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Three-step Mann iterations for a general system of variational inequalities and an infinite family of nonexpansive mappings in Banach spaces

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Abstract

In this paper, let X be a uniformly convex Banach space which either is uniformly smooth or has a weakly continuous duality map. We introduce and consider three-step Mann iterations for finding a common solution of a general system of variational inequalities (GSVI) and a fixed point problem (FPP) of an infinite family of nonexpansive mappings in X . Here three-step Mann iterations are based on Korpelevich's extragradient method, the viscosity approximation method and the Mann iteration method. We prove the strong convergence of this method to a common solution of the GSVI and the FPP, which solves a variational inequality on their common solution set. We also give a weak convergence theorem for three-step Mann iterations involving the GSVI and the FPP in a Hilbert space. The results presented in this paper improve, extend, supplement and develop the corresponding results announced in the earlier and very recent literature.

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1 Introduction

Let X be a real Banach space whose dual space is denoted by X^* . The normalized duality mapping $J : X \rightarrow 2^{X^*}$ is defined by

$$J(x) = \{x^* \in X^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}, \quad \forall x \in X,$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. It is an immediate consequence of the Hahn-Banach theorem that $J(x)$ is nonempty for each $x \in X$. Let $U = \{x \in X : \|x\| = 1\}$ denote the unite sphere of X . A Banach space X is said to be uniformly convex if for each $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that for all $x, y \in U$,

$$\|x - y\| \geq \epsilon \quad \Rightarrow \quad \|x + y\|/2 \leq 1 - \delta.$$

It is known that a uniformly convex Banach space is reflexive and strictly convex. A Banach space X is said to be smooth if the limit

$$\lim_{t \rightarrow 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for all $x, y \in U$; in this case, X is also said to have a Gateaux differentiable norm. X is said to have a uniformly Gateaux differentiable norm if for each $y \in U$, the limit is attained uniformly for $x \in U$. Moreover, it is said to be uniformly smooth if this limit is attained uniformly for $x, y \in U$. The norm of X is said to be the Frechet differential if for each $x \in U$, this limit is attained uniformly for $y \in U$.

Let C be a nonempty closed convex subset of X , and let $T : C \rightarrow C$ be a nonlinear mapping. Denote by $\text{Fix}(T)$ the set of fixed points of T , i.e., $\text{Fix}(T) = \{x \in C : Tx = x\}$. Recall that T is nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. A mapping $f : C \rightarrow C$ is said to be a contraction on C if there exists a constant ρ in $(0, 1)$ such that $\|f(x) - f(y)\| \leq \rho\|x - y\|$ for all $x, y \in C$. A mapping $A : C \rightarrow X$ is said to be accretive if for each $x, y \in C$ there exists $j(x - y) \in J(x - y)$ such that $\langle Ax - Ay, j(x - y) \rangle \geq 0$.

Recently, Yao *et al.* [1] combined the viscosity approximation method and the Mann iteration method and gave the following hybrid viscosity approximation method.

Let C be a nonempty closed convex subset of a real uniformly smooth Banach space X , let $T : C \rightarrow C$ be a nonexpansive mapping such that $\text{Fix}(T) \neq \emptyset$ and $f \in \mathcal{E}_C$ with a contractive coefficient $\rho \in (0, 1)$, where \mathcal{E}_C is the collection of all contractive self-mappings on C . For an arbitrary $x_0 \in C$, define $\{x_n\}$ in the following way:

$$\begin{cases} y_n = \alpha_n x_n + (1 - \alpha_n)Tx_n, \\ x_{n+1} = \beta_n f(x_n) + (1 - \beta_n)y_n, \quad \forall n \geq 0, \end{cases} \quad (\text{YCY})$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0, 1)$. They proved under certain control conditions on the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ that $\{x_n\}$ converges strongly to a fixed point of T . Subsequently, Ceng and Yao [2] under the convergence of no parameter sequences to zero proved that the sequence $\{x_n\}$ generated by (YCY) converges strongly to a fixed point of T . Such a result includes [1, Theorem 1] as a special case.

Theorem 1.1 (see [2, Theorem 3.1]) *Let C be a nonempty closed convex subset of a uniformly smooth Banach space X . Let $T : C \rightarrow C$ be a nonexpansive mapping with $\text{Fix}(T) \neq \emptyset$ and $f \in \mathcal{E}_C$ with a contractive coefficient $\rho \in (0, 1)$. Given sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in $[0, 1]$, the following control conditions are satisfied:*

- (i) $0 \leq \beta_n \leq 1 - \rho, \forall n \geq n_0$ for some integer $n_0 \geq 0$;
- (ii) $\sum_{n=0}^{\infty} \beta_n = \infty$;
- (iii) $0 < \liminf_{n \rightarrow \infty} \alpha_n \leq \limsup_{n \rightarrow \infty} \alpha_n < 1$;
- (iv) $\lim_{n \rightarrow \infty} \left(\frac{\beta_{n+1}}{1 - (1 - \beta_{n+1})\alpha_{n+1}} - \frac{\beta_n}{1 - (1 - \beta_n)\alpha_n} \right) = 0$.

For an arbitrary $x_0 \in C$, let $\{x_n\}$ be generated by (YCY). Then

$$x_n \rightarrow q \iff \beta_n(f(x_n) - x_n) \rightarrow 0,$$

where $q \in \text{Fix}(T)$ solves the VIP

$$\langle q - f(q), J(q - p) \rangle \leq 0, \quad \forall p \in \text{Fix}(T).$$

Let C be a nonempty closed convex subset of a real Banach space X , and $f \in \Xi_C$ with a contractive coefficient $\rho \in (0, 1)$. Let $\{T_n\}_{n=0}^\infty$ be an infinite family of nonexpansive self-mappings on C , and let $\{\lambda_n\}_{n=0}^\infty$ be a sequence of nonnegative numbers in $[0, 1]$. For any $n \geq 0$, define a self-mapping W_n on C as follows:

$$\left\{ \begin{array}{l} U_{n,n+1} = I, \\ U_{n,n} = \lambda_n T_n U_{n,n+1} + (1 - \lambda_n)I, \\ U_{n,n-1} = \lambda_{n-1} T_{n-1} U_{n,n} + (1 - \lambda_{n-1})I, \\ \dots, \\ U_{n,k} = \lambda_k T_k U_{n,k+1} + (1 - \lambda_k)I, \\ U_{n,k-1} = \lambda_{k-1} T_{k-1} U_{n,k} + (1 - \lambda_{k-1})I, \\ \dots, \\ U_{n,1} = \lambda_1 T_1 U_{n,2} + (1 - \lambda_1)I, \\ W_n = U_{n,0} = \lambda_0 T_0 U_{n,1} + (1 - \lambda_0)I. \end{array} \right. \quad (\text{CWY})$$

Such a mapping W_n is called the W -mapping generated by T_n, T_{n-1}, \dots, T_0 and $\lambda_n, \lambda_{n-1}, \dots, \lambda_0$; see [3].

In 2012, Ceng *et al.* [4] introduced and analyzed the following hybrid viscosity approximation method for finding a common fixed point of an infinite family of nonexpansive mappings in a strictly convex and reflexive Banach space, which either is uniformly smooth or has a weakly continuous duality map J_φ with gauge φ .

Theorem 1.2 (see [4, Theorem 3.3]) *Let C be a nonempty closed convex subset of a reflexive and strictly convex Banach space X . Assume, in addition, that X either is uniformly smooth or has a weakly continuous duality map J_φ with gauge φ . Let $\{T_n\}_{n=0}^\infty$ be an infinite family of nonexpansive self-mappings on C such that the common fixed point set $F := \bigcap_{n=0}^\infty \text{Fix}(T_n) \neq \emptyset$ and $f \in \Xi_C$ with a contractive coefficient $\rho \in (0, 1)$. Given sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\lambda_n\}$ in $[0, 1]$, the following conditions are satisfied:*

- (i) $0 \leq \beta_n \leq 1 - \rho$, $\forall n \geq n_0$ for some $n_0 \geq 0$, and $\sum_{n=0}^\infty \beta_n = \infty$;
- (ii) $0 < \liminf_{n \rightarrow \infty} \alpha_n \leq \limsup_{n \rightarrow \infty} \alpha_n < 1$;
- (iii) $\lim_{n \rightarrow \infty} \left(\frac{\beta_{n+1}}{1 - (1 - \beta_{n+1})\alpha_{n+1}} - \frac{\beta_n}{1 - (1 - \beta_n)\alpha_n} \right) = 0$;
- (iv) $0 < \lambda_n \leq b < 1$, $\forall n \geq 0$ for some constant $b \in (0, 1)$.

For an arbitrary $x_0 \in C$, let $\{x_n\}$ be generated by

$$\left\{ \begin{array}{l} y_n = \alpha_n x_n + (1 - \alpha_n) W_n x_n, \\ x_{n+1} = \beta_n f(x_n) + (1 - \beta_n) y_n, \quad \forall n \geq 0, \end{array} \right.$$

where W_n is the W -mapping generated by T_n, T_{n-1}, \dots, T_0 and $\lambda_n, \lambda_{n-1}, \dots, \lambda_0$. Then

$$x_n \rightarrow q \in F \iff \beta_n (f(x_n) - x_n) \rightarrow 0.$$

In this case,

(i) if X is uniformly smooth, then $q \in F$ solves the VIP

$$\langle q - f(q), J(q - p) \rangle, \quad \forall p \in F;$$

(ii) if X has a weakly continuous duality map J_φ with gauge φ , then $q \in F$ solves the VIP

$$\langle q - f(q), J_\varphi(q - p) \rangle, \quad \forall p \in F.$$

On the other hand, Cai and Bu [5] considered the following general system of variational inequalities (GSVI) in a real smooth Banach space X , which involves finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \mu_1 B_1 y^* + x^* - y^*, J(x - x^*) \rangle \geq 0, & \forall x \in C, \\ \langle \mu_2 B_2 x^* + y^* - x^*, J(x - y^*) \rangle \geq 0, & \forall x \in C, \end{cases} \quad (1.1)$$

where C is a nonempty, closed and convex subset of X , $B_1, B_2 : C \rightarrow X$ are two nonlinear mappings and μ_1 and μ_2 are two positive constants. Here, the set of solutions of GSVI (1.1) is denoted by $\text{GSVI}(C, B_1, B_2)$. In particular, if $X = H$, a real Hilbert space, then GSVI (1.1) reduces to the following GSVI of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \mu_1 B_1 y^* + x^* - y^*, x - x^* \rangle \geq 0, & \forall x \in C, \\ \langle \mu_2 B_2 x^* + y^* - x^*, x - y^* \rangle \geq 0, & \forall x \in C, \end{cases} \quad (1.2)$$

which is studied in Ceng *et al.* [6]. The set of solutions of problem (1.2) is still denoted by $\text{GSVI}(C, B_1, B_2)$. In particular, if $B_1 = B_2 = A$, then problem (1.2) reduces to the new system of variational inequalities (NSVI), introduced and studied by Verma [7]. Further, if $x^* = y^*$ additionally, then the NSVI reduces to the classical variational inequality problem (VIP) of finding $x^* \in C$ such that

$$\langle Ax^*, x - x^* \rangle \geq 0, \quad \forall x \in C. \quad (1.3)$$

The solution set of VIP (1.3) is denoted by $\text{VI}(C, A)$. Variational inequality theory has been studied quite extensively and has emerged as an important tool in the study of a wide class of obstacle, unilateral, free, moving, equilibrium problems. It is now well known that the variational inequalities are equivalent to the fixed point problems, the origin of which can be traced back to Lions and Stampacchia [8]. This alternative formulation has been used to suggest and analyze the projection iterative method for solving variational inequalities under the conditions that the involved operator must be strongly monotone and Lipschitz continuous.

