1 and WHCF-Theorem applied to the interval

$$\mathbb{I}_0 = \{u \in \mathbb{I} \mid u \geq s_0\},$$

because $f$ is convex for $u \in \mathbb{I}_0$, $u \leq s$. Also, in the case $s \leq s_0$, WPCF-Theorem follows immediately from Lemma 2.2 and WHCF-Theorem applied to the interval

$$\mathbb{I}_0 = \{u \in \mathbb{I} \mid u \leq s_0\},$$

because $f$ is convex for $u \in \mathbb{I}_0$, $u \geq s$.

**Remark 2.3** According to Remark 1.2, it suffices to consider in PCF-Theorem and WPCF-Theorem that

$$x \geq s \geq y$$

when $f$ is convex on $[s_0, s]$, and

$$x \le s \le y$$

when $f$ is convex on $[s, s_0]$. Also, it suffices to consider in PCF-Corollary and WPCF-Corollary that

$$a \ge r \ge b$$

when $f$ is convex for $r_0 \le e^u \le r$, and

$$a \le r \le b$$

when $f$ is convex for $r \le e^u \le r_0$.

**Remark 2.4** Let us denote

$$g(u) = \frac{f(u) - f(s)}{u - s}, \qquad h(x, y) = \frac{g(x) - g(y)}{x - y}.$$

In many applications, it is useful to replace the hypothesis

$$pf(x) + (1 - p)f(y) \ge f(s)$$

in WHCF-Theorem and WPCF-Theorem by the equivalent condition:

$$h(x, y) \ge 0 \ \forall x, y \in \mathbb{I}, \quad px + (1 - p)y = s.$$

This equivalence is true since

$$\begin{aligned} pf(x) + (1 - p)f(y) - f(s) &= p\big[f(x) - f(s)\big] + (1 - p)\big[f(y) - f(s)\big] \\ &= p(x - s)g(x) + (1 - p)(y - s)g(y) \\ &= p(1 - p)(x - y)\big[g(x) - g(y)\big] \\ &= p(1 - p)(x - y)^2 h(x, y). \end{aligned}$$

In the particular case $p_1 = p_2 = \cdots = p_n = 1/n$, this condition becomes

$$h(x, y) \ge 0 \ \forall x, y \in \mathbb{I}, \quad x + (n - 1)y = ns.$$

**Remark 2.5** The required inequalities in WHCF-Theorem and WPCF-Theorem turn into equalities for $x_1 = x_2 = \cdots = x_n = s$. In addition, on the assumption that

$$p = \min\{p_1, p_2, \dots, p_n\},$$

the equality also holds for $x_1 = x$ and $x_2 = \cdots = x_n = y$ if there exist $x, y \in \mathbb{I}$, $x \ne y$, such that

$$px + (1 - p)y = s, \qquad pf(x) + (1 - p)f(y) = f(s).$$

## 3 Proof of lemmas

*Proof of Lemma* 2.1 For $i = 1, 2, \dots, n$, define the numbers $y_i \in \mathbb{I}$ as

$$y_i = \begin{cases} s_0, & x_i \le s_0, \\ x_i, & x_i > s_0. \end{cases}$$

Since $y_i \ge x_i$ for $i = 1, 2, \dots, n$, we have

$$p_1y_1 + p_2y_2 + \cdots + p_ny_n \ge p_1x_1 + p_2x_2 + \cdots + p_nx_n = s.$$

In addition, since $f(y_i) \le f(x_i)$ for $x_i \le s_0$ and $f(y_i) = f(x_i)$ for $x_i > s_0$, we have $f(y_i) \le f(x_i)$ for $i = 1, 2, \dots, n$, and hence

$$p_1f(y_1) + p_2f(y_2) + \cdots + p_nf(y_n) \le p_1f(x_1) + p_2f(x_2) + \cdots + p_nf(x_n).$$

Thus, it suffices to show that

$$p_1f(y_1) + p_2f(y_2) + \cdots + p_nf(y_n) \ge f(s)$$

for all $y_1, y_2, \dots, y_n \ge s_0$ such that $p_1y_1 + p_2y_2 + \cdots + p_ny_n \ge s$. By hypothesis, this inequality is true for $y_1, y_2, \dots, y_n \ge s_0$ and $p_1y_1 + p_2y_2 + \cdots + p_ny_n = s$. Since $f$ is increasing for $u \in \mathbb{I}$, $u \ge s_0$, the more we have $p_1f(y_1) + p_2f(y_2) + \cdots + p_nf(y_n) \ge f(s)$ for $y_1, y_2, \dots, y_n \ge s_0$ and $p_1y_1 + p_2y_2 + \cdots + p_ny_n \ge s$. □

