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Certain integral representations of Stieltjes constants γ_n

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Abstract

A remarkably large number of integral formulas for the Euler-Mascheroni constant γ have been presented. The Stieltjes constants (or generalized Euler-Mascheroni constants) γ_n and $\gamma_0 = \gamma$, which arise from the coefficients of the Laurent series expansion of the Riemann zeta function $\zeta(s)$ at $s = 1$, have been investigated in various ways, especially for their integral representations. Here we aim at presenting certain integral representations for γ_n by choosing to use three known integral representations for the Riemann zeta function $\zeta(s)$. Our method used here is similar to those in some earlier works, but our results seem a little simpler. Some relevant connections of some special cases of our results presented here with those in earlier works are also pointed out.

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1 Introduction and preliminaries

The Riemann zeta function $\zeta(s)$ is defined by (see, *e.g.*, [1, Section 2.3])

$$\zeta(s) := \begin{cases} \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1-2^{-s}} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^s} & (\Re(s) > 1), \\ \frac{1}{1-2^{1-s}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} & (\Re(s) > 0; s \neq 1), \end{cases} \tag{1.1}$$

which is an obvious special case of the Hurwitz (or generalized) zeta function $\zeta(s,a)$ defined by

$$\zeta(s,a) := \sum_{k=0}^{\infty} (k+a)^{-s} \quad \left(\Re(s) > 1; a \in \mathbb{C} \setminus \mathbb{Z}_0^-\right), \tag{1.2}$$

where $\mathbb{C}$ and $\mathbb{Z}_0^-$ denote the sets of complex numbers and nonpositive integers, respectively. Both the Riemann zeta function $\zeta(s)$ and the Hurwitz zeta function $\zeta(s,a)$ can be continued meromorphically to the whole complex s-plane, except for a simple pole only at $s = 1$, with their respective residue 1, in many different ways. The Stieltjes constants γ_n for $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{N} := \{1, 2, 3, \ldots\}$, arise from the following Laurent expansion of the

Riemann zeta function $\zeta(s)$ about $s = 1$ (see, *e.g.*, [2, pp.166-169], [3, p.255] and [1, p.165]):

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \gamma_n (s-1)^n, \tag{1.3}$$

where

$$\begin{aligned} \gamma_n &= \lim_{m\to\infty} \left\{ \sum_{k=1}^{m} \frac{(\log k)^n}{k} - \int_1^m \frac{(\log x)^n}{x}\, dx \right\} \\ &= \lim_{m\to\infty} \left\{ \sum_{k=1}^{m} \frac{(\log k)^n}{k} - \frac{(\log m)^{n+1}}{n+1} \right\} \quad (n \in \mathbb{N}_0) \end{aligned} \tag{1.4}$$

and, in particular, γ_0 (denoted by γ) is the Euler-Mascheroni constant (see, for details, [2, Section 1.5] and [1, Section 1.2]):

$$\gamma := \lim_{m\to\infty} \left(\sum_{k=1}^{m} \frac{1}{k} - \log m \right) \cong 0.5772156649\cdots. \tag{1.5}$$

The Stieltjes constants γ_n are named after Thomas Jan Stieltjes and often referred to as generalized Euler-Mascheroni constants. Liang and Todd [4] computed numerical approximations of the first 20 Stieltjes constants in 1972. In 1985, using contour integration, Ainsworth and Howell [5] showed that

$$\gamma_n = 2\Re \left\{ \int_0^{\infty} \frac{(x-i)(\log(1-ix))^n}{(1+x^2)(e^{2\pi x}-1)}\, dx \right\} \quad (n \in \mathbb{N}) \tag{1.6}$$

and

$$\begin{aligned} \gamma = \gamma_0 &= \frac{1}{2} + 2\Re \left\{ \int_0^{\infty} \frac{(x-i)}{(1+x^2)(e^{2\pi x}-1)}\, dx \right\} \\ &= \frac{1}{2} + 2 \int_0^{\infty} \frac{x}{(1+x^2)(e^{2\pi x}-1)}\, dx. \end{aligned} \tag{1.7}$$