In 1976, Korpelevich [9] proposed an iterative algorithm for solving VIP (1.3) in Euclidean space \mathbf{R}^n :

$$\begin{cases} y_n = P_C(x_n - \tau Ax_n), \\ x_{n+1} = P_C(x_n - \tau Ay_n), \quad \forall n \geq 0, \end{cases} \quad (1.4)$$

with $\tau > 0$ a given number, which is known as the extragradient method (see also [10, 11]). The literature on the VIP is vast and Korpelevich's extragradient method has received great attention from many authors, who improved it in various ways; see, e.g., [5, 8, 12–29] and references therein, to name but a few.

In particular, whenever X is still a real smooth Banach space, $B_1 = B_2 = A$ and $x^* = y^*$, then GSVI (1.1) reduces to the variational inequality problem (VIP) of finding $x^* \in C$ such that

$$\langle Ax^*, J(x - x^*) \rangle \geq 0, \quad \forall x \in C, \tag{1.5}$$

which was considered by Aoyama *et al.* [30]. Note that VIP (1.5) is connected with the fixed point problem for a nonlinear mapping (see, e.g., [31]), the problem of finding a zero point of a nonlinear operator (see, e.g., [32]) and so on. It is clear that VIP (1.5) extends VIP (1.3) from Hilbert spaces to Banach spaces.

In order to find a solution of VIP (1.5), Aoyama *et al.* [30] introduced the following Mann iterative scheme for an accretive operator A :

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) \Pi_C(x_n - \lambda_n A x_n), \quad \forall n \geq 1, \tag{1.6}$$

where Π_C is a sunny nonexpansive retraction from X onto C . Then they proved a weak convergence theorem. For related work, please see [33] and the references therein.

Beyond doubt, it is an interesting and valuable problem of constructing some algorithms with strong convergence for solving GSVI (1.1) which contains VIP (1.5) as a special case. Very recently, Cai and Bu [5] constructed an iterative algorithm for solving GSVI (1.1) and a fixed point problem of an infinite family of nonexpansive mappings in a uniformly convex and 2-uniformly smooth Banach space. They proved the strong convergence of the proposed algorithm by virtue of the following inequality in a 2-uniformly smooth Banach space X .

Lemma 1.1 (see [34]) *Let X be a 2-uniformly smooth Banach space. Then*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, J(x) \rangle + 2\|\kappa y\|^2, \quad \forall x, y \in X,$$

where κ is the 2-uniformly smooth constant of X and J is the normalized duality mapping from X into X^* .

Theorem 1.3 (see [5, Theorem 3.1]) *Let C be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space X . Let Π_C be a sunny nonexpansive retraction from X onto C . Let the mapping $B_i : C \rightarrow X$ be β_i -inverse-strongly accretive with $0 < \mu_i < \frac{\beta_i}{\kappa^2}$ for $i = 1, 2$. Let f be a contraction of C into itself with a coefficient $\rho \in (0, 1)$. Let $\{T_n\}_{n=1}^\infty$ be an infinite family of nonexpansive mappings of C into itself such that $F = \bigcap_{i=1}^\infty \text{Fix}(T_i) \cap \Omega \neq \emptyset$, where Ω is a fixed point set of the mapping $G := \Pi_C(I - \mu_1 B_1) \Pi_C(I - \mu_2 B_2)$. For arbitrarily given $x_1 \in C$, let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} y_n = \alpha_n f(x_n) + (1 - \alpha_n) G x_n, \\ x_{n+1} = \beta_n x_n + (1 - \beta_n) T_n y_n, \end{cases} \quad \forall n \geq 1.$$

Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0, 1)$ satisfying the following conditions:

- (i) $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;
- (ii) $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$.

Assume that $\sum_{n=1}^{\infty} \sup_{x \in D} \|T_{n+1}x - T_nx\| < \infty$ for any bounded subset D of C , and let T be a mapping of C into X defined by $Tx = \lim_{n \rightarrow \infty} T_nx$ for all $x \in C$ and suppose that $\text{Fix}(T) = \bigcap_{n=1}^{\infty} \text{Fix}(T_n)$. Then $\{x_n\}$ converges strongly to $q \in F$, which solves the following VIP:

$$\langle q - f(q), J(q - p) \rangle \leq 0, \quad \forall p \in F.$$

For the convenience of implementing the argument techniques in [6], the authors [5] used the following inequality in a real smooth and uniform convex Banach space X .

Proposition 1.1 (see [35]) *Let X be a real smooth and uniform convex Banach space, and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, 2r] \rightarrow \mathbf{R}$, $g(0) = 0$ such that*

$$g(\|x - y\|) \leq \|x\|^2 - 2\langle x, J(y) \rangle + \|y\|^2, \quad \forall x, y \in B_r,$$

where $B_r = \{x \in X : \|x\| \leq r\}$.

In this paper, let X be a uniformly convex Banach space which either is uniformly smooth or has a weakly continuous duality map. Let C be a nonempty closed convex subset of X , Π_C be a sunny nonexpansive retraction from X onto C and $f \in \mathcal{E}_C$ with a contractive coefficient $\rho \in (0, 1)$. Motivated and inspired by the research going on in this area, we introduce and analyze three-step Mann iterations for finding a common solution of GSVI (1.1) and a fixed point problem (FPP) of an infinite family of nonexpansive self-mappings on C . Here, three-step Mann iterations are based on Korpelevich's extragradient method, the viscosity approximation method and the Mann iteration method. We prove the strong convergence of this method to a common solution of GSVI (1.1) and the FPP, which solves a variational inequality on their common solution set. We also give a weak convergence theorem for three-step Mann iterations involving GSVI (1.2) and the FPP in the case of $X = H$, a Hilbert space. The results presented in this paper improve, extend, supplement and develop the corresponding results announced in the earlier and very recent literature; see, e.g., [2, 4–6, 29].

2 Preliminaries

Let X be a real Banach space. We define a function $\rho : [0, \infty) \rightarrow [0, \infty)$ called the modulus of smoothness of X as follows:

$$\rho(\tau) = \sup \left\{ \frac{1}{2} (\|x + y\| + \|x - y\|) - 1 : x, y \in X, \|x\| = 1, \|y\| = \tau \right\}.$$

It is known that X is uniformly smooth if and only if $\lim_{\tau \rightarrow 0} \rho(\tau)/\tau = 0$. Let q be a fixed real number with $1 < q \leq 2$. Then a Banach space X is said to be q -uniformly smooth if there exists a constant $c > 0$ such that $\rho(\tau) \leq c\tau^q$ for all $\tau > 0$. As pointed out in [36], no Banach space is q -uniformly smooth for $q > 2$. In addition, it is also known that J is single-valued if and only if X is smooth, whereas if X is uniformly smooth, then the mapping J is norm-to-norm uniformly continuous on bounded subsets of X . If X has a uniformly Gateaux

differentiable norm, then the duality mapping J is norm-to-weak* uniformly continuous on bounded subsets of X . We use the notation \rightharpoonup to indicate the weak convergence and \rightarrow to indicate the strong convergence.

Let C be a nonempty closed convex subset of X . Recall that a mapping $A : C \rightarrow X$ is said to be

- (i) α -strongly accretive if for each $x, y \in C$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y) \rangle \geq \alpha \|x - y\|^2$$

for some $\alpha \in (0, 1)$;

- (ii) β -inverse-strongly-accretive if for each $x, y \in C$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y) \rangle \geq \beta \|Ax - Ay\|^2$$

for some $\beta > 0$;

- (iii) λ -strictly pseudocontractive [37] (see also [38]) if for each $x, y \in C$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y) \rangle \leq \|x - y\|^2 - \lambda \|x - y - (Ax - Ay)\|^2$$

for some $\lambda \in (0, 1)$.

It is worth emphasizing that the definition of the inverse strongly accretive mapping is based on that of the inverse strongly monotone mapping, which was studied by so many authors; see, e.g., [17, 39, 40].

We list some lemmas that will be used in the sequel. Lemma 2.1 can be found in [41]. Lemma 2.2 is an immediate consequence of the subdifferential inequality of the function $\frac{1}{2} \|\cdot\|^2$.

Lemma 2.1 *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying the condition*

$$s_{n+1} \leq (1 - \mu_n)s_n + \mu_n v_n, \quad \forall n \geq 0,$$

where $\{\mu_n\}$ and $\{v_n\}$ are sequences of real numbers such that

- (i) $\{\mu_n\} \subset [0, 1]$ and $\sum_{n=0}^{\infty} \mu_n = \infty$, or equivalently,

$$\prod_{n=0}^{\infty} (1 - \mu_n) := \lim_{n \rightarrow \infty} \prod_{k=0}^n (1 - \mu_k) = 0;$$

- (ii) $\limsup_{n \rightarrow \infty} v_n \leq 0$, or $\sum_{n=0}^{\infty} \mu_n |v_n| < \infty$.

Then $\limsup_{n \rightarrow \infty} s_n = 0$.

Lemma 2.2 (see [42]) *Let X be a real Banach space and J be the normalized duality map on X . Then, for any given $x, y \in X$, the following inequality holds:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y) \rangle, \quad \forall j(x + y) \in J(x + y).$$

Let D be a subset of C , and let Π be a mapping of C into D . Then Π is said to be sunny if

$$\Pi[\Pi(x) + t(x - \Pi(x))] = \Pi(x),$$

whenever $\Pi(x) + t(x - \Pi(x)) \in C$ for $x \in C$ and $t \geq 0$. A mapping Π of C into itself is called a retraction if $\Pi^2 = \Pi$. If a mapping Π of C into itself is a retraction, then $\Pi(z) = z$ for every $z \in R(\Pi)$, where $R(\Pi)$ is the range of Π . A subset D of C is called a sunny nonexpansive retract of C if there exists a sunny nonexpansive retraction from C onto D .

The following lemma concerns the sunny nonexpansive retraction.

Lemma 2.3 (see [43]) *Let C be a nonempty closed convex subset of a real smooth Banach space X . Let D be a nonempty subset of C . Let Π be a retraction of C onto D . Then the following are equivalent:*

- (i) Π is sunny and nonexpansive;
- (ii) $\|\Pi(x) - \Pi(y)\|^2 \leq \langle x - y, J(\Pi(x) - \Pi(y)) \rangle$, $\forall x, y \in C$;
- (iii) $\langle x - \Pi(x), J(y - \Pi(x)) \rangle \leq 0$, $\forall x \in C, y \in D$.