*Proof of Lemma* 2.2 For $i = 1, 2, \dots, n$, define the numbers $y_i \in \mathbb{I}$ as follows:

$$y_i = \begin{cases} x_i, & x_i \le s_0, \\ s_0, & x_i > s_0. \end{cases}$$

We have $y_i \le s_0$, $y_i \le x_i$ and $f(y_i) \le f(x_i)$ for $i = 1, 2, \dots, n$. Therefore,

$$p_1y_1 + p_2y_2 + \cdots + p_ny_n \le p_1x_1 + p_2x_2 + \cdots + p_nx_n = s$$

and

$$p_1f(y_1) + p_2f(y_2) + \cdots + p_nf(y_n) \le p_1f(x_1) + p_2f(x_2) + \cdots + p_nf(x_n).$$

Thus, it suffices to show that

$$p_1f(y_1) + p_2f(y_2) + \cdots + p_nf(y_n) \ge f(s)$$

for all $y_1, y_2, \dots, y_n \le s_0$ such that $p_1y_1 + p_2y_2 + \cdots + p_ny_n \le s$. By hypothesis, this inequality is true for $y_1, y_2, \dots, y_n \le s_0$ and $p_1y_1 + p_2y_2 + \cdots + p_ny_n = s$. Since $f$ is decreasing for $u \in \mathbb{I}$, $u \le s_0$, we have also $p_1f(y_1) + p_2f(y_2) + \cdots + p_nf(y_n) \ge f(s)$ for $y_1, y_2, \dots, y_n \le s_0$ and $p_1y_1 + p_2y_2 + \cdots + p_ny_n \le s$. □

## 4 Applications

**Application 4.1** *Let $x_1, x_2, \dots, x_n \geq \frac{-n}{n-2}$ $(n \geq 3)$ such that*

$$x_1 + x_2 + \cdots + x_n = n.$$

*If $k \geq \frac{n(3n-4)}{(n-2)^2}$, then*

$$\frac{1-x_1}{k+x_1^2} + \frac{1-x_2}{k+x_2^2} + \cdots + \frac{1-x_n}{k+x_n^2} \geq 0,$$

*with equality for $x_1 = x_2 = \cdots = x_n = 1$, and also for $x_1 = \frac{-n}{n-2}$ and $x_2 = \cdots = x_n = \frac{n}{n-2}$ (or any cyclic permutation).*

*Proof* Rewrite the desired inequality as

$$f(x_1) + f(x_2) + \cdots + f(x_n) \geq nf(s),$$

where

$$s = 1,$$
$$f(u) = \frac{1-u}{k+u^2}, \quad u \in \mathbb{I} = \left[\frac{-n}{n-2}, \frac{n(2n-3)}{n-2}\right].$$

We have

$$f'(u) = \frac{u^2-2u-k}{(u^2+k)^2},$$
$$f''(u) = \frac{2f_1(u)}{(u^2+k)^3},$$

where

$$f_1(u) = -u^3 + 3u^2 + 3ku - k = (k+1)(3u-1) - (u-1)^3.$$

There are two cases to consider.

Case 1: $\sqrt{k+1} \geq \frac{2(n-1)^2}{n-2}$. For $u \in \mathbb{I}$, $u \geq 1$, we have

$$f_1(u) > (k+1)(u-1) - (u-1)^3 = (u-1)\left[k+1-(u-1)^2\right] \geq 0,$$

since

$$u - 1 \leq \frac{n(2n-3)}{n-2} - 1 = \frac{2(n-1)^2}{n-2} \leq \sqrt{k+1}.$$

Therefore, $f$ is convex for $u \in \mathbb{I}$, $u \geq 1$. By HCF-Theorem, we only need to show that $f(x) + (n-1)f(y) \geq nf(1)$ for all $x, y \in \mathbb{I}$ which satisfy $x + (n-1)y = n$. According to Remark 2.4, this is true if $h(x,y) \geq 0$ for $x, y \in \mathbb{I}$ and $x + (n-1)y = n$, where

$$h(x,y) = \frac{g(x)-g(y)}{x-y}, \qquad g(u) = \frac{f(u)-f(1)}{u-1}.$$

Indeed, we have

$$g(u) = \frac{-1}{u^2 + k}$$

and

$$h(x,y) = \frac{x + y}{(x^2 + k)(y^2 + k)} = \frac{n + (n-2)x}{(n-1)(x^2 + k)(y^2 + k)} \geq 0.$$

Case 2: $\frac{2(n-1)}{n-2} \leq \sqrt{k+1} < \frac{2(n-1)^2}{n-2}$. Since

$$1 - \sqrt{1+k} \leq \frac{-n}{n-2}$$

and

$$1 + \sqrt{1+k} < 1 + \frac{2(n-1)^2}{n-2} = \frac{n(2n-3)}{n-2},$$

from the expression of $f'$ it follows that $f$ is decreasing on $[\frac{-n}{n-2}, s_0]$ and increasing on $[s_0, \frac{n(2n-3)}{n-2}]$, where

$$s_0 = 1 + \sqrt{k+1} > 1.$$

On the other hand, for $u \in [1, s_0]$, we have

$$f_1(u) > (k+1)(u-1) - (u-1)^3 = (u-1)\left[k + 1 - (u-1)^2\right] \geq 0,$$

since

$$u - 1 \leq s_0 - 1 = \sqrt{k+1}.$$

Thus, $f$ is convex on $[1, s_0]$. By PCF-Theorem, we only need to show that $f(x) + (n-1)f(y) \geq nf(1)$ for all $x, y \in \mathbb{I}$ such that $x + (n-1)y = n$. We have proved this before (at Case 1). □