By using binomial theorem, we have

$$\begin{aligned} \bigl(\log(1-ix)\bigr)^{2m} &= \left\{ \frac{1}{2} \log\bigl(1+x^2\bigr) - i \arctan x \right\}^{2m} \\ &= \mathcal{A}_m(x) + i\mathcal{B}_m(x) \quad (m \in \mathbb{N}), \end{aligned} \tag{1.8}$$

where, for convenience and simplicity,

$$\mathcal{A}_m(x) := \sum_{k=0}^{m} \frac{(-1)^k}{2^{2m-2k}} \binom{2m}{2k} (\arctan x)^{2k} \bigl(\ln\bigl(1+x^2\bigr)\bigr)^{2m-2k}$$

and

$$\mathcal{B}_m(x) := \sum_{k=0}^{m-1} \frac{(-1)^{k+1}}{2^{2m-2k-1}} \binom{2m}{2k+1} (\arctan x)^{2k+1} \bigl(\ln\bigl(1+x^2\bigr)\bigr)^{2m-2k-1}.$$

From (1.6) and (1.8), we obtain a more explicit integral representation for the Stieltjes constants γ_{2m}:

$$\gamma_{2m} = 2\int_0^\infty \frac{x\mathcal{A}_m(x) + \mathcal{B}_m(x)}{(1+x^2)(e^{2\pi x}-1)}\,dx \quad (m \in \mathbb{N}), \tag{1.9}$$

where $\mathcal{A}_m(x)$ and $\mathcal{B}_m(x)$ are given in (1.8). Similarly, we have

$$\big(\log(1-ix)\big)^{2m+1} = \mathcal{C}_m(x) + i\mathcal{D}_m(x) \quad (m \in \mathbb{N}_0), \tag{1.10}$$

where, for convenience and simplicity,

$$\mathcal{C}_m(x) := \sum_{k=0}^{m} \frac{(-1)^k}{2^{2m+1-2k}}\binom{2m+1}{2k}(\arctan x)^{2k}\big(\ln\big(1+x^2\big)\big)^{2m+1-2k}$$

and

$$\mathcal{D}_m(x) := \sum_{k=0}^{m} \frac{(-1)^{k+1}}{2^{2m-2k}}\binom{2m+1}{2k+1}(\arctan x)^{2k+1}\big(\ln\big(1+x^2\big)\big)^{2m-2k}.$$

From (1.6) and (1.10), we get a more explicit integral representation for the Stieltjes constants γ_{2m+1}:

$$\gamma_{2m+1} = 2\int_0^\infty \frac{x\mathcal{C}_m(x) + \mathcal{D}_m(x)}{(1+x^2)(e^{2\pi x}-1)}\,dx \quad (m \in \mathbb{N}_0), \tag{1.11}$$

where $\mathcal{C}_m(x)$ and $\mathcal{D}_m(x)$ are given in (1.10). Connon (see, *e.g.*, *cf.*, [6, Eq. (4.3)]; see also [7, Eq. (1.5)]) presented an integral representation of the Stieltjes constants γ_n of a similar nature in (1.6):

$$\gamma_n = i\int_0^\infty \frac{(1-ix)(\log(1+ix))^n - (1+ix)(\log(1-ix))^n}{(1+x^2)(e^{2\pi x}-1)}\,dx \quad (n \in \mathbb{N}). \tag{1.12}$$

We recall the polygamma functions $\psi^{(n)}(s)$ $(n \in \mathbb{N})$ defined by

$$\psi^{(n)}(s) := \frac{d^{n+1}}{dz^{n+1}}\log\Gamma(s) = \frac{d^n}{ds^n}\psi(s) \quad \big(n \in \mathbb{N}_0; s \in \mathbb{C}\setminus\mathbb{Z}_0^-\big), \tag{1.13}$$

where $\psi(s)$ denotes the psi (or digamma) function defined by

$$\psi(s) := \frac{d}{ds}\log\Gamma(s) \quad \text{and} \quad \psi^{(0)}(s) = \psi(s) \quad \big(s \in \mathbb{C}\setminus\mathbb{Z}_0^-\big). \tag{1.14}$$