It is well known that if $X = H$, a Hilbert space, then a sunny nonexpansive retraction Π_C is coincident with the metric projection from X onto C ; that is, $\Pi_C = P_C$. If C is a nonempty closed convex subset of a strictly convex and uniformly smooth Banach space X and if $T : C \rightarrow C$ is a nonexpansive mapping with the fixed point set $\text{Fix}(T) \neq \emptyset$, then the set $\text{Fix}(T)$ is a sunny nonexpansive retract of C .

Lemma 2.4 *Let C be a nonempty closed convex subset of a smooth Banach space X . Let Π_C be a sunny nonexpansive retraction from X onto C , and let $B_1, B_2 : C \rightarrow X$ be nonlinear mappings. For given $x^*, y^* \in C$, (x^*, y^*) is a solution of GSVI (1.1) if and only if $x^* = \Pi_C(y^* - \mu_1 B_1 y^*)$, where $y^* = \Pi_C(x^* - \mu_2 B_2 x^*)$.*

Proof We can rewrite GSVI (1.1) as

$$\begin{cases} \langle x^* - (y^* - \mu_1 B_1 y^*), J(x - x^*) \rangle \geq 0, & \forall x \in C, \\ \langle y^* - (x^* - \mu_2 B_2 x^*), J(x - y^*) \rangle \geq 0, & \forall x \in C, \end{cases}$$

which is obviously equivalent to

$$\begin{cases} x^* = \Pi_C(y^* - \mu_1 B_1 y^*), \\ y^* = \Pi_C(x^* - \mu_2 B_2 x^*) \end{cases}$$

because of Lemma 2.3. This completes the proof. □

In terms of Lemma 2.4, we observe that

$$x^* = \Pi_C[\Pi_C(x^* - \mu_2 B_2 x^*) - \mu_1 B_1 \Pi_C(x^* - \mu_2 B_2 x^*)],$$

which implies that x^* is a fixed point of the mapping G . Throughout this paper, the set of fixed points of the mapping G is denoted by Ω .

Lemma 2.5 (see [44]) *Let X be a uniformly smooth Banach space, C be a nonempty closed convex subset of X , $T : C \rightarrow C$ be a nonexpansive mapping with $\text{Fix}(T) \neq \emptyset$, and $f \in \mathcal{E}_C$. Then the net $\{x_t\}$ defined by $x_t = tf(x_t) + (1-t)Tx_t, \forall t \in (0,1)$, converges strongly to a point in $\text{Fix}(T)$. If we define a mapping $Q : \mathcal{E}_C \rightarrow \text{Fix}(T)$ by $Q(f) := s - \lim_{t \rightarrow 0} x_t, \forall f \in \mathcal{E}_C$, then $Q(f)$ solves the VIP*

$$\langle (I-f)Q(f), J(Q(f)-p) \rangle \leq 0, \quad \forall p \in \text{Fix}(T).$$

In particular, if $f = u \in C$ is a constant, then the map $u \mapsto Q(u)$ is reduced to the sunny nonexpansive retraction of Reich type from C onto $\text{Fix}(T)$, i.e.,

$$\langle Q(u) - u, J(Q(u) - p) \rangle \leq 0, \quad \forall p \in \text{Fix}(T).$$

Recall that a gauge is a continuous strictly increasing function $\varphi : [0, \infty) \rightarrow [0, \infty)$ such that $\varphi(0) = 0$ and $\varphi(t) \rightarrow \infty$ as $t \rightarrow \infty$. Associated to the gauge φ is the duality map $J_\varphi : X \rightarrow 2^{X^*}$ defined by

$$J_\varphi(x) = \{x^* \in X^* : \langle x, x^* \rangle = \|x\|\varphi(\|x\|), \|x^*\| = \varphi(\|x\|)\}, \quad \forall x \in X.$$

We say that a Banach space X has a weakly continuous duality map if there exists a gauge φ for which the duality map J_φ is single-valued and weak-to-weak* sequentially continuous. It is known that ℓ^p has a weakly continuous duality map with gauge $\varphi(t) = t^{p-1}$ for all $1 < p < \infty$. Set

$$\Phi(t) = \int_0^t \varphi(\tau) d\tau, \quad \forall t \geq 0.$$

Then $J_\varphi(x) = \partial\Phi(\|x\|)$ for all $x \in X$, where ∂ denotes the subdifferential in the sense of convex analysis; see [42] for more details.

The first part of the following lemma is an immediate consequence of the subdifferential inequality, and the proof of the second part can be found in [45].

Lemma 2.6 *Assume that X has a weakly continuous duality map J_φ with gauge φ .*

(i) *For all $x, y \in X$, the following inequality holds:*

$$\Phi(\|x+y\|) \leq \Phi(\|x\|) + \langle y, J_\varphi(x+y) \rangle.$$

(ii) *Assume that a sequence $\{x_n\}$ in X is weakly convergent to a point x . Then the following identity holds:*

$$\limsup_{n \rightarrow \infty} \Phi(\|x_n - y\|) = \limsup_{n \rightarrow \infty} \Phi(\|x_n - x\|) + \Phi(\|y - x\|), \quad \forall y \in X.$$

Lemma 2.7 ([31, Theorem 3.1 and its proof]) *Let X be a reflexive Banach space and have a weakly continuous duality map J_φ with gauge φ , let C be a nonempty closed convex subset of X , let $T : C \rightarrow C$ be a nonexpansive mapping with $\text{Fix}(T) \neq \emptyset$, and let $f \in \mathcal{E}_C$. Then $\{x_t\}$ defined by $x_t = tf(x_t) + (1-t)Tx_t, \forall t \in (0,1)$, converges strongly to a point in $\text{Fix}(T)$ as*

$t \rightarrow 0^+$. Define $Q: E_C \rightarrow \text{Fix}(T)$ by $Q(f) := s - \lim_{t \rightarrow 0^+} x_t$. Then $Q(f)$ solves the variational inequality

$$\langle (I - f)Q(f), J_\varphi(Q(f) - p) \rangle \leq 0, \quad \forall p \in \text{Fix}(T).$$

In particular, if $f = u \in C$ is a constant, then the map $u \mapsto Q(u)$ is reduced to the sunny nonexpansive retraction of Reich type from C onto $\text{Fix}(T)$, i.e.,

$$\langle Q(u) - u, J_\varphi(Q(u) - p) \rangle \leq 0, \quad \forall p \in \text{Fix}(T).$$

Recall that X satisfies Opial's property [46] provided, for each sequence $\{x_n\}$ in X , the condition $x_n \rightharpoonup x$ implies

$$\limsup_{n \rightarrow \infty} \|x_n - x\| < \limsup_{n \rightarrow \infty} \|x_n - y\|, \quad \forall y \in X, y \neq x.$$

It is known in [46] that each l^p ($1 \leq p < \infty$) enjoys this property, while L^p does not unless $p = 2$. It is known in [47] that every separable Banach space can be equivalently renormed so that it satisfies Opial's property. We denote by $\omega_w(x_n)$ the weak ω -limit set of $\{x_n\}$, i.e.,

$$\omega_w(x_n) = \{ \bar{x} \in X : x_{n_i} \rightharpoonup \bar{x} \text{ for some subsequence } \{x_{n_i}\} \text{ of } \{x_n\} \}. \quad (2.1)$$

Also, recall that in a Hilbert space H , the following equality holds:

$$\| \lambda x + (1 - \lambda)y \|^2 = \lambda \|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2, \quad \forall x, y \in H, \forall \lambda \in [0, 1]. \quad (2.2)$$

Lemma 2.8 (see [48]) *Let C be a nonempty closed convex subset of a strictly convex Banach space X . Let $\{T_n\}_{n=0}^\infty$ be a sequence of nonexpansive mappings on C . Suppose that $\bigcap_{n=0}^\infty \text{Fix}(T_n)$ is nonempty. Let $\{\lambda_n\}$ be a sequence of positive numbers with $\sum_{n=0}^\infty \lambda_n = 1$. Then a mapping S on C defined by $Sx = \sum_{n=0}^\infty \lambda_n T_n x$ for $x \in C$ is defined well, nonexpansive and $\text{Fix}(S) = \bigcap_{n=0}^\infty \text{Fix}(T_n)$ holds.*

Lemma 2.9 (see [30]) *Let C be a nonempty closed convex subset of a smooth Banach space X . Let Π_C be a sunny nonexpansive retraction from X onto C , and let $A: C \rightarrow X$ be an accretive mapping. Then, for all $\lambda > 0$,*

$$\text{VI}(C, A) = \text{Fix}(\Pi_C(I - \lambda A)).$$

Lemma 2.10 (see [49, Lemma 3.2]) *Let C be a nonempty closed convex subset of a strictly convex Banach space X . Let $\{T_n\}_{n=0}^\infty$ be a sequence of nonexpansive self-mappings on C such that $\bigcap_{n=0}^\infty \text{Fix}(T_n) \neq \emptyset$, and let $\{\lambda_n\}_{n=0}^\infty$ be a sequence of positive numbers in $(0, b]$ for some $b \in (0, 1)$. Then, for every $x \in C$ and $k \geq 0$, the limit $\lim_{n \rightarrow \infty} U_{n,k}x$ exists.*

Using Lemma 2.10, one can define a mapping $W: C \rightarrow C$ as follows:

$$Wx = \lim_{n \rightarrow \infty} W_n x = \lim_{n \rightarrow \infty} U_{n,0}x, \quad \forall x \in C. \quad (2.3)$$

Such W is called the W -mapping generated by the sequences $\{T_n\}_{n=0}^\infty$ and $\{\lambda_n\}_{n=0}^\infty$. Throughout this paper, we always assume that $\{\lambda_n\}_{n=0}^\infty$ is a sequence of positive numbers in $(0, b]$ for some $b \in (0, 1)$.

Lemma 2.11 (see [49, Lemma 3.3]) *Let C be a nonempty closed convex subset of a strictly convex Banach space X . Let $\{T_n\}_{n=0}^\infty$ be a sequence of nonexpansive self-mappings on C such that $\bigcap_{n=0}^\infty \text{Fix}(T_n) \neq \emptyset$, and let $\{\lambda_n\}_{n=0}^\infty$ be a sequence of positive numbers in $(0, b]$ for some $b \in (0, 1)$. Then $\text{Fix}(W) = \bigcap_{n=0}^\infty \text{Fix}(T_n)$.*

Lemma 2.12 (see [50, Lemma 2]) *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space X , and let $\{\beta_n\}$ be a sequence of nonnegative numbers in $[0, 1]$ with $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$. Suppose that $x_{n+1} = \beta_n x_n + (1 - \beta_n)z_n$ for all integers $n \geq 0$ and $\limsup_{n \rightarrow \infty} (\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then $\lim_{n \rightarrow \infty} \|x_n - z_n\| = 0$.*

Lemma 2.13 (see [34]) *Given a number $r > 0$. A real Banach space X is uniformly convex if and only if there exists a continuous strictly increasing function $g : [0, \infty) \rightarrow [0, \infty)$, $g(0) = 0$ such that*

$$\|\lambda x + (1 - \lambda)y\|^2 \leq \lambda \|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)g(\|x - y\|)$$

for all $\lambda \in [0, 1]$ and $x, y \in X$ such that $\|x\| \leq r$ and $\|y\| \leq r$.