**Application 4.2** *If $x_1, x_2, \ldots, x_n$ $(n \geq 3)$ are real numbers such that*

$$x_1 + x_2 + \cdots + x_n = n,$$

*then* [6]

$$\sum_{i=1}^{n} \frac{n(n+1) - 2x_i}{n^2 + (n-2)x_i^2} \leq n,$$

*with equality for $x_1 = x_2 = \cdots = x_n = 1$, and also for $x_1 = n$ and $x_2 = \cdots = x_n = 0$ (or any cyclic permutation).*

*Proof* The desired inequality is true for $x_1 > \frac{n(n+1)}{2}$ since

$$\frac{n(n+1)-2x_1}{n^2+(n-2)x_1^2} < 0$$

and

$$\frac{n(n+1)-2x_i}{n^2+(n-2)x_i^2} < \frac{n}{n-1}, \quad i = 2,3,\dots,n.$$

Consider further that $x_1, x_2, \dots, x_n \le \frac{n(n+1)}{2}$ and rewrite the desired inequality as

$$f(x_1) + f(x_2) + \cdots + f(x_n) \ge nf(s),$$

where

$$s = 1,$$
$$f(u) = \frac{2u - n(n+1)}{(n-2)u^2 + n^2}, \quad u \in \mathbb{I} = \left[\frac{n(3-n^2)}{2}, \frac{n(n+1)}{2}\right].$$

We have

$$\frac{f'(u)}{2(n-2)} = \frac{n^2 + n(n+1)u - u^2}{[(n-2)u^2+n^2]^2}$$

and

$$\frac{f''(u)}{2(n-2)} = \frac{f_1(u)}{[(n-2)u^2+n^2]^3},$$

where

$$f_1(u) = 2(n-2)u^3 - 3n(n+1)(n-2)u^2 - 2n^2(2n-3)u + n^3(n+1).$$

From the expression of $f'$, it follows that $f$ is decreasing on $[\frac{n(3-n^2)}{2}, s_0]$ and increasing on $[s_0, \frac{n(n+1)}{2}]$, where

$$s_0 = \frac{n}{2}\left(n+1-\sqrt{n^2+2n+5}\right) \in (-1,0).$$

On the other hand, for $-1 \le u \le 1$, we have

$$\begin{aligned} f_1(u) &> -2(n-2) - 3n(n+1)(n-2) - 2n^2(2n-3) + n^3(n+1) \\ &= n^2(n-3)^2 + 4(n+1) > 0, \end{aligned}$$

and hence $f''(u) > 0$. Since $[s_0, s] \subset [-1,1]$, $f$ is convex on $[s_0, s]$. By PCF-Theorem, we only need to show that $f(x) + (n-1)f(y) \ge nf(1)$ for all $x, y \in \mathbb{I}$ which satisfy $x + (n-1)y = n$. According to Remark 2.4, this is true if $h(x,y) \ge 0$ for $x, y \in \mathbb{I}$ and $x + (n-1)y = n$, where

$$h(x,y) = \frac{g(x)-g(y)}{x-y}, \qquad g(u) = \frac{f(u)-f(1)}{u-1}.$$

Indeed, we have

$$g(u) = \frac{(n-2)u+n}{(n-2)u^2+n^2}$$

and

$$\frac{h(x,y)}{n-2} = \frac{n^2-n(x+y)-(n-2)xy}{[(n-2)x^2+n^2][(n-2)y^2+n^2]}$$
$$= \frac{(n-1)(n-2)y^2}{[(n-2)x^2+n^2][(n-2)y^2+n^2]} \ge 0.$$ □

**Application 4.3** *Let* $x_1, x_2, \ldots, x_n$ $(n \ge 2)$ *be positive real numbers such that*

$$x_1 + x_2 + \cdots + x_n \ge n.$$

*If* $k > 1$, *then* [6]

$$\frac{x_1}{x_1^k + x_2 + \cdots + x_n} + \frac{x_2}{x_1 + x_2^k + \cdots + x_n} + \cdots + \frac{x_n}{x_1 + x_2 + \cdots + x_n^k} \le 1,$$

*with equality for* $x_1 = x_2 = \cdots = x_n = 1$.