Connon [8, Eq. (4.27)] also obtained an integral representation of the Stieltjes constants γ_n:

$$\begin{aligned}\gamma_n = (-1)^n \sum_{k=0}^{n}\binom{n}{k} Y_k\big(-\psi(1), -\psi^{(1)}(1), \ldots, -\psi^{(k-1)}(1)\big) \\ \cdot \int_0^\infty (\log t)^{n-k}\cdot\left(\frac{1}{e^t-1} - \frac{1}{t}\right)e^{-t}\,dt \quad (n \in \mathbb{N}),\end{aligned} \tag{1.15}$$

where $Y_n(x_1,\ldots,x_n)$ are the complete Bell polynomials defined by $Y_0 = 1$ and

$$Y_n(x_1,\ldots,x_n) = \sum_{\pi(n)} \frac{n!}{k_1!k_2!\cdots k_n!}\left(\frac{x_1}{1!}\right)^{k_1}\left(\frac{x_2}{2!}\right)^{k_2}\cdots\left(\frac{x_n}{n!}\right)^{k_n} \quad (n \in \mathbb{N}), \tag{1.16}$$

the sum being taken over all partitions $\pi(n)$ of n, *i.e.*, over all sets of $k_j \in \mathbb{N}_0$ such that

$$k_1 + 2k_2 + \cdots + nk_n = n.$$

Connon [8, Eq. (4.11)] presented an integral representation of γ_1 which may be a special case of (1.15):

$$\gamma_1 = \gamma - \gamma^2 - \int_0^\infty \log t \cdot \left(\frac{1}{e^t - 1} - \frac{1}{t}\right)e^{-t}\,dt. \tag{1.17}$$

A remarkably large number of integral formulas for the Euler-Mascheroni constant γ have been presented (see, *e.g.*, [9, 10], and [1, Section 1.2]). The Stieltjes constants γ_n ($n \in \mathbb{N}_0$) have been investigated in various ways, especially for their integral representations (see, *e.g.*, [4–8]; see also [2, Section 2.21] and the references cited therein). Here we aim at presenting certain integral representations for γ_n by choosing to use three known integral representations for the Riemann zeta function $\zeta(s)$. Our method used here is similar to those in some earlier works, but our results seem a little simpler. Some relevant connections of some special cases of our results presented here with those in earlier works are also pointed out.

To do this, we first observe a simple property asserted in the following lemma.

Lemma 1 *If some representations of the Riemann zeta function* $\zeta(s)$ *are analytic in a deleted neighborhood of* $s = 1$, *except for a simple pole at* $s = 1$ *with its residue* 1, *then the following function* $Z(s)$ *defined by*

$$Z(s) := \zeta(s) - \frac{1}{s-1} \tag{1.18}$$

is analytic at $s = 1$ *if we define*

$$Z(1) := \gamma = \lim_{s\to 1} \zeta(s) - \frac{1}{s-1}. \tag{1.19}$$

Furthermore, we have

$$Z^{(n)}(1) = (-1)^n \gamma_n \quad (n \in \mathbb{N}_0). \tag{1.20}$$

Proof We prove only (1.20). If the above-defined $Z(s)$ is analytic at $s = 1$, then the Taylor series expansion of $Z(s)$ is given as follows:

$$Z(s) = \sum_{n=0}^\infty \frac{Z^{(n)}(1)}{n!}(s-1)^n$$

in a neighborhood of $s = 1$. In view of (1.3), by uniqueness of Taylor (or Laurent) series expansion of a function, (1.20) is proved. The other argument is obvious from a well-known property of the Riemann zeta function $\zeta(s)$. □