We will also use the following elementary lemmas in the sequel.

Lemma 2.14 (see [51]) *Let $\{a_n\}$ and $\{b_n\}$ be the sequences of nonnegative real numbers such that $\sum_{n=0}^\infty b_n < \infty$ and $a_{n+1} \leq a_n + b_n$ for all $n \geq 0$. Then $\lim_{n \rightarrow \infty} a_n$ exists.*

Lemma 2.15 (Demiclosedness principle [42]) *Assume that T is a nonexpansive self-mapping of a nonempty closed convex subset C of a Hilbert space H . If T has a fixed point, then $I - T$ is demiclosed. That is, whenever $x_n \rightarrow x$ in C and $(I - T)x_n \rightarrow y$ in H , it follows that $(I - T)x = y$. Here, I is the identity operator of H .*

3 Main results

In this section, in order to prove our main results, we will use the following useful lemmas whose proofs will be omitted since they can be proved by standard arguments.

Lemma 3.1 *Let C be a nonempty closed convex subset of a smooth Banach space X , and let the mapping $B_i : C \rightarrow X$ be ζ_i -strictly pseudocontractive and η_i -strongly accretive with $\zeta_i + \eta_i \geq 1$ for $i = 1, 2$. Then, for $\mu_i \in (0, 1]$, we have*

$$\|(I - \mu_i B_i)x - (I - \mu_i B_i)y\| \leq \left\{ \sqrt{\frac{1 - \eta_i}{\zeta_i}} + (1 - \mu_i) \left(1 + \frac{1}{\zeta_i} \right) \right\} \|x - y\|, \quad \forall x, y \in C$$

for $i = 1, 2$. In particular, if $1 - \frac{\zeta_i}{1 + \zeta_i} (1 - \sqrt{\frac{1 - \eta_i}{\zeta_i}}) \leq \mu_i \leq 1$, then $I - \mu_i B_i$ is nonexpansive for $i = 1, 2$.

Lemma 3.2 *Let C be a nonempty closed convex subset of a smooth Banach space X . Let Π_C be a sunny nonexpansive retraction from X onto C , and let the mapping $B_i : C \rightarrow X$ be ζ_i -strictly pseudocontractive and η_i -strongly accretive with $\zeta_i + \eta_i \geq 1$ for $i = 1, 2$. Let $G : C \rightarrow C$ be the mapping defined by*

$$G(x) = \Pi_C[\Pi_C(x - \mu_2 B_2 x) - \mu_1 B_1 \Pi_C(x - \mu_2 B_2 x)], \quad \forall x \in C.$$

If $1 - \frac{\zeta_i}{1+\zeta_i}(1 - \sqrt{\frac{1-\eta_i}{\zeta_i}}) \leq \mu_i \leq 1$, then $G : C \rightarrow C$ is nonexpansive.

We now state and prove the main result of this paper.

Theorem 3.1 *Let C be a nonempty closed convex subset of a uniformly convex Banach space X . Assume, in addition, that X either is uniformly smooth or has a weakly continuous duality map J_φ with gauge φ . Let Π_C be a sunny nonexpansive retraction from X onto C . Let the mapping $B_i : C \rightarrow X$ be ζ_i -strictly pseudocontractive and η_i -strongly accretive with $\zeta_i + \eta_i \geq 1$ for $i = 1, 2$. Let $f \in \mathcal{E}_C$ with a contractive coefficient $\rho \in (0, 1)$. Let $\{\lambda_n\}_{n=0}^\infty$ be a sequence of positive numbers in $(0, b]$ for some $b \in (0, 1)$ and $\{T_n\}_{n=0}^\infty$ be an infinite family of nonexpansive self-mappings on C such that $F := \bigcap_{n=0}^\infty \text{Fix}(T_n) \cap \Omega \neq \emptyset$, where Ω is the fixed point set of the mapping $G = \Pi_C(I - \mu_1 B_1)\Pi_C(I - \mu_2 B_2)$ with $1 - \frac{\zeta_i}{1+\zeta_i}(1 - \sqrt{\frac{1-\eta_i}{\zeta_i}}) \leq \mu_i \leq 1$ for $i = 1, 2$. For an arbitrary $x_0 \in C$, let $\{x_n\}$ be generated by*

$$\begin{cases} z_n = \sigma_n x_n + (1 - \sigma_n) G x_n, \\ y_n = \alpha_n x_n + (1 - \alpha_n) W_n z_n, \\ x_{n+1} = \beta_n f(x_n) + (1 - \beta_n) y_n, \quad \forall n \geq 0, \end{cases} \quad (3.1)$$

where W_n is the W -mapping generated by T_n, T_{n-1}, \dots, T_0 and $\lambda_n, \lambda_{n-1}, \dots, \lambda_0$, and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\sigma_n\}$ are sequences in $[0, 1]$. Suppose that the following conditions hold:

- (i) $0 \leq \beta_n \leq 1 - \rho$, $\forall n \geq n_0$ for some $n_0 \geq 0$, and $\sum_{n=0}^\infty \beta_n = \infty$;
- (ii) $\lim_{n \rightarrow \infty} \left| \frac{\beta_{n+1}}{1 - (1 - \beta_{n+1})\alpha_{n+1}} - \frac{\beta_n}{1 - (1 - \beta_n)\alpha_n} \right| = 0$ and $\lim_{n \rightarrow \infty} |\sigma_{n+1} - \sigma_n| = 0$;
- (iii) $0 < \liminf_{n \rightarrow \infty} \alpha_n \leq \limsup_{n \rightarrow \infty} \alpha_n < 1$;
- (iv) $0 < \liminf_{n \rightarrow \infty} \sigma_n \leq \limsup_{n \rightarrow \infty} \sigma_n < 1$.

Then

$$x_n \rightarrow q \in F \iff \beta_n(f(x_n) - x_n) \rightarrow 0.$$

In this case,

- (i) if X is uniformly smooth, then $q \in F$ solves the VIP

$$\langle q - f(q), J(q - p) \rangle, \quad \forall p \in F;$$

- (ii) if X has a weakly continuous duality map J_φ with gauge φ , then $q \in F$ solves the VIP

$$\langle q - f(q), J_\varphi(q - p) \rangle \leq 0, \quad \forall p \in F.$$

Proof First of all, let us show that $\{x_n\}$ is bounded. Indeed, taking an element $p \in F = \bigcap_{n=0}^\infty \text{Fix}(T_n) \cap \Omega$ arbitrarily, we obtain that $Gp = p$ and $p = W_n p$ for all $n \geq 0$. By

Lemma 3.2, we know that G is nonexpansive. It follows from the nonexpansivity of G and W_n that

$$\begin{aligned} \|z_n - p\| &\leq \sigma_n \|x_n - p\| + (1 - \sigma_n) \|Gx_n - p\| \leq \|x_n - p\|, \\ \|y_n - p\| &\leq \alpha_n \|x_n - p\| + (1 - \alpha_n) \|W_n z_n - p\| \\ &\leq \alpha_n \|x_n - p\| + (1 - \alpha_n) \|z_n - p\| \\ &\leq \alpha_n \|x_n - p\| + (1 - \alpha_n) \|x_n - p\| \\ &= \|x_n - p\| \end{aligned}$$

and

$$\begin{aligned} \|x_{n+1} - p\| &\leq \beta_n \|f(x_n) - p\| + (1 - \beta_n) \|y_n - p\| \\ &\leq \beta_n (\|f(x_n) - f(p)\| + \|f(p) - p\|) + (1 - \beta_n) \|x_n - p\| \\ &\leq \beta_n (\rho \|x_n - p\| + \|f(p) - p\|) + (1 - \beta_n) \|x_n - p\| \\ &= (1 - (1 - \rho)\beta_n) \|x_n - p\| + \beta_n \|f(p) - p\| \\ &\leq \max \left\{ \|x_n - p\|, \frac{\|f(p) - p\|}{1 - \rho} \right\}. \end{aligned}$$

By induction, we have

$$\|x_n - p\| \leq \max \left\{ \|x_0 - p\|, \frac{\|f(p) - p\|}{1 - \rho} \right\}, \quad \forall n \geq 0.$$

Hence $\{x_n\}$ is bounded, and so are the sequences $\{y_n\}$, $\{z_n\}$, $\{Gx_n\}$, $\{W_n z_n\}$ and $\{f(x_n)\}$.

Suppose that $x_n \rightarrow q \in F$ as $n \rightarrow \infty$. Then $q = Gq$ and $q = W_n q$ for all $n \geq 0$. From (3.1) it follows that

$$\begin{aligned} \|z_n - q\| &\leq \sigma_n \|x_n - q\| + (1 - \sigma_n) \|Gx_n - q\| \\ &\leq \sigma_n \|x_n - q\| + (1 - \sigma_n) \|x_n - q\| \\ &= \|x_n - q\| \end{aligned}$$

and

$$\begin{aligned} \|y_n - q\| &\leq \alpha_n \|x_n - q\| + (1 - \alpha_n) \|W_n z_n - q\| \\ &\leq \alpha_n \|x_n - q\| + (1 - \alpha_n) \|z_n - q\| \\ &\leq \alpha_n \|x_n - q\| + (1 - \alpha_n) \|x_n - q\| \\ &= \|x_n - q\| \rightarrow 0 \quad (n \rightarrow \infty), \end{aligned}$$

that is, $x_n \rightarrow q$. Again from (3.1) we obtain that

$$\begin{aligned} \|\beta_n (f(x_n) - x_n)\| &= \|x_{n+1} - x_n - (1 - \beta_n)(y_n - x_n)\| \\ &\leq \|x_{n+1} - x_n\| + (1 - \beta_n) \|y_n - x_n\| \rightarrow 0. \end{aligned}$$

Conversely, suppose that $\beta_n(f(x_n) - x_n) \rightarrow 0$ ($n \rightarrow \infty$). Put $\gamma_n = (1 - \beta_n)\alpha_n$ for each $n \geq 0$. Then it follows from conditions (i) and (iii) that

$$\alpha_n \geq \gamma_n = (1 - \beta_n)\alpha_n \geq (1 - (1 - \rho))\alpha_n = \rho\alpha_n, \quad \forall n \geq n_0,$$

and hence

$$0 < \liminf_{n \rightarrow \infty} \gamma_n \leq \limsup_{n \rightarrow \infty} \gamma_n < 1. \tag{3.2}$$

Define \hat{z}_n by

$$x_{n+1} = \gamma_n x_n + (1 - \gamma_n)\hat{z}_n, \quad \forall n \geq 0. \tag{3.3}$$