*Proof* Using the substitutions

$$s = \frac{x_1 + x_2 + \cdots + x_n}{n},$$

and

$$y_1 = \frac{x_1}{s}, \qquad y_2 = \frac{x_2}{s}, \qquad \ldots, \qquad y_n = \frac{x_n}{s},$$

the desired inequality becomes

$$\frac{y_1}{s^{k-1}y_1^k + y_2 + \cdots + y_n} + \cdots + \frac{y_n}{y_1 + y_2 + \cdots + s^{k-1}y_n^k} \le 1,$$

where $s \ge 1$ and $y_1 + y_2 + \cdots + y_n = n$. Clearly, if this inequality holds for $s = 1$, then it holds for any $s \ge 1$. Therefore, we need only to consider the case $s = 1$, when $x_1 + x_2 + \cdots + x_n = n$, and the desired inequality is equivalent to

$$\frac{x_1}{x_1^k - x_1 + n} + \frac{x_2}{x_2^k - x_2 + n} + \cdots + \frac{x_n}{x_n^k - x_n + n} \le 1.$$

There are two cases to consider: $1 < k \le n+1$ and $k > n+1$.

Case 1: $1 < k \le n+1$. By Bernoulli's inequality, we have

$$x_1^k \ge 1 + k(x_1 - 1),$$

and hence

$$x_1^k - x_1 + n \ge n - k + 1 + (k-1)x_1 \ge 0.$$

Consequently, it suffices to prove that

$$\sum_{i=1}^{n} \frac{x_i}{n-k+1+(k-1)x_i} \leq 1.$$

For $k = n+1$, this inequality is an equality. Otherwise, for $1 < k < n+1$, we rewrite the inequality as

$$\sum_{i=1}^{n} \frac{1}{n-k+1+(k-1)x_i} \geq 1,$$

which follows from the AM-HM inequality as follows:

$$\sum_{i=1}^{n} \frac{1}{n-k+1+(k-1)x_i} \geq \frac{n^2}{\sum_{i=1}^{n}[n-k+1+(k-1)x_i]} = 1.$$

Case 2: $k > n+1$. Write the desired inequality as

$$f(x_1) + f(x_2) + \cdots + f(x_n) \geq nf(s),$$

where

$$s = 1,$$
$$f(u) = \frac{-u}{u^k - u + n}, \quad u \in \mathbb{I} = (0, n).$$

We have

$$f'(u) = \frac{(k-1)u^k - n}{(u^k - u + n)^2}$$

and

$$f''(u) = \frac{f_1(u)}{(u^k - u + n)^3},$$

where

$$f_1(u) = k(k-1)u^{k-1}\left(u^k - u + n\right) - 2\left(ku^{k-1} - 1\right)\left[(k-1)u^k - n\right].$$

From the expression of $f'$, it follows that $f$ is decreasing on $(0, s_0]$ and increasing on $[s_0, n)$, where

$$s_0 = \left(\frac{n}{k-1}\right)^{1/k} < 1.$$

On the other hand, for $u \in [s_0, s]$, we have

$$(k-1)u^k - n \geq (k-1)s_0^k - n = 0,$$

and hence

$$\begin{aligned} f_1(u) &\geq k(k-1)u^{k-1}\left(u^k - u + n\right) - 2ku^{k-1}\left[(k-1)u^k - n\right] \\ &= ku^{k-1}\left[-(k-1)\left(u^k + u\right) + n(k+1)\right] \\ &\geq ku^{k-1}\left[-2(k-1) + 2(k+1)\right] = 4ku^{k-1} > 0. \end{aligned}$$

Thus, $f''(u) > 0$, and hence $f$ is convex on $[s_0, s]$. By PCF-Theorem and Remark 2.3, we need to show that $f(x) + (n-1)f(y) \geq nf(1)$ for all positive $x$, $y$ which satisfy $x \geq 1 \geq y > 0$ and $x + (n-1)y = n$. Consider the nontrivial case where $x > 1 > y > 0$ and write the inequality $f(x) + (n-1)f(y) \geq nf(1)$ as follows:

$$\begin{aligned} &\frac{x}{x^k - x - n} + \frac{(n-1)y}{y^k - y + n} \leq 1, \\ &x^k - x + n \geq \frac{x(y^k - y + n)}{y^k - ny + n}, \\ &x^k - x \geq \frac{(n-1)y(y - y^k)}{y^k - ny + n}. \end{aligned}$$

Since $y < 1$ and $y^k - ny + n > y^k - y + 1 > 0$, it suffices to show that

$$x^k - x \geq \frac{(n-1)(y - y^k)}{y^k - y + 1},$$

which is equivalent to

$$h(x) \geq \frac{y - y^k}{(1-y)(y^k - y + 1)},$$

where

$$h(x) = \frac{x^k - x}{x - 1}.$$

By the weighted AM-GM inequality, we have

$$h'(x) = \frac{(k-1)x^k + 1 - kx^{k-1}}{(x-1)^2} > 0,$$

and hence $h$ is strictly increasing. Since $x - 1 = (n-1)(1-y) \geq 1 - y$, we get

$$h(x) \geq h(2-y) = \frac{(2-y)^k - 2 + y}{1 - y}.$$

Thus, it suffices to show that

$$(2-y)^k - 2 + y \geq \frac{y - y^k}{y^k - y + 1}.$$

Putting $1-y=t$, $0<t<1$, we write this inequality as

$$(2-y)^k - 1 + y \geq \frac{1}{y^k - y + 1},$$
$$(1+t)^k - t \geq \frac{1}{(1-t)^k + t},$$
$$\left(1-t^2\right)^k + t(1+t)^k \geq 1 + t^2 + t(1-t)^k.$$