A well-known (and potentially useful) relationship between the polygamma functions $\psi^{(n)}(s)$ and the generalized zeta function $\zeta(s,a)$ is also given by

$$\psi^{(n)}(s) = (-1)^{n+1} n! \sum_{k=0}^{\infty} \frac{1}{(k+s)^{n+1}} = (-1)^{n+1} n! \zeta(n+1, s) \quad \bigl(n \in \mathbb{N}; s \in \mathbb{C} \setminus \mathbb{Z}_0^-\bigr). \tag{1.21}$$

In particular, we have

$$\psi^{(n)}(1) = (-1)^{n+1} n! \zeta(n+1) \quad (n \in \mathbb{N}). \tag{1.22}$$

2 Integral representations for γ_n

We begin by presenting an integral representation for the Stieltjes constants γ_n given in the following theorem.

Theorem 1 *The following integral representation for* γ_n *holds true*:

$$\begin{aligned}\gamma_n = (-1)^n \int_0^{\infty} \left(\psi'(1+t) - \frac{1}{1+t}\right) \\ \cdot \left\{\sum_{k=0}^{n} \binom{n}{k} (-1)^{[k/2]+1} \frac{1-(-1)^{k+1}}{2} \frac{\pi^k}{k+1} (\log t)^{n-k}\right\} dt \quad (n \in \mathbb{N}_0).\end{aligned} \tag{2.1}$$

We note that $Z(s)$ in (2.7) below is analytic in a neighborhood of $s = 1$. So we can use the relation (1.20) for the integral representation of $Z(s)$. In this regard, we first try to get the following formulas asserted by Lemma 2 below.

Lemma 2 *Each of the following formulas holds true*:

$$\lim_{s \to 1} \left(\frac{d^n}{ds^n} \frac{1}{t^{1-s}}\right) = (\log t)^n \quad (n \in \mathbb{N}_0) \tag{2.2}$$

and

$$\lim_{s \to 1} \left\{\frac{d^n}{ds^n} \left(\frac{\sin(\pi s)}{\pi(s-1)}\right)\right\} = (-1)^{[n/2]+1} \frac{1-(-1)^{n+1}}{2} \frac{\pi^n}{n+1} \quad (n \in \mathbb{N}_0), \tag{2.3}$$

where $[x]$ *denotes the greatest integer less than or equal to a real number* x.

Proof The formula (2.2) is obvious. For (2.3), we recall the Maclaurin series expansion of $\sin t$:

$$\sin t = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} t^{2k+1} \quad \bigl(|t| < \infty\bigr). \tag{2.4}$$

By using (2.4), we have

$$\begin{aligned}&\lim_{s\to 1}\left\{\frac{d^{2n}}{ds^{2n}}\left(\frac{\sin(\pi s)}{\pi(s-1)}\right)\right\}\\ &\quad=-\lim_{t\to 0}\left\{\frac{d^{2n}}{dt^{2n}}\left(\frac{\sin(\pi t)}{\pi t}\right)\right\}\\ &\quad=\frac{(-1)^{n+1}}{(2n+1)!}\lim_{t\to 0}\frac{d^{2n}}{dt^{2n}}(\pi t)^{2n}=(-1)^{n+1}\frac{\pi^{2n}}{2n+1}\quad(n\in\mathbb{N}_0).\end{aligned}\tag{2.5}$$

Similarly, we obtain

$$\lim_{s\to 1}\left\{\frac{d^{2n-1}}{ds^{2n-1}}\left(\frac{\sin(\pi s)}{\pi(s-1)}\right)\right\}=0\quad(n\in\mathbb{N}).\tag{2.6}$$

Now it is not difficult to combine the two formulas (2.5) and (2.6) to see the unified formula (2.3). □

Proof of Theorem 1 We choose to recall the following integral representation of $\zeta(s)$ (see, *e.g.*, [1, p.172, Eq. (47)]):

$$Z(s)=\zeta(s)-\frac{1}{s-1}=\frac{\sin(\pi s)}{\pi(s-1)}\int_0^\infty\left(\psi'(1+t)-\frac{1}{1+t}\right)\frac{dt}{t^{1-s}}\quad\bigl(0<\Re(s)<2\bigr).\tag{2.7}$$