Observe that

$$\begin{aligned} \hat{z}_{n+1} - \hat{z}_n &= \frac{x_{n+2} - \gamma_{n+1}x_{n+1}}{1 - \gamma_{n+1}} - \frac{x_{n+1} - \gamma_n x_n}{1 - \gamma_n} \\ &= \frac{\beta_{n+1}f(x_{n+1}) + (1 - \beta_{n+1})\gamma_{n+1} - \gamma_{n+1}x_{n+1}}{1 - \gamma_{n+1}} - \frac{\beta_n f(x_n) + (1 - \beta_n)\gamma_n - \gamma_n x_n}{1 - \gamma_n} \\ &= \left(\frac{\beta_{n+1}f(x_{n+1})}{1 - \gamma_{n+1}} - \frac{\beta_n f(x_n)}{1 - \gamma_n} \right) - \frac{(1 - \beta_n)[\alpha_n x_n + (1 - \alpha_n)W_n z_n] - \gamma_n x_n}{1 - \gamma_n} \\ &\quad + \frac{(1 - \beta_{n+1})[\alpha_{n+1}x_{n+1} + (1 - \alpha_{n+1})W_{n+1}z_{n+1}] - \gamma_{n+1}x_{n+1}}{1 - \gamma_{n+1}} \\ &= \left(\frac{\beta_{n+1}f(x_{n+1})}{1 - \gamma_{n+1}} - \frac{\beta_n f(x_n)}{1 - \gamma_n} \right) \\ &\quad + \frac{(1 - \beta_{n+1})(1 - \alpha_{n+1})W_{n+1}z_{n+1}}{1 - \gamma_{n+1}} - \frac{(1 - \beta_n)(1 - \alpha_n)W_n z_n}{1 - \gamma_n} \\ &= \left(\frac{\beta_{n+1}f(x_{n+1})}{1 - \gamma_{n+1}} - \frac{\beta_n f(x_n)}{1 - \gamma_n} \right) + (W_{n+1}z_{n+1} - W_n z_n) \\ &\quad - \frac{\beta_{n+1}}{1 - \gamma_{n+1}}W_{n+1}z_{n+1} + \frac{\beta_n}{1 - \gamma_n}W_n z_n \\ &= \left(\frac{\beta_{n+1}}{1 - \gamma_{n+1}} - \frac{\beta_n}{1 - \gamma_n} \right)f(x_{n+1}) + \frac{\beta_n}{1 - \gamma_n}(f(x_{n+1}) - f(x_n)) + (W_{n+1}z_{n+1} - W_n z_n) \\ &\quad - \left(\frac{\beta_{n+1}}{1 - \gamma_{n+1}} - \frac{\beta_n}{1 - \gamma_n} \right)W_{n+1}z_{n+1} - (W_{n+1}z_{n+1} - W_n z_n)\frac{\beta_n}{1 - \gamma_n} \\ &= \left(\frac{\beta_{n+1}}{1 - \gamma_{n+1}} - \frac{\beta_n}{1 - \gamma_n} \right)(f(x_{n+1}) - W_{n+1}z_{n+1}) + \frac{\beta_n}{1 - \gamma_n}(f(x_{n+1}) - f(x_n)) \\ &\quad + \frac{1 - \gamma_n - \beta_n}{1 - \gamma_n}(W_{n+1}z_{n+1} - W_n z_n). \end{aligned}$$

It follows that

$$\begin{aligned} \|\hat{z}_{n+1} - \hat{z}_n\| &\leq \left| \frac{\beta_{n+1}}{1 - \gamma_{n+1}} - \frac{\beta_n}{1 - \gamma_n} \right| \|f(x_{n+1}) - W_{n+1}z_{n+1}\| + \frac{\beta_n}{1 - \gamma_n} \|f(x_{n+1}) - f(x_n)\| \\ &\quad + \frac{1 - \gamma_n - \beta_n}{1 - \gamma_n} \|W_{n+1}z_{n+1} - W_n z_n\| \end{aligned}$$

$$\begin{aligned}
 &\leq \left| \frac{\beta_{n+1}}{1-\gamma_{n+1}} - \frac{\beta_n}{1-\gamma_n} \right| (\|f(x_{n+1})\| + \|W_{n+1}z_{n+1}\|) + \frac{\rho\beta_n}{1-\gamma_n} \|x_{n+1} - x_n\| \\
 &\quad + \frac{1-\gamma_n-\beta_n}{1-\gamma_n} (\|W_{n+1}z_{n+1} - W_{n+1}z_n\| + \|W_{n+1}z_n - W_nz_n\|) \\
 &\leq \left| \frac{\beta_{n+1}}{1-\gamma_{n+1}} - \frac{\beta_n}{1-\gamma_n} \right| (\|f(x_{n+1})\| + \|W_{n+1}z_{n+1}\|) + \frac{\rho\beta_n}{1-\gamma_n} \|x_{n+1} - x_n\| \\
 &\quad + \frac{1-\gamma_n-\beta_n}{1-\gamma_n} (\|z_{n+1} - z_n\| + \|W_{n+1}z_n - W_nz_n\|). \tag{3.4}
 \end{aligned}$$

In the meantime, simple calculations show that

$$z_{n+1} - z_n = \sigma_{n+1}(x_{n+1} - x_n) + (1 - \sigma_{n+1})(Gx_{n+1} - Gx_n) + (\sigma_{n+1} - \sigma_n)(x_n - Gx_n),$$

which hence yields

$$\begin{aligned}
 \|z_{n+1} - z_n\| &\leq \sigma_{n+1}\|x_{n+1} - x_n\| + (1 - \sigma_{n+1})\|Gx_{n+1} - Gx_n\| + |\sigma_{n+1} - \sigma_n|\|x_n - Gx_n\| \\
 &\leq \sigma_{n+1}\|x_{n+1} - x_n\| + (1 - \sigma_{n+1})\|x_{n+1} - x_n\| + |\sigma_{n+1} - \sigma_n|\|x_n - Gx_n\| \\
 &= \|x_{n+1} - x_n\| + |\sigma_{n+1} - \sigma_n|\|x_n - Gx_n\|. \tag{3.5}
 \end{aligned}$$

Taking into account the nonexpansivity of T_k and $U_{n,k}$, from (CWY) we have

$$\begin{aligned}
 \|W_{n+1}z_n - W_nz_n\| &= \|\lambda_0 T_0 U_{n+1,1}z_n - \lambda_0 T_0 U_{n,1}z_n\| \\
 &\leq \lambda_0 \|U_{n+1,1}z_n - U_{n,1}z_n\| \\
 &= \lambda_0 \|\lambda_1 T_1 U_{n+1,2}z_n - \lambda_1 T_1 U_{n,2}z_n\| \\
 &\leq \lambda_0 \lambda_1 \|U_{n+1,2}z_n - U_{n,2}z_n\| \\
 &\leq \dots \\
 &\leq \lambda_0 \lambda_1 \dots \lambda_n \|U_{n+1,n+1}z_n - U_{n,n+1}z_n\| \\
 &= \lambda_0 \lambda_1 \dots \lambda_{n+1} \|T_{n+1}z_n - z_n\| \\
 &\leq M_0 \prod_{k=0}^{n+1} \lambda_k, \tag{3.6}
 \end{aligned}$$

where $\sup_{n \geq 0} \|T_{n+1}z_n - z_n\| \leq M_0$ for some $M_0 > 0$. Thus, from (3.4), (3.5) and (3.6), we get

$$\begin{aligned}
 &\|\hat{z}_{n+1} - \hat{z}_n\| \\
 &\leq \left| \frac{\beta_{n+1}}{1-\gamma_{n+1}} - \frac{\beta_n}{1-\gamma_n} \right| (\|f(x_{n+1})\| + \|W_{n+1}z_{n+1}\|) + \frac{\rho\beta_n}{1-\gamma_n} \|x_{n+1} - x_n\| \\
 &\quad + \frac{1-\gamma_n-\beta_n}{1-\gamma_n} \left(\|x_{n+1} - x_n\| + |\sigma_{n+1} - \sigma_n|\|x_n - Gx_n\| + M_0 \prod_{k=0}^{n+1} \lambda_k \right) \\
 &\leq \|x_{n+1} - x_n\| + \left| \frac{\beta_{n+1}}{1-\gamma_{n+1}} - \frac{\beta_n}{1-\gamma_n} \right| (\|f(x_{n+1})\| + \|W_{n+1}z_{n+1}\|)
 \end{aligned}$$

$$\begin{aligned}
 & + |\sigma_{n+1} - \sigma_n| \|x_n - Gx_n\| + M_0 \prod_{k=0}^{n+1} \lambda_k \\
 & \leq \|x_{n+1} - x_n\| + M \left(\left| \frac{\beta_{n+1}}{1 - \gamma_{n+1}} - \frac{\beta_n}{1 - \gamma_n} \right| + |\sigma_{n+1} - \sigma_n| + \prod_{k=0}^{n+1} \lambda_k \right),
 \end{aligned}$$

where $\sup_{n \geq 0} \{ \|f(x_n)\| + \|W_n z_n\| + \|Gx_n\| + \|x_n\| + M_0 \} \leq M$ for some $M > 0$. Then it immediately follows that

$$\|\hat{z}_{n+1} - \hat{z}_n\| - \|x_{n+1} - x_n\| \leq M \left(\left| \frac{\beta_{n+1}}{1 - \gamma_{n+1}} - \frac{\beta_n}{1 - \gamma_n} \right| + |\sigma_{n+1} - \sigma_n| + \prod_{k=0}^{n+1} \lambda_k \right).$$

From condition (ii) and $0 < \lambda_k \leq b < 1, \forall k \geq 0$, we deduce that

$$\limsup_{n \rightarrow \infty} (\|\hat{z}_{n+1} - \hat{z}_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Hence by Lemma 2.12 we have

$$\lim_{n \rightarrow \infty} \|\hat{z}_n - x_n\| = 0.$$

It follows from (3.2) and (3.3) that

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = \lim_{n \rightarrow \infty} (1 - \gamma_n) \|\hat{z}_n - x_n\| = 0.$$

From (3.1) we have

$$x_{n+1} - x_n = \beta_n (f(x_n) - x_n) + (1 - \beta_n)(y_n - x_n).$$

This implies that

$$\begin{aligned}
 \rho \|y_n - x_n\| & \leq (1 - \beta_n) \|y_n - x_n\| \\
 & = \|x_{n+1} - x_n - \beta_n (f(x_n) - x_n)\| \\
 & \leq \|x_{n+1} - x_n\| + \|\beta_n (f(x_n) - x_n)\|.
 \end{aligned}$$