By Bernoulli's inequality,

$$\left(1-t^2\right)^k + t(1+t)^k > 1 - kt^2 + t(1+kt) = 1 + t.$$

So, we only need to show that $t(1-t) \geq t(1-t)^k$, which is clearly true. □

**Application 4.4** *Let $x_1, x_2, \dots, x_n$ $(n \geq 2)$ be positive real numbers such that*

$$x_1 + x_2 + \cdots + x_n = n.$$

*If* $0 < k \leq \frac{n}{n-1}$, *then* [6]

$$x_1^{k/x_1} + x_2^{k/x_2} + \cdots + x_n^{k/x_n} \leq n$$

*with equality for* $x_1 = x_2 = \cdots = x_n = 1$.

*Proof* According to the power mean inequality, we have

$$\left(\frac{x_1^{p/x_1} + x_2^{p/x_2} + \cdots + x_n^{p/x_n}}{n}\right)^{1/p} \geq \left(\frac{x_1^{q/x_1} + x_2^{q/x_2} + \cdots + x_n^{q/x_n}}{n}\right)^{1/q}$$

for all $p \geq q > 0$. Thus, it suffices to prove the desired inequality for

$$k = \frac{n}{n-1}, \quad 1 < k \leq 2.$$

Rewrite the desired inequality as

$$f(x_1) + f(x_2) + \cdots + f(x_n) \geq nf(s),$$

where

$$s = 1,$$
$$f(u) = -u^{k/u}, \quad u \in \mathbb{I} = (0, n).$$

We have

$$f'(u) = ku^{\frac{k}{u}-2}(\ln u - 1),$$
$$f''(u) = ku^{\frac{k}{u}-4}\left[u + (1-\ln u)(2u - k + k\ln u)\right].$$

From the expression of $f'$, it follows that $f$ is decreasing on $(0, s_0]$ and increasing on $[s_0, n)$, where

$$s_0 = e.$$

In addition, we claim that $f$ is convex on $[s, s_0]$. Indeed, since $1 - \ln u \geq 0$ and

$$2u - k + k\ln u \geq 2 - k \geq 0,$$

we have $f'' > 0$ for $u \in [s, s_0]$. Therefore, by PCF-Theorem and Remark 2.3, we only need to show that

$$x^{k/x} + (n-1)y^{k/y} \leq n$$

for $0 < x \leq 1 \leq y$ and $x + (n-1)y = n$. We have

$$\frac{k}{x} \geq k > 1.$$

Also, from

$$\frac{k}{y} = \frac{n}{(n-1)y} > \frac{n}{x + (n-1)y} = 1, \qquad \frac{k}{y} \leq \frac{2}{y} \leq 2,$$

we get

$$0 < \frac{k}{y} - 1 \leq 1.$$

Therefore, by Bernoulli's inequality, we have

$$\begin{aligned}
&x^{k/x} + (n-1)y^{k/y} - n \\
&\quad = \frac{1}{(\frac{1}{x})^{k/x}} + (n-1)y \cdot y^{k/y-1} - n \\
&\quad \leq \frac{1}{1 + \frac{k}{x}(\frac{1}{x} - 1)} + (n-1)y\left[1 + \left(\frac{k}{y} - 1\right)(y-1)\right] - n \\
&\quad = \frac{x^2}{x^2 - kx + k} - (k-1)x^2 - (2-k)x \\
&\quad = \frac{-(x-1)^2[(k-1)x + k(2-k)]}{x^2 - kx + k} \leq 0.
\end{aligned}$$

□

**Application 4.5** *If $a$, $b$, $c$ are positive real numbers such that $abc = 1$, then*

$$\frac{1-a}{17 + 4a + 6a^2} + \frac{1-b}{17 + 4b + 6b^2} + \frac{1-c}{17 + 4c + 6c^2} \geq 0,$$

*with equality for $a = b = c = 1$ and also for $8a = b = c = 2$ (or any cyclic permutation).*

*Proof* Write the desired inequality as

$$g(a) + g(b) + g(c) \geq 3g(r),$$

where $r = 1$ and

$$g(t) = \frac{1-t}{1+pt+qt^2}, \quad t \in \mathbb{I} = (0, \infty),$$

where $p = \frac{4}{17}$, $q = \frac{6}{17}$. From

$$g'(t) = \frac{qt^2 - 2qt - p - 1}{(1+pt+qt^2)^2},$$

it follows that $g$ is decreasing on $(0, r_0]$ and increasing on $[r_0, \infty)$, where

$$r_0 = 1 + \sqrt{1 + \frac{p+1}{q}} > 1.$$

We have

$$f(u) = g(e^u) = \frac{1-e^u}{1+pe^u+qe^{2u}},$$

$$f''(u) = \frac{t \cdot h(t)}{(1+pt+qt^2)^3},$$

where

$$t = e^u, \quad h(t) = -q^2t^4 + q(p+4q)t^3 + 3q(p+2)t^2 + (p - 4q + p^2)t - p - 1.$$