To get the nth derivative of a product of the two involved functions in (2.7),

$$\frac{\sin(\pi s)}{\pi(s-1)}\cdot\frac{1}{t^{1-s}},$$

we apply Leibniz's generalization of the product rule for differentiation and use the results in Lemma 2, in view of (1.20), to yield (2.1). □

Recall an integral representation for $\zeta(s)$ (see, *e.g.*, [1, p.169, Eq. (31)]):

$$Z(s)=\zeta(s)-\frac{1}{s-1}=\frac{1}{2}+\frac{1}{\Gamma(s)}\int_0^\infty\left(\frac{1}{e^t-1}-\frac{1}{t}+\frac{1}{2}\right)e^{-t}t^{s-1}\,dt\quad\bigl(\Re(s)>-1\bigr).\tag{2.8}$$

In order to use (2.8) to get an integral representation for γ_n, we first find the following formula given in Lemma 3.

Lemma 3 *If we define* α_j *by*

$$\alpha_j:=\lim_{s\to 1}\left(\frac{1}{\Gamma(s)}\right)^{(j)}\quad(j\in\mathbb{N}_0),\tag{2.9}$$

then we have a recurrence formula for α_j

$$\alpha_{k+1}=\sum_{j=0}^{k-1}(-1)^{k-j}\frac{k!}{j!}\zeta(k+1-j)\alpha_j+\gamma\alpha_k\quad(k\in\mathbb{N}_0),\tag{2.10}$$

where

$$\alpha_0 = 1 \quad \textit{and} \quad \alpha_1 = -\psi(1) = \gamma, \tag{2.11}$$

and an empty sum (*as usual*) *is understood to be nil throughout this paper.*

In addition to the formulas in (2.11), the next several α_j are given as follows:

$$\begin{aligned} &\alpha_2 = \gamma^2 - \zeta(2), \qquad \alpha_3 = \gamma^3 - 3\gamma\zeta(2) + 2\zeta(3), \\ &\alpha_4 = \gamma^4 - 6\gamma^2\zeta(2) + 8\gamma\zeta(3) + 3\big(\zeta(2)\big)^2 - 6\zeta(4), \\ &\alpha_5 = \gamma^5 - 10\gamma^3\zeta(2) + 20\gamma^2\zeta(3) - 20\zeta(2)\zeta(3) \\ &\qquad + 15\gamma\big(\zeta(2)\big)^2 - 30\gamma\zeta(4) + 24\zeta(5). \end{aligned} \tag{2.12}$$

Proof of Lemma 3 Taking the logarithmic derivative of $1/\Gamma(s)$, we have

$$\left(\frac{1}{\Gamma(s)}\right)' = -\frac{1}{\Gamma(s)} \cdot \psi(s).$$

Using Leibniz's generalization of the product rule for differentiation when we differentiate the last formula k times and taking the limit $s \to 1$ on the resulting identity, and applying (1.22) to the last resulting formula, we obtain

$$\begin{aligned} \alpha_{k+1} &= -\sum_{j=0}^{k-1} \binom{k}{j} \psi^{(k-j)}(1) + \gamma\alpha_k \\ &= \sum_{j=0}^{k-1} (-1)^{k-j} \frac{k!}{j!} \zeta(k+1-j)\alpha_j + \gamma\alpha_k. \end{aligned}$$

This completes the proof of Lemma 3. □

Using Leibniz's generalization of the product rule for differentiation to differentiate both sides of $Z(s)$ in (2.8) with respect to s, n times, and taking the limit $s \to 1$, $Z(s)$ being analytic at $s = 1$ on the resulting identity, and finally using the α_j in (2.9) and the relation (1.20), we obtain an integral formula for γ_n asserted by Theorem 2 below.