Since $x_{n+1} - x_n \rightarrow 0$ and $\beta_n (f(x_n) - x_n) \rightarrow 0$, we get

$$\lim_{n \rightarrow \infty} \|y_n - x_n\| = 0. \tag{3.7}$$

Observe that

$$y_n - x_n = (1 - \alpha_n)(W_n z_n - x_n). \tag{3.8}$$

It follows from condition (iii), (3.7) and (3.8) that

$$\lim_{n \rightarrow \infty} \|x_n - W_n z_n\| = 0. \tag{3.9}$$

Also, utilizing Lemma 2.13, we obtain from (3.1) that for $p \in F$

$$\begin{aligned} \|z_n - p\|^2 &= \|\sigma_n(x_n - p) + (1 - \sigma_n)(Gx_n - p)\|^2 \\ &\leq \sigma_n \|x_n - p\|^2 + (1 - \sigma_n) \|Gx_n - p\|^2 - \sigma_n(1 - \sigma_n)g(\|x_n - Gx_n\|) \\ &\leq \sigma_n \|x_n - p\|^2 + (1 - \sigma_n) \|x_n - p\|^2 - \sigma_n(1 - \sigma_n)g(\|x_n - Gx_n\|) \\ &= \|x_n - p\|^2 - \sigma_n(1 - \sigma_n)g(\|x_n - Gx_n\|), \end{aligned}$$

and hence

$$\begin{aligned} \|y_n - p\|^2 &\leq \alpha_n \|x_n - p\|^2 + (1 - \alpha_n) \|W_n z_n - p\|^2 \\ &\leq \alpha_n \|x_n - p\|^2 + (1 - \alpha_n) \|z_n - p\|^2 \\ &\leq \alpha_n \|x_n - p\|^2 + (1 - \alpha_n) [\|x_n - p\|^2 - \sigma_n(1 - \sigma_n)g(\|x_n - Gx_n\|)] \\ &= \|x_n - p\|^2 - (1 - \alpha_n)\sigma_n(1 - \sigma_n)g(\|x_n - Gx_n\|). \end{aligned}$$

Thus, we get

$$\begin{aligned} (1 - \alpha_n)\sigma_n(1 - \sigma_n)g(\|x_n - Gx_n\|) &\leq \|x_n - p\|^2 - \|y_n - p\|^2 \\ &\leq (\|x_n - p\| + \|y_n - p\|) \|x_n - y_n\|. \end{aligned}$$

From (3.7), conditions (iii), (iv) and the boundedness of $\{x_n\}$ and $\{y_n\}$, it follows that

$$\lim_{n \rightarrow \infty} g(\|x_n - Gx_n\|) = 0.$$

Utilizing the properties of g , we have

$$\lim_{n \rightarrow \infty} \|x_n - Gx_n\| = 0. \tag{3.10}$$

This immediately implies that

$$\lim_{n \rightarrow \infty} \|z_n - x_n\| = \lim_{n \rightarrow \infty} (1 - \sigma_n) \|Gx_n - x_n\| = 0. \tag{3.11}$$

Note that

$$\|x_n - W_n x_n\| \leq \|x_n - W_n z_n\| + \|W_n z_n - W_n x_n\| \leq \|x_n - W_n z_n\| + \|z_n - x_n\|,$$

which together with (3.9) and (3.11) implies that

$$\lim_{n \rightarrow \infty} \|x_n - W_n x_n\| = 0.$$

Also, note that

$$\|Wx_n - x_n\| \leq \|Wx_n - W_n x_n\| + \|W_n x_n - x_n\|.$$

From [52, Remark 2.2] (see also [53, Remark 3.1]), we have

$$\lim_{n \rightarrow \infty} \|Wx_n - W_n x_n\| = 0.$$

It follows that

$$\lim_{n \rightarrow \infty} \|Wx_n - x_n\| = 0. \tag{3.12}$$

In terms of (2.3) and Lemma 2.11, $W : C \rightarrow C$ is a nonexpansive mapping such that $\text{Fix}(W) = \bigcap_{n=0}^{\infty} \text{Fix}(T_n)$. Define a mapping $Vx = (1 - \theta)Wx + \theta Gx$, where θ is a constant in $(0, 1)$. Then by Lemma 2.8, we have that $\text{Fix}(V) = \text{Fix}(W) \cap \text{Fix}(G) = F$. Moreover, from (3.10) and (3.12), we get

$$\|Vx_n - x_n\| \leq (1 - \theta)\|Wx_n - x_n\| + \theta\|Gx_n - x_n\| \rightarrow 0,$$

that is,

$$\lim_{n \rightarrow \infty} \|Vx_n - x_n\| = 0. \tag{3.13}$$

In the following, we discuss two cases.

(i) Firstly, suppose that X is uniformly smooth. Let x_t be the unique fixed point of the contraction mapping T_t given by

$$T_t x = tf(x)(1 - t)Vx, \quad \forall t \in (0, 1).$$

By Lemma 2.5, we can define $q := s - \lim_{t \rightarrow 0^+} x_t$, and $q \in \text{Fix}(V) = F$ solves the VIP

$$\langle q - f(q), J(q - p) \rangle, \quad \forall p \in F.$$

Let us show that

$$\limsup_{n \rightarrow \infty} \langle f(q) - q, J(x_n - q) \rangle \leq 0. \tag{3.14}$$

Note that

$$x_t - x_n = t(f(x_t) - x_n) + (1 - t)(Vx_t - x_n).$$

Applying Lemma 2.2, we derive

$$\begin{aligned} \|x_t - x_n\|^2 &\leq (1 - t)^2 \|Vx_t - x_n\|^2 + 2t \langle f(x_t) - x_n, J(x_t - x_n) \rangle \\ &\leq (1 - t)^2 (\|Vx_t - Vx_n\| + \|Vx_n - x_n\|)^2 + 2t \langle f(x_t) - x_t, J(x_t - x_n) \rangle + 2t \|x_t - x_n\|^2 \\ &\leq (1 - t)^2 \|x_t - x_n\|^2 + a_n(t) + 2t \langle f(x_t) - x_t, J(x_t - x_n) \rangle + 2t \|x_t - x_n\|^2, \end{aligned}$$

where

$$a_n(t) = \|Vx_n - x_n\| (2\|x_t - x_n\| + \|Vx_n - x_n\|) \rightarrow 0 \quad (\text{due to (3.13)}).$$

The last inequality implies

$$\langle x_t - f(x_t), J(x_t - x_n) \rangle \leq \frac{t}{2} \|x_t - x_n\|^2 + \frac{1}{2t} a_n(t).$$

It follows that

$$\limsup_{n \rightarrow \infty} \langle x_t - f(x_t), J(x_t - x_n) \rangle \leq M_1 \frac{t}{2}, \tag{3.15}$$

where $M_1 > 0$ is a constant such that $M_1 \geq \|x_t - x_n\|^2$ for all $n \geq 0$ and small enough $t \in (0, 1)$. Taking the \limsup as $t \rightarrow 0^+$ in (3.15) and noticing the fact that the two limits are interchangeable due to the fact that the duality map J is uniformly norm-to-norm continuous on any bounded subset of X , we get (3.14).

Now, let us show that $x_n \rightarrow q$ as $n \rightarrow \infty$.

Indeed, utilizing Lemma 2.2, we obtain from (3.1) that

$$\begin{aligned} \|z_n - q\|^2 &\leq \sigma_n \|x_n - q\|^2 + (1 - \sigma_n) \|Gx_n - q\|^2 \leq \|x_n - q\|^2, \\ \|y_n - q\|^2 &\leq \alpha_n \|x_n - q\|^2 + (1 - \alpha_n) \|W_n z_n - q\|^2 \\ &\leq \alpha_n \|x_n - q\|^2 + (1 - \alpha_n) \|z_n - q\|^2 \\ &\leq \alpha_n \|x_n - q\|^2 + (1 - \alpha_n) \|x_n - q\|^2 \\ &= \|x_n - q\|^2, \end{aligned} \tag{3.16}$$

and hence

$$\begin{aligned} \|x_{n+1} - q\|^2 &= \|\beta_n(f(x_n) - f(q)) + (1 - \beta_n)(y_n - q) + \beta_n(f(q) - q)\|^2 \\ &\leq \|\beta_n(f(x_n) - f(q)) + (1 - \beta_n)(y_n - q)\|^2 + 2\beta_n \langle f(q) - q, J(x_{n+1} - q) \rangle \\ &\leq \beta_n \|f(x_n) - f(q)\|^2 + (1 - \beta_n) \|y_n - q\|^2 + 2\beta_n \langle f(q) - q, J(x_{n+1} - q) \rangle \\ &\leq \beta_n \rho \|x_n - q\|^2 + (1 - \beta_n) \|x_n - q\|^2 + 2\beta_n \langle f(q) - q, J(x_{n+1} - q) \rangle \\ &= (1 - (1 - \rho)\beta_n) \|x_n - q\|^2 + 2\beta_n \langle f(q) - q, J(x_{n+1} - q) \rangle \\ &= (1 - (1 - \rho)\beta_n) \|x_n - q\|^2 + (1 - \rho)\beta_n \frac{2 \langle f(q) - q, J(x_{n+1} - q) \rangle}{1 - \rho}. \end{aligned} \tag{3.17}$$

Therefore, applying Lemma 2.1 to (3.17), we conclude from (3.14) and condition (i) that $x_n \rightarrow q$ as $n \rightarrow \infty$.

(ii) Secondly, suppose that X has a weakly continuous duality map J_φ with gauge φ . Let x_t be the unique fixed point of the contraction mapping T_t given by

$$T_t x = t f(x) + (1 - t) Vx, \quad \forall t \in (0, 1).$$

By Lemma 2.7, we can define $q := s - \lim_{t \rightarrow 0^+} x_t$, and $q \in \text{Fix}(V) = F$ solves the VIP

$$\langle q - f(q), J_\varphi(q - p) \rangle \leq 0, \quad \forall p \in F. \tag{3.18}$$

Let us show that

$$\limsup_{n \rightarrow \infty} \langle f(q) - q, J_\varphi(x_n - q) \rangle \leq 0. \tag{3.19}$$

We take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n \rightarrow \infty} \langle f(q) - q, J_\varphi(x_n - q) \rangle = \lim_{k \rightarrow \infty} \langle f(q) - q, J_\varphi(x_{n_k} - q) \rangle. \tag{3.20}$$

Since X is reflexive and $\{x_n\}$ is bounded, we may further assume that $x_{n_k} \rightharpoonup \bar{x}$ for some $\bar{x} \in C$. Since J_φ is weakly continuous, utilizing Lemma 2.6(ii), we have

$$\limsup_{k \rightarrow \infty} \Phi(\|x_{n_k} - x\|) = \limsup_{k \rightarrow \infty} \Phi(\|x_{n_k} - \bar{x}\|) + \Phi(\|x - \bar{x}\|), \quad \forall x \in X.$$

Put $\Gamma(x) = \limsup_{k \rightarrow \infty} \Phi(\|x_{n_k} - x\|)$, $\forall x \in X$. It follows that

$$\Gamma(x) = \Gamma(\bar{x}) + \Phi(\|x - \bar{x}\|), \quad \forall x \in X.$$

From (3.13), we have

$$\begin{aligned} \Gamma(V\bar{x}) &= \limsup_{k \rightarrow \infty} \Phi(\|x_{n_k} - V\bar{x}\|) \\ &= \limsup_{k \rightarrow \infty} \Phi(\|Vx_{n_k} - V\bar{x}\|) \\ &\leq \limsup_{k \rightarrow \infty} \Phi(\|x_{n_k} - \bar{x}\|) = \Gamma(\bar{x}). \end{aligned} \tag{3.21}$$

Furthermore, observe that

$$\Gamma(V\bar{x}) = \Gamma(\bar{x}) + \Phi(\|V\bar{x} - \bar{x}\|). \tag{3.22}$$

Combining (3.21) with (3.22), we obtain

$$\Phi(\|V\bar{x} - \bar{x}\|) \leq 0.$$

Hence $V\bar{x} = \bar{x}$ and $\bar{x} \in \text{Fix}(V) = F$. Thus, from (3.18) and (3.20), it is easy to see that

$$\limsup_{n \rightarrow \infty} \langle f(q) - q, J_\varphi(x_n - q) \rangle = \langle f(q) - q, J_\varphi(\bar{x} - q) \rangle \leq 0.$$

Therefore, we deduce that (3.19) holds.