We will show that $h(t) > 0$ for $t \in [r, r_0]$, and hence $f$ is convex for

$$e^u \in [r, r_0] = \left[1, 1 + \sqrt{1 + \frac{p+1}{q}}\right].$$

We have

$$h'(t) = -4q^2t^3 + 3q(p+4q)t^2 + 6q(p+2)t + p - 4q + p^2,$$

$$h''(t) = 6q\left[-2qt^2 + (p+4q)t + p + 2\right].$$

Since

$$h''(t) = 6q\left[2\left(-qt^2 + 2qt + p + 1\right) + p(t-1)\right] \geq 12q\left(-qt^2 + 2qt + p + 1\right) \geq 0,$$

$h'(t)$ is increasing,

$$h'(t) \geq h'(1) = p^2 + 9pq + 8q^2 + p + 8q > 0,$$

$h$ is increasing, and hence

$$h(t) \geq h(1) = p^2 + 4pq + 3q^2 + 2q - 1 = (p + 2q)^2 - (q - 1)^2$$
$$= (p + q + 1)(p + 3q - 1) > 0.$$

By PCF-Corollary, we only need to prove that $g(a) + 2g(b) \geq 3g(1)$ for $ab^2 = 1$; that is,

$$\frac{1-a}{1+pa+qa^2} + \frac{2(1-b)}{1+pb+qb^2} \geq 0,$$

$$\frac{b^2(b^2-1)}{b^4+pb^2+q} + \frac{2(1-b)}{1+pb+qb^2} \geq 0,$$

$$pA + qB \geq C,$$

where

$$A = b^2(b-1)^2(b+2),$$
$$B = (b-1)^2\left(b^4 + 2b^3 + 2b^2 + 2b + 2\right),$$
$$C = b^2(b-1)^2(2b+1).$$

Indeed, we have

$$17(pA + qB - C) = 3(b-1)^2(b-2)^2\left(2b^2 + 2b + 1\right) \geq 0. \qquad \square$$

**Application 4.6** *If $a$, $b$, $c$ are positive real numbers such that $abc = 1$, then* [6]

$$\frac{7-6a}{2+a^2} + \frac{7-6b}{2+b^2} + \frac{7-6c}{2+c^2} \geq 1,$$

*with equality for $a = b = c = 1$ and also for $8a = b = c = 2$ (or any cyclic permutation).*

*Proof* Write the desired inequality as

$$g(a) + g(b) + g(c) \geq 3g(r),$$

where $r = 1$ and

$$g(t) = \frac{7-6t}{2+t^2}, \quad t \in \mathbb{I} = (0, \infty).$$

From

$$g'(t) = \frac{2(3t+2)(t-3)}{(2+t^2)^2},$$

it follows that $g$ is decreasing on $(0, r_0]$ and increasing on $[r_0, \infty)$, where

$$r_0 = 3.$$

We have

$$f(u) = g\left(e^u\right) = \frac{7 - 6e^u}{2 + e^{2u}},$$
$$f''(u) = \frac{2t \cdot h(t)}{(2 + t^2)^3},$$

where

$$t = e^u, \quad h(t) = -3t^4 + 14t^3 + 36t^2 - 28t - 12.$$

We will show that $h(t) > 0$ for $t \in [r, r_0]$, and hence $f$ is convex for

$$e^u \in [r, r_0] = [1, 3].$$

We have

$$h(t) = 3\left(t^2 - 1\right)\left(9 - t^2\right) + 14t^3 + 6t^2 - 28t + 15$$
$$= 3\left(t^2 - 1\right)\left(9 - t^2\right) + 14t^2(t - 1) + 14(t - 1)^2 + 6t^2 + 1 > 0.$$

By PCF-Corollary, we only need to prove that $g(a) + 2g(b) \geq 3g(1)$ for $ab^2 = 1$; that is,

$$\frac{7 - 6a}{2 + a^2} + \frac{2(7 - 6b)}{2 + b^2} \geq 1,$$
$$\frac{b^2(7b^2 - 6)}{2b^4 + 1} + \frac{2(7 - 6b)}{2 + b^2} \geq 1,$$
$$(b - 1)^2(b - 2)^2\left(5b^2 + 6b + 3\right) \geq 0.$$

Since the last inequality is true, the proof is completed. □

**Application 4.7** *Let $a$, $b$, $c$ be positive real numbers such that $abc = 1$. If*

$$k \in \left[\frac{-13}{3\sqrt{3}}, \frac{13}{3\sqrt{3}}\right],$$

*then* [6]

$$\frac{a + k}{a^2 + 1} + \frac{b + k}{b^2 + 1} + \frac{c + k}{c^2 + 1} \leq \frac{3(1 + k)}{2},$$