Theorem 2 *The following integral representation for* γ_n *holds true*:

$$\begin{aligned} \gamma = \gamma_0 &= 1 + \int_0^\infty \left(\frac{1}{e^t - 1} - \frac{1}{t}\right) e^{-t}\, dt \\ &= \int_0^\infty \left(\frac{1}{1 - e^{-t}} - \frac{1}{t}\right) e^{-t}\, dt \end{aligned} \tag{2.13}$$

and

$$\gamma_n = \int_0^\infty \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2}\right) e^{-t} \left(\sum_{k=0}^{n} \binom{n}{k} \alpha_k (\log t)^{n-k}\right) dt \quad (n \in \mathbb{N}), \tag{2.14}$$

where α_k *are given in Lemma* 3.

The first three of γ_n in (2.14) are given in Corollary 1 below.

Corollary 1 *Each of the following integral formulas holds true*:

$$\gamma_1 = \gamma - \gamma^2 - \int_0^\infty \left(\frac{1}{e^t - 1} - \frac{1}{t}\right) e^{-t} \log t \, dt; \tag{2.15}$$

$$\begin{aligned} \gamma_2 = {} & \gamma^3 - \gamma^2 - \gamma\zeta(2) + \zeta(2) \\ & + \int_0^\infty \left(\frac{1}{e^t - 1} - \frac{1}{t}\right) e^{-t} \{2\gamma \log t + (\log t)^2\} \, dt; \end{aligned} \tag{2.16}$$

$$\begin{aligned} \gamma_3 = {} & \gamma^3 - \gamma^4 + 3\gamma^2\zeta(2) - 3\gamma\zeta(2) - 2\gamma\zeta(3) + 2\zeta(3) \\ & - \int_0^\infty \left(\frac{1}{e^t - 1} - \frac{1}{t}\right) e^{-t} \{3(\gamma^2 - \zeta(2)) \log t + 3\gamma(\log t)^2 + (\log t)^3\} \, dt. \end{aligned} \tag{2.17}$$

Proof It is enough to apply (2.13) and a known recurrence formula (see, *e.g.*, [1, pp.369-371]) for

$$\Gamma^{(n)}(1) = \int_0^\infty e^{-t} (\log t)^n \, dt \quad (n \in \mathbb{N}_0) \tag{2.18}$$

to the first three of γ_n in (2.14). For easy reference, we record here the first three of $\Gamma^{(n)}(1)$:

$$\Gamma'(1) = -\gamma; \qquad \Gamma^{(2)}(1) = \gamma^2 + \zeta(2); \qquad \Gamma^{(3)}(1) = -\gamma^3 - 3\gamma\zeta(2) - 2\zeta(3). \tag{2.19}$$

□

We recall Hermite's integral formula for $\zeta(s)$ (see, *e.g.*, [1, p.169, Eq. (34)]):

$$Z(s) = \zeta(s) - \frac{1}{s-1} = \frac{1}{2} + 2 \int_0^\infty \frac{\sin(s \arctan t)}{(1+t^2)^{\frac{1}{2}s}} \frac{dt}{e^{2\pi t} - 1}. \tag{2.20}$$

Applying Leiniz's generalization of the product rule for differentiation to (2.20), similarly as in Theorems 1 and 2, we get an integral representation for γ_n given in Theorem 3 below.

Theorem 3 *The following integral representation for γ_n holds true*:

$$\gamma = \gamma_0 = \frac{1}{2} + 2 \int_0^\infty \frac{t}{(1+t^2)(e^{2\pi t} - 1)} \, dt \tag{2.21}$$

and

$$\begin{aligned} \gamma_n = (-1)^n \int_0^\infty & \left\{ \sum_{k=0}^n \binom{n}{k} \left(-\frac{1}{2}\right)^{n-k} (\arctan t)^k \sin\left(\arctan t + \frac{k\pi}{2}\right) \right. \\ & \left. \cdot \frac{(\log(1+t^2))^{n-k}}{\sqrt{1+t^2}} \right\} \frac{dt}{e^{2\pi t} - 1} \quad (n \in \mathbb{N}), \end{aligned} \tag{2.22}$$

where

$$\sin(\arctan t) = \frac{t}{\sqrt{1+t^2}} \quad \text{and} \quad \cos(\arctan t) = \frac{1}{\sqrt{1+t^2}}. \tag{2.23}$$

The first three of γ_n in (2.22) are given in Corollary 2 below.