Next, let us show that $x_n \rightarrow q$ as $n \rightarrow \infty$. Indeed, utilizing Lemma 2.6(i), we obtain from (3.1) that

$$\begin{aligned} \|z_n - q\| &\leq \sigma_n \|x_n - q\| + (1 - \sigma_n) \|Gx_n - q\| \leq \|x_n - q\|, \\ \|y_n - q\| &\leq \alpha_n \|x_n - q\| + (1 - \alpha_n) \|W_n z_n - q\| \\ &\leq \alpha_n \|x_n - q\| + (1 - \alpha_n) \|z_n - q\| \\ &\leq \|x_n - q\|, \end{aligned}$$

and hence

$$\begin{aligned} &\Phi(\|x_{n+1} - q\|) \\ &= \Phi(\|\beta_n(f(x_n) - q) + (1 - \beta_n)(y_n - q)\|) \\ &= \Phi(\|\beta_n(f(x_n) - f(q)) + (1 - \beta_n)(y_n - q) + \beta_n(f(q) - q)\|) \\ &\leq \Phi(\|\beta_n(f(x_n) - f(q)) + (1 - \beta_n)(y_n - q)\|) + \beta_n \langle f(q) - q, J_\varphi(x_{n+1} - q) \rangle \\ &\leq \Phi(\beta_n \|f(x_n) - f(q)\| + (1 - \beta_n) \|y_n - q\|) + \beta_n \langle f(q) - q, J_\varphi(x_{n+1} - q) \rangle \\ &\leq \Phi(\beta_n \rho \|x_n - q\| + (1 - \beta_n) \|x_n - q\|) + \beta_n \langle f(q) - q, J_\varphi(x_{n+1} - q) \rangle \\ &\leq (1 - (1 - \rho)\beta_n) \Phi(\|x_n - q\|) + \beta_n \langle f(q) - q, J_\varphi(x_{n+1} - q) \rangle \\ &= (1 - (1 - \rho)\beta_n) \Phi(\|x_n - q\|) + (1 - \rho)\beta_n \frac{\langle f(q) - q, J_\varphi(x_{n+1} - q) \rangle}{1 - \rho}. \end{aligned} \tag{3.23}$$

Applying Lemma 2.1 to (3.23), we conclude from (3.19) and condition (i) that

$$\Phi(\|x_n - q\|) \rightarrow 0 \quad (n \rightarrow \infty),$$

which implies that $\|x_n - q\| \rightarrow 0$ ($n \rightarrow \infty$), i.e., $x_n \rightarrow q$ ($n \rightarrow \infty$). This completes the proof. \square

Corollary 3.1 *The conclusion in Theorem 3.1 still holds, provided the conditions (i)-(iv) are replaced by the following:*

- (i) $0 \leq \beta_n \leq 1 - \rho, \forall n \geq n_0$ for some $n_0 \geq 0$;
- (ii) $\lim_{n \rightarrow \infty} |\beta_n - \beta_{n+1}| = 0$ and $\sum_{n=0}^{\infty} \beta_n = \infty$;
- (iii) $\lim_{n \rightarrow \infty} |\alpha_n - \alpha_{n+1}| = 0$ and $0 < \liminf_{n \rightarrow \infty} \alpha_n \leq \limsup_{n \rightarrow \infty} \alpha_n < 1$;
- (iv) $\lim_{n \rightarrow \infty} |\sigma_n - \sigma_{n+1}| = 0$ and $0 < \liminf_{n \rightarrow \infty} \sigma_n \leq \limsup_{n \rightarrow \infty} \sigma_n < 1$.

Proof Observe that

$$\begin{aligned} &\frac{\beta_{n+1}}{1 - (1 - \beta_{n+1})\alpha_{n+1}} - \frac{\beta_n}{1 - (1 - \beta_n)\alpha_n} \\ &= \frac{(\beta_{n+1} - \beta_n) - \beta_{n+1}\alpha_n + \beta_n\alpha_{n+1} + \beta_{n+1}\beta_n\alpha_n - \beta_n\beta_{n+1}\alpha_{n+1}}{(1 - (1 - \beta_{n+1})\alpha_{n+1})(1 - (1 - \beta_n)\alpha_n)} \\ &= \frac{(\beta_{n+1} - \beta_n) - \beta_{n+1}(\alpha_n - \alpha_{n+1}) - \alpha_{n+1}(\beta_{n+1} - \beta_n) + \beta_n\beta_{n+1}(\alpha_n - \alpha_{n+1})}{(1 - (1 - \beta_{n+1})\alpha_{n+1})(1 - (1 - \beta_n)\alpha_n)} \\ &= \frac{(\beta_{n+1} - \beta_n)(1 - \alpha_{n+1}) - \beta_{n+1}(\alpha_n - \alpha_{n+1})(1 - \beta_n)}{(1 - (1 - \beta_{n+1})\alpha_{n+1})(1 - (1 - \beta_n)\alpha_n)}. \end{aligned}$$

Since $\lim_{n \rightarrow \infty} |\beta_n - \beta_{n+1}| = 0$ and $\lim_{n \rightarrow \infty} |\alpha_n - \alpha_{n+1}| = 0$, it follows that

$$\lim_{n \rightarrow \infty} \left| \frac{\beta_{n+1}}{1 - (1 - \beta_{n+1})\alpha_{n+1}} - \frac{\beta_n}{1 - (1 - \beta_n)\alpha_n} \right| = 0.$$

Consequently, all the conditions of Theorem 3.1 are satisfied. So, utilizing Theorem 3.1, we obtain the desired result. \square

Corollary 3.2 *Let C be a nonempty closed convex subset of a uniformly convex Banach space X . Assume, in addition, that X either is uniformly smooth or has a weakly continuous duality map J_φ with gauge φ . Let Π_C be a sunny nonexpansive retraction from X onto C . Let the mapping $B_i : C \rightarrow X$ be ζ_i -strictly pseudocontractive and η_i -strongly accretive with $\zeta_i + \eta_i \geq 1$ for $i = 1, 2$. Let $f \in \mathcal{E}_C$ with a contractive coefficient $\rho \in (0, 1)$. Let $\{\lambda_n\}_{n=0}^\infty$ be a sequence of positive numbers in $(0, b]$ for some $b \in (0, 1)$ and $\{T_n\}_{n=0}^\infty$ be an infinite family of nonexpansive self-mappings on C such that $F := \bigcap_{n=0}^\infty \text{Fix}(T_n) \cap \Omega \neq \emptyset$, where Ω is the fixed point set of the mapping $G = \Pi_C(I - \mu_1 B_1)\Pi_C(I - \mu_2 B_2)$ with $1 - \frac{\zeta_i}{1 + \zeta_i} (1 - \sqrt{\frac{1 - \eta_i}{\zeta_i}}) \leq \mu_i \leq 1$ for $i = 1, 2$. Suppose that $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\sigma_n\}$ are sequences in $[0, 1]$ satisfying the following conditions:*

- (i) $\lim_{n \rightarrow \infty} \beta_n = 0$ and $\sum_{n=0}^\infty \beta_n = \infty$;
- (ii) $\lim_{n \rightarrow \infty} |\sigma_{n+1} - \sigma_n| = 0$;
- (iii) $0 < \liminf_{n \rightarrow \infty} \alpha_n \leq \limsup_{n \rightarrow \infty} \alpha_n < 1$;
- (iv) $0 < \liminf_{n \rightarrow \infty} \sigma_n \leq \limsup_{n \rightarrow \infty} \sigma_n < 1$.

Then, for an arbitrary but fixed $x_0 \in C$, the sequence $\{x_n\}$ defined by (3.1) converges strongly to some $q \in F$. Moreover,

- (i) if X is uniformly smooth, then $q \in F$ solves the VIP

$$\langle q - f(q), J(q - p) \rangle, \quad \forall p \in F;$$

- (ii) if X has a weakly continuous duality map J_φ with gauge φ , then $q \in F$ solves the VIP

$$\langle q - f(q), J_\varphi(q - p) \rangle \leq 0, \quad \forall p \in F.$$

Proof Repeating the same arguments as those in the proof of Theorem 3.1, we know that $\{x_n\}$ is bounded, and so are the sequences $\{y_n\}$, $\{z_n\}$, $\{Gx_n\}$, $\{W_n z_n\}$ and $\{f(x_n)\}$. Since $\lim_{n \rightarrow \infty} \beta_n = 0$, it is easy to see that the following hold:

- (i) $\beta_n(f(x_n) - x_n) \rightarrow 0$ ($n \rightarrow \infty$);
- (ii) $0 \leq \beta_n \leq 1 - \rho$, $\forall n \geq n_0$ for some integer $n_0 \geq 0$;
- (iii) $\lim_{n \rightarrow \infty} \left| \frac{\beta_{n+1}}{1 - (1 - \beta_{n+1})\alpha_{n+1}} - \frac{\beta_n}{1 - (1 - \beta_n)\alpha_n} \right| = 0$.

Therefore, all the conditions of Theorem 3.1 are satisfied. So, utilizing Theorem 3.1, we derive the desired result. \square

To end this paper, we give a weak convergence theorem for three-step Mann iterations (3.1) involving GSVI (1.2) and an infinite family of nonexpansive mappings T_0, T_1, \dots in a Hilbert space H .

Theorem 3.2 *Let C be a nonempty closed convex subset of a Hilbert space H . Let the mapping $B_i : C \rightarrow H$ be β_i -inverse strongly monotone for $i = 1, 2$. Let $f \in \mathcal{E}_C$ with a contractive coefficient $\rho \in (0, 1)$. Let $\{\lambda_n\}_{n=0}^\infty$ be a sequence of positive numbers in $(0, b]$ for*

some $b \in (0, 1)$, and let $\{T_n\}_{n=0}^\infty$ be an infinite family of nonexpansive self-mappings on C such that $F := \bigcap_{n=0}^\infty \text{Fix}(T_n) \cap \Omega \neq \emptyset$, where Ω is the fixed point set of the mapping $G = P_C(I - \mu_1 B_1)P_C(I - \mu_2 B_2)$ with $0 < \mu_i \leq 2\beta_i$ for $i = 1, 2$. Suppose that $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\sigma_n\}$ are sequences in $[0, 1]$ satisfying the following conditions:

- (i) $\sum_{n=0}^\infty \beta_n < \infty$;
- (ii) $0 < \liminf_{n \rightarrow \infty} \alpha_n \leq \limsup_{n \rightarrow \infty} \alpha_n < 1$;
- (iii) $0 < \liminf_{n \rightarrow \infty} \sigma_n \leq \limsup_{n \rightarrow \infty} \sigma_n < 1$.

Then, for an arbitrary but fixed $x_0 \in C$, the sequence $\{x_n\}$ defined by (3.1) converges weakly to a point in F .