*with equality for $a = b = c = 1$. If $k = \frac{13}{3\sqrt{3}}$, then the equality holds also for $a = 7 + 4\sqrt{3}$ and $b = c = 2 - \sqrt{3}$ (or any cyclic permutation). If $k = \frac{-13}{3\sqrt{3}}$, then the equality holds also for $a = 7 - 4\sqrt{3}$ and $b = c = 2 + \sqrt{3}$ (or any cyclic permutation).*

*Proof* The desired inequality is equivalent to

$$\sum \frac{(a - 1)^2}{a^2 + 1} \geq k\left(\sum \frac{2}{a^2 + 1} - 3\right).$$

Thus, it suffices to prove this inequality for only $|k| = \frac{13}{3\sqrt{3}}$. On the other hand, replacing $a$, $b$, $c$ by $1/a$, $1/b$, $1/c$, the inequality becomes

$$\sum \frac{(a-1)^2}{a^2+1} \ge k\left(3 - \sum \frac{2}{a^2+1}\right).$$

Thus, we only need to prove the desired inequality for $k = \frac{13}{3\sqrt{3}}$. Write this inequality as

$$g(a) + g(b) + g(c) \ge 3g(r),$$

where $r = 1$ and

$$g(t) = \frac{-t-k}{t^2+1}, \quad t \in \mathbb{I} = (0, \infty).$$

From

$$g'(t) = \frac{t^2 + 2kt - 1}{(t^2+1)^2},$$

it follows that $g$ is decreasing on $(0, r_0]$ and increasing on $[r_0, \infty)$, where

$$r_0 = \frac{\sqrt{3}}{9}.$$

We have

$$f(u) = g\left(e^u\right) = \frac{-e^u - k}{e^{2u}+1},$$
$$f''(u) = \frac{t \cdot h(t)}{(t^2+1)^3},$$

where

$$t = e^u, \quad h(t) = -t^4 - 4kt^3 + 6t^2 + 4kt - 1.$$

We will show that $h(t) > 0$ for $t \in [r_0, r]$, and hence $f$ is convex for

$$e^u \in [r_0, r] = \left[\frac{\sqrt{3}}{9}, 1\right].$$

Indeed, since

$$4kt = \frac{52t}{3\sqrt{3}} = \frac{52}{27} > 1,$$

we have

$$h(t) = -t^4 + 6t^2 - 1 + 4kt\left(1 - t^2\right) \ge -t^4 + 6t^2 - 1 + \left(1 - t^2\right) = t^2\left(5 - t^2\right) > 0.$$

By PCF-Corollary, we only need to prove that $g(a) + 2g(b) \geq 3g(1)$ for $ab^2 = 1$; that is,

$$\frac{a+k}{a^2+1} + \frac{2(b+k)}{b^2+1} \leq \frac{3(1+k)}{2},$$

$$\frac{b^2(kb^2+1)}{b^4+1} + \frac{2(b+k)}{b^2+1} \leq \frac{3(1+k)}{2},$$

$$3b^6 - 4b^5 + b^4 + b^2 - 4b + 3 - k\left(1-b^2\right)^3 \geq 0,$$

$$(b-1)^2\left[(3+k)b^4 + 2(1+k)b^3 + 2b^2 + 2(1-k)b + 3 - k\right] \geq 0,$$

$$(b-1)^2(b-2+\sqrt{3})^2\left[(27+13\sqrt{3})b^2 + 24(2+\sqrt{3})b + 33 + 17\sqrt{3}\right] \geq 0.$$

The last inequality is clearly true, and the proof is completed. □

**Application 4.8** *If* $a, b, c$ *are positive real numbers and* $0 < k \leq 2 + 2\sqrt{2}$, *then* [6]

$$\frac{a^3}{ka^2+bc} + \frac{b^3}{kb^2+ca} + \frac{c^3}{kc^2+ab} \geq \frac{a+b+c}{k+1},$$

*with equality for* $a = b = c = 1$. *If* $k = 2 + 2\sqrt{2}$, *then the equality holds also for* $\frac{a}{\sqrt{2}} = b = c$ (*or any cyclic permutation*).

*Proof* For $k < 2 + 2\sqrt{2}$, the proof is similar to the one of the main case $k = 2 + 2\sqrt{2}$. For this reason, we consider further only the case where

$$k = 2 + 2\sqrt{2}.$$

Due to homogeneity, we may assume that $abc = 1$. On this hypothesis,

$$\sum \frac{a^3}{ka^2+bc} - \frac{1}{k+1}\sum a = \sum\left(\frac{a^4}{ka^3+1} - \frac{a}{k+1}\right) = \frac{1}{k+1}\sum \frac{a^4-a}{ka^3+1}.$$