Corollary 2 *Each of the following integral formulas holds true*:

$$\gamma_1 = -\int_0^\infty \frac{2\arctan t - t\log(1+t^2)}{1+t^2}\frac{dt}{e^{2\pi t}-1}; \tag{2.24}$$

$$\gamma_2 = \int_0^\infty \frac{-4t\arctan^2 t - 4\arctan t\log(1+t^2) + t(\log(1+t^2))^2}{2(1+t^2)}\frac{dt}{e^{2\pi t}-1}; \tag{2.25}$$

$$\begin{aligned}\gamma_3 = \int_0^\infty &\bigl\{8\arctan^3 t - 12t\arctan^2 t\log\bigl(1+t^2\bigr) \\ &- 6\arctan t\bigl(\log\bigl(1+t^2\bigr)\bigr)^2 + t\bigl(\log\bigl(1+t^2\bigr)\bigr)^3\bigr\}\frac{1}{4(1+t^2)}\frac{dt}{e^{2\pi t}-1}.\end{aligned} \tag{2.26}$$

Remark Setting $n = 0$ in (2.1), in view of relation (1.21), we obtain an integral representation for γ:

$$\begin{aligned}\gamma = \gamma_0 &= \int_0^\infty \left(\frac{1}{1+t} - \psi'(1+t)\right)dt \\ &= \int_0^\infty \left(\frac{1}{1+t} - \zeta(2,1+t)\right)dt,\end{aligned} \tag{2.27}$$

which is a known formula (see, *e.g.*, [9, Eq. (3.67)]). Equation (1.7) is equal to Equation (2.21), which is recorded, for example, in [1, p.17, Eq. (31)]. Equation (2.13) is a known result (see, *e.g.*, [10, p.355, Entry 3.427-2]). The result (2.24) is equal to the special case of (1.11) when $m = 0$. Connon's result (1.17) is equal to the integral representation (2.15) for γ_1. It is also interesting to compare Connon's result with our one (2.14).

Competing interests
The author declares that he has no competing interests.

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References
1. Srivastava, HM, Choi, J: Zeta and q-Zeta Functions and Associated Series and Integrals. Elsevier, Amsterdam (2012)
2. Finch, SR: Mathematical Constants. Encyclopedia of Mathematics and Its Applications, vol. 94. Cambridge University Press, Cambridge (2003)
3. Furdui, O: Limits, Series, and Fractional Part Integrals: Problems in Mathematical Analysis. Springer, New York (2013)
4. Liang, JJ, Todd, J: The Stieltjes constants. J. Res. Natl. Bur. Stand. B, Math. Sci. **76**, 161-178 (1972)
5. Ainsworth, OR, Howell, LW: An integral representation of the generalized Euler-Mascheroni constants. NASA Centre for AeroSpace Information (CASI) NASA-TP-2456; NAS 1.60.2456 (1985)
6. Connon, DF: Some applications of the Stieltjes constants. January 14 (2009). arXiv:0901.2083
7. Coffey, MW: The Stieltjes constants, their relation to the η_j coefficients, and representation of the Hurwitz zeta function. Analysis **99**, 1001-1021 (2010)
8. Connon, DF: Some integrals involving the Stieltjes constants. April 11 (2011). http://www.researchgate.net/publication/51887271_Some_integrals_involving_the_Stieltjes_constantsPart_II
9. Choi, J, Srivastava, HM: Integral representations for the Euler-Mascheroni constant γ. Integral Transforms Spec. Funct. **21**, 675-690 (2010)
10. Gradshteyn, IS, Ryzhik, IM: Tables of Integrals, Series, and Products. (Corrected and enlarged edition prepared by A. Jeffrey.) Academic Press, New York (1980). 6th edn. (2000)

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