Proof First of all, by Lemma 1.1, we know that $G : C \rightarrow C$ is nonexpansive. Take an arbitrary $p \in F$. Repeating the same arguments as those in the proof of Theorem 3.1, we know that $\{x_n\}$ is bounded, and so are the sequences $\{y_n\}$, $\{z_n\}$, $\{Gx_n\}$, $\{W_n z_n\}$ and $\{f(x_n)\}$.

It follows from (2.2) and (3.1) that

$$\begin{aligned} \|z_n - p\|^2 &= \sigma_n \|x_n - p\|^2 + (1 - \sigma_n) \|Gx_n - p\|^2 - \sigma_n(1 - \sigma_n) \|x_n - Gx_n\|^2 \\ &\leq \sigma_n \|x_n - p\|^2 + (1 - \sigma_n) \|x_n - p\|^2 - \sigma_n(1 - \sigma_n) \|x_n - Gx_n\|^2 \\ &= \|x_n - p\|^2 - \sigma_n(1 - \sigma_n) \|x_n - Gx_n\|^2, \\ \|y_n - p\|^2 &= \alpha_n \|x_n - p\|^2 + (1 - \alpha_n) \|W_n z_n - p\|^2 - \alpha_n(1 - \alpha_n) \|x_n - W_n z_n\|^2 \\ &\leq \alpha_n \|x_n - p\|^2 + (1 - \alpha_n) \|z_n - p\|^2 - \alpha_n(1 - \alpha_n) \|x_n - W_n z_n\|^2 \\ &\leq \alpha_n \|x_n - p\|^2 + (1 - \alpha_n) [\|x_n - p\|^2 - \sigma_n(1 - \sigma_n) \|x_n - Gx_n\|^2] \\ &\quad - \alpha_n(1 - \alpha_n) \|x_n - W_n z_n\|^2 \\ &= \|x_n - p\|^2 - (1 - \alpha_n)\sigma_n(1 - \sigma_n) \|x_n - Gx_n\|^2 - \alpha_n(1 - \alpha_n) \|x_n - W_n z_n\|^2, \end{aligned}$$

and hence

$$\begin{aligned} \|x_{n+1} - p\|^2 &\leq (1 - \beta_n) \|y_n - p\|^2 + \beta_n \|f(x_n) - p\|^2 \\ &\leq \|y_n - p\|^2 + \beta_n \|f(x_n) - p\|^2 \\ &\leq \|x_n - p\|^2 - (1 - \alpha_n)\sigma_n(1 - \sigma_n) \|x_n - Gx_n\|^2 \\ &\quad - \alpha_n(1 - \alpha_n) \|x_n - W_n z_n\|^2 + \beta_n \|f(x_n) - p\|^2 \\ &\leq \|x_n - p\|^2 + \beta_n \|f(x_n) - p\|^2. \end{aligned} \tag{3.24}$$

Since $\sum_{n=0}^\infty \beta_n < \infty$ and $\{f(x_n)\}$ is bounded, we obtain $\sum_{n=0}^\infty \beta_n \|f(x_n) - p\|^2 < \infty$. Utilizing Lemma 2.14, we conclude that $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists. Furthermore, it follows from (3.24) that for all $e \geq 0$,

$$\begin{aligned} &(1 - \alpha_n)\sigma_n(1 - \sigma_n) \|x_n - Gx_n\|^2 + \alpha_n(1 - \alpha_n) \|x_n - W_n z_n\|^2 \\ &\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \beta_n \|f(x_n) - p\|^2. \end{aligned} \tag{3.25}$$

Since $\beta_n \rightarrow 0$, $0 < \liminf_{n \rightarrow \infty} \alpha_n \leq \limsup_{n \rightarrow \infty} \alpha_n < 1$ and

$$0 < \liminf_{n \rightarrow \infty} \sigma_n \leq \limsup_{n \rightarrow \infty} \sigma_n < 1,$$

we deduce from (3.25) that

$$\lim_{n \rightarrow \infty} \|x_n - Gx_n\| = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \|x_n - W_n z_n\| = 0. \tag{3.26}$$

It immediately follows that

$$\lim_{n \rightarrow \infty} \|z_n - x_n\| = \lim_{n \rightarrow \infty} (1 - \sigma_n) \|Gx_n - x_n\| = 0. \tag{3.27}$$

Note that

$$\|x_n - W_n x_n\| \leq \|x_n - W_n z_n\| + \|W_n z_n - W_n x_n\| \leq \|x_n - W_n z_n\| + \|z_n - x_n\|.$$

Thus, from (3.26) and (3.27) we have

$$\lim_{n \rightarrow \infty} \|x_n - W_n x_n\| = 0. \tag{3.28}$$

Also, observe that

$$\|Wx_n - x_n\| \leq \|Wx_n - W_n x_n\| + \|W_n x_n - x_n\|.$$

From [52, Remark 2.2] (see also [53, Remark 3.1]), we have

$$\lim_{n \rightarrow \infty} \|Wx_n - W_n x_n\| = 0.$$

This implies immediately that

$$\lim_{n \rightarrow \infty} \|Wx_n - x_n\| = 0.$$

Now, let us show that $\omega_w(x_n) \subset F$ (see (2.1)). Indeed, let $\bar{x} \in \omega_w(x_n)$. Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup \bar{x}$. Since $x_n - Wx_n \rightarrow 0$ and $x_n - Gx_n \rightarrow 0$, by Lemma 2.15, we know that $\bar{x} \in \text{Fix}(W)$ and $\bar{x} \in \text{Fix}(G)$. Thus, $\bar{x} \in \text{Fix}(W) \cap \text{Fix}(G) = F$ according to Lemmas 1.1 and 2.11.

Finally, let us show that $\omega_w(x_n)$ is a singleton. Indeed, let $\{x_{m_l}\}$ be another subsequence of $\{x_n\}$ such that $x_{m_l} \rightharpoonup \hat{x}$. Then \hat{x} also lies in F . If $\bar{x} \neq \hat{x}$, by Opial's property of H , we reach the following contradiction:

$$\begin{aligned} \lim_{n \rightarrow \infty} \|x_n - \bar{x}\| &= \lim_{k \rightarrow \infty} \|x_{n_k} - \bar{x}\| \\ &< \lim_{k \rightarrow \infty} \|x_{n_k} - \hat{x}\| = \lim_{l \rightarrow \infty} \|x_{m_l} - \hat{x}\| \\ &< \lim_{l \rightarrow \infty} \|x_{m_l} - \bar{x}\| = \lim_{n \rightarrow \infty} \|x_n - \bar{x}\|. \end{aligned}$$

This implies that $\omega_w(x_n)$ is a singleton. Consequently, $\{x_n\}$ converges weakly to a point in F . □

Remark 3.1 Our Theorem 3.1 improves, extends, supplements and develops Ceng *et al.* [4, Theorem 3.3], Cai and Bu [5, Theorem 3.1] and Ceng and Yao [2, Theorem 3.1] in the following aspects.

(i) The problem of finding a point $q \in \bigcap_{i=0}^{\infty} \text{Fix}(T_i) \cap \Omega$ in our Theorem 3.1 is more general and more subtle than the problem of finding a point $q \in \bigcap_{i=0}^{\infty} \text{Fix}(T_i)$ in [4, Theorem 3.3], and the problem of finding a point $q \in \text{Fix}(T)$ in [2, Theorem 3.1].

(ii) The iterative scheme in [4, Theorems 3.3] is extended to develop the iterative scheme (3.1) of our Theorem 3.1 by virtue of the iterative schemes of [5, Theorem 3.1] and [2, Theorem 3.1]. The iterative scheme (3.1) of our Theorem 3.1 is more advantageous and more flexible than the iterative schemes of [2, Theorem 3.1] and [4, Theorem 3.3] because it can be applied to solving two problems (*i.e.*, GSVI (1.1), fixed point problem of infinitely many nonexpansive mappings) and involves several parameter sequences $\{\sigma_n\}$, $\{\alpha_n\}$ and $\{\beta_n\}$.

(iii) Our Theorem 3.1 extends and generalizes Ceng and Yao [2, Theorem 3.1] from a nonexpansive mapping to a countable family of nonexpansive mappings, and Ceng and Yao [4, Theorems 3.3] to the setting of infinitely many nonexpansive mappings and GSVI (1.1) for two strictly pseudocontractive and strongly accretive mappings. In the meantime, our Theorem 3.1 drops the following restrictions in Cai and Bu [5, Theorem 3.1]:

Assume that $\sum_{n=1}^{\infty} \sup_{x \in D} \|T_{n+1}x - T_nx\| < \infty$ for any bounded subset D of C , and let T be a mapping of C into X defined by $Tx = \lim_{n \rightarrow \infty} T_nx$ for all $x \in C$ and suppose that $\text{Fix}(T) = \bigcap_{n=1}^{\infty} \text{Fix}(T_n)$.

(iv) The iterative scheme (3.1) in our Theorem 3.1 is very different from every one in [4, Theorem 3.3], [5, Theorem 3.1] and [2, Theorem 3.1] because the mapping T_n in [5, Theorem 3.1] and the mapping T in [2, Theorem 3.1] are replaced by the same W -mapping W_n in the iterative scheme (3.1) of our Theorem 3.1, and (3.1) in our Theorem 3.1 is three-step Mann iterations for finding a point $q \in \bigcap_{i=0}^{\infty} \text{Fix}(T_i) \cap \Omega$ in comparison with two-step Mann iterations for finding a point $q \in \bigcap_{i=0}^{\infty} \text{Fix}(T_i)$ in [4, Theorem 3.3].

(v) Cai and Bu's proof in [5, Theorem 3.1] depends on the argument techniques in [6], the inequality in 2-uniformly smooth Banach spaces (see Lemma 1.1) and the inequality in smooth and uniform convex Banach spaces (see Proposition 1.1). However, the proof of our Theorem 3.1 does not depend on the argument techniques in [6], the inequality in 2-uniformly smooth Banach spaces (see Lemma 1.1), and the inequality in smooth and uniform convex Banach spaces (see Proposition 1.1). It depends only on the inequality in uniform convex Banach spaces (see Lemma 2.13 in Section 2 of this paper), the properties of reflexive Banach space having a weakly continuous duality map J_{φ} with gauge φ , and the properties of the W -mapping (see Lemmas 2.6-2.7 and 2.10-2.11 in Section 2 of this paper).

(vi) The assumption of the uniformly convex and 2-uniformly smooth Banach space X in [5, Theorem 3.1] is weakened to the one of the uniformly convex Banach space X which either has a weakly continuous duality map J_{φ} with gauge φ or is uniformly smooth in our Theorem 3.1. Moreover, our Theorem 3.1 shows that the assumption of the uniformly smooth Banach space X in [2, Theorem 3.1] can be replaced by the assumption of the uniformly convex Banach space X having a weakly continuous duality map J_{φ} with gauge φ in our Theorem 3.1. It is worth emphasizing that there is no assumption on the convergence of parameter sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\sigma_n\}$ to the zero in our Theorem 3.1.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to this work. All authors read and approved the final manuscript.

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