Thus, we can write the inequality as

$$g(a) + g(b) + g(c) \geq 3g(r),$$

where $r = 1$ and

$$g(t) = \frac{t^4-t}{kt^3+1}, \quad t \in \mathbb{I} = (0,\infty).$$

From

$$g'(t) = \frac{kt^6 + 2(k+2)t^3 - 1}{(kt^3+1)^2},$$

it follows that $g$ is decreasing on $(0, r_0]$ and increasing on $[r_0, \infty)$, where

$$r_0 = \sqrt[3]{\frac{-k-2+\sqrt{(k+1)(k+4)}}{k}} \approx 0.4149.$$

We have

$$f(u) = g(e^u) = \frac{e^{4u} - e^u}{ke^{3u} + 1},$$
$$f''(u) = \frac{t \cdot h(t)}{(kt^3 + 1)^3},$$

where

$$t = e^u, \quad h(t) = k^2t^9 - k(4k + 1)t^6 + (13k + 16)t^3 - 1.$$

We have $h(t) \geq 0$ for $t \in [t_1, t_2]$, where $t_1 \approx 0.2345$ and $t_2 \approx 1.02$. Since

$$[r_0, r] \subset [t_1, t_2],$$

$f$ is convex for $e^u \in [r_0, r]$. Then, by PCF-Corollary, it suffices to show that $g(a) + 2g(b) \geq 3g(1)$ for $ab^2 = 1$. This is true if the original inequality holds for $b = c = 1$. Thus, we need to show that

$$\frac{a^3}{ka^2 + 1} + \frac{2}{a + k} \geq \frac{a + 2}{k + 1},$$

which is equivalent to the obvious inequality

$$(a - 1)^2(a - \sqrt{2})^2 \geq 0.$$ □

**Application 4.9** *If $a_1$, $a_2$, $a_3$, $a_4$, $a_5$ are positive real numbers such that*

$$a_1a_2a_3a_4a_5 = 1,$$

*then* [6]

$$\frac{1 - a_1}{1 + a_1^2} + \frac{1 - a_2}{1 + a_2^2} + \frac{1 - a_3}{1 + a_3^2} + \frac{1 - a_4}{1 + a_4^2} + \frac{1 - a_5}{1 + a_5^2} \geq 0,$$

*with equality for $a_1 = a_2 = a_3 = a_4 = a_5 = 1$.*

*Proof* Write the inequality as

$$g(a_1) + g(a_2) + g(a_3) + g(a_4) + g(a_5) \geq 3g(r),$$

where $r = 1$ and

$$g(t) = \frac{1 - t}{1 + t^2}, \quad t \in \mathbb{I} = (0, \infty).$$

From

$$g'(t) = \frac{t^2 - 2t - 1}{(t^2 + 1)^2},$$

it follows that $g$ is decreasing on $(0, r_0]$ and increasing on $[r_0, \infty)$, where

$$r_0 = 1 + \sqrt{2}.$$

We have

$$f(u) = g(e^u) = \frac{1 - e^u}{1 + e^{2u}},$$
$$f''(u) = \frac{t \cdot h(t)}{(t^2 + 1)^3},$$

where

$$t = e^u, \quad h(t) = -t^4 + 4t^3 + 6t^2 - 4t - 1.$$

We will show that $h(t) > 0$ for $t \in [r, r_0]$, and hence $f$ is convex for

$$e^u \in [r, r_0] = [1, 1 + \sqrt{2}].$$

Indeed,

$$h(t) \geq -t^4 + 6t^2 - 1 = 8 - (3 - t^2)^2 \geq 4.$$

By PCF-Corollary, we only need to prove that $g(a) + 4g(b) \geq 5g(1)$ for $ab^4 = 1$; that is,

$$\frac{1 - a}{1 + a^2} + \frac{4(1 - b)}{1 + b^2} \geq 0,$$
$$\frac{b^4(b^4 - 1)}{1 + b^8} + \frac{4(1 - b)}{1 + b^2} \geq 0,$$
$$1 + \frac{4(1 - b)}{1 + b^2} \geq \frac{1 + b^4}{1 + b^8}.$$

Since

$$\frac{1 + b^4}{1 + b^8} \leq \frac{2}{1 + b^4} \leq \frac{4}{(1 + b^2)^2},$$

it suffices to show that

$$1 + \frac{4(1 - b)}{1 + b^2} \geq \frac{4}{(1 + b^2)^2}.$$

This inequality is equivalent to $(1 - b)^4 \geq 0$, and the proof is completed. □

**Remark 4.1** The inequality

$$\frac{1 - a_1}{1 + a_1^2} + \frac{1 - a_2}{1 + a_2^2} + \frac{1 - a_3}{1 + a_3^2} + \frac{1 - a_4}{1 + a_4^2} + \frac{1 - a_5}{1 + a_5^2} + \frac{1 - a_6}{1 + a_6^2} \geq 0$$

is not true for any positive numbers $a_1, a_2, a_3, a_4, a_5, a_6$ satisfying $a_1a_2a_3a_4a_5a_6 = 1$. Indeed, for $a_2 = a_3 = a_4 = a_5 = a_6 = 2$, the inequality becomes

$$\frac{-a_1(a_1+1)}{1+a_1^2} \geq 0,$$

which is false for

$$a_1 = \frac{1}{a_2a_3a_4a_5a_6} = \frac{1}{32}.$$

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The author declares that he has no competing interests.

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