Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Existence of a solution of integral equations via fixed point theorem

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Abstract

In this paper, we establish a solution to the following integral equation:

$$u(t)=\int_0^T G(t,s)f\big(s,u(s)\big)\,ds \quad \text{for all } t\in[0,T], \tag{1}$$

where $T>0$, $f:[0,T]\times\mathbb{R}\to\mathbb{R}$ and $G:[0,T]\times[0,T]\to[0,\infty)$ are continuous functions. For this purpose, we also obtain some auxiliary fixed point results which generalize, improve and unify some fixed point theorems in the literature.

MSC: 47H10; 54H25

Keywords: cyclic representation; fixed point

1 Introduction and preliminaries

Fixed point theory is one of the most efficient tools in nonlinear functional analysis to solve the nonlinear differential and integral equations. The existence/uniqueness of a solution of differential/integral equations turns into the existence/uniqueness of a (common) fixed point of the operators which are obtained after suitable substitutions and elementary calculations; see, *e.g.*, [1–14].

In this paper, we first obtain some fixed point theorems to solve the integral equation mentioned above. For the sake of completeness, we recollect some basic definitions and elementary results. Let X be a nonempty set and T be a self-mapping on X. Then, the set of all fixed points of T on X is denoted by $Fix(T)_X$. Let Ψ be the set of all functions $\psi:[0,\infty)\times[0,\infty)\to[0,\infty)$ satisfying the following conditions:

(1) ψ is continuous,
(2) $\psi(t_1,t_2)=0$ if and only if $t_1=t_2=0$,
(3) $\psi(t_1,t_2)\le\frac{1}{2}(t_1+t_2)$.

Cyclic mapping and cyclic contraction were introduced by Kirk-Srinavasan-Veeramani to improve the well-known Banach fixed point theorem. Later, various types of cyclic contraction have been investigated by a number of authors; see, *e.g.*, [6, 15–17].

Definition 1.1 [18] Suppose that (X,d) is a metric space and T is a self-mapping on X. Let m be a natural number and X_i, $i=1,\dots,m$, be nonempty sets. Then $Y=\bigcup_{i=1}^m X_i$ is called a cyclic representation of X with respect to T if

$$T(X_1) \subset X_2, \qquad \dots, \qquad T(X_{m-1}) \subset X_m, \qquad T(X_m) \subset X_{m+1},$$

where $X_{m+1} = X_1$.

Definition 1.2 [17] Let $T : X \to X$, $r > 0$ and $\eta, \xi : X \to [0, +\infty)$ be two functions. We say that T is r-(η, ξ)-admissible if

(i) $\eta(x) \geq r$ for some $x \in X$ implies $\eta(Tx) \geq r$,

(ii) $\xi(x) \leq r$ for some $x \in X$ implies $\xi(Tx) \leq r$.

Definition 1.3 Let (X, d) be a metric space and $T : Y \to Y$ be a self-mapping, where $Y = \bigcup_{i=1}^m X_i$ is a cyclic representation of Y with respect to T. Let $\eta, \xi : Y \to [0, +\infty)$ be two functions. An operator $T : Y \to Y$ is called:

- a cyclic weak r-(η, ξ)-C-contractive mapping of the first kind if

$$\eta(x)\eta(y)d(Tx, Ty) \leq \xi(x)\xi(y)\left[\frac{1}{2}\left[d(x, Ty) + d(y, Tx)\right] - \psi\left(d(x, Ty), d(y, Tx)\right)\right] \tag{2}$$

  holds for all $x \in X_i$ and $y \in X_{i+1}$, where $\psi \in \Psi$.

- a cyclic weak r-(η, ξ)-C-contractive mapping of the second kind if

$$\left[\eta(x)\eta(y) + r\right]^{d(Tx,Ty)} \leq \left[\xi(x)\xi(y) + r\right]^{[\frac{1}{2}[d(x,Ty)+d(y,Tx)]-\psi(d(x,Ty),d(y,Tx))]} \tag{3}$$

  such that $r^2 + r > 1$ holds for all $x \in X_i$ and $y \in X_{i+1}$, where $\psi \in \Psi$.

2 Auxiliary fixed point results

We state the main result of this section as follows.

Theorem 2.1 *Let (X, d) be a complete metric space, $m \in \mathbb{N}$, $X_1, X_2, \dots, X_m$ be nonempty closed subsets of (X, d) and $Y = \bigcup_{i=1}^m X_i$. Suppose that $T : Y \to Y$ is a cyclic weak r-(η, ξ)-C-contractive mapping of the first kind such that*

(i) *T is r-(η, ξ)-admissible;*

(ii) *there exists $x_0 \in Y$ such that $\eta(x_0) \geq r$ and $\xi(x_0) \leq r$;*

(iii) *if $\{x_n\}$ is a sequence in Y such that $\eta(x_n) \geq r$ and $\xi(x_n) \leq r$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$, then $\eta(x) \geq r$ and $\xi(x) \leq r$.*

Then T has a fixed point $x \in \bigcap_{i=1}^n X_i$. Moreover, if $\eta(x) \geq r$, $\eta(y) \geq r$, $\xi(x) \leq r$, $\xi(y) \leq r$ for all $x, y \in Fix(T)_Y$, then T has a unique fixed point.

Proof Let there exist $x_0 \in Y$ such that $\eta(x_0) \geq r$ and $\xi(x_0) \leq r$. Since T is r-(η, ξ)-admissible, then $\eta(Tx_0) \geq r$ and $\xi(Tx_0) \leq r$. Again, since T is r-(η, ξ)-admissible, then $\eta(T^2x_0) \geq r$ and $\xi(T^2x_0) \leq r$. By continuing this process, we get

$$\eta\left(T^n x_0\right) \geq r \quad \text{and} \quad \xi\left(T^n x_0\right) \leq r \quad \text{for all } n \in \mathbb{N}. \tag{4}$$

On the other hand, since $x_0 \in Y$, there exists some i_0 such that $x_0 \in X_{i_0}$. Now $T(X_{i_0}) \subseteq X_{i_0+1}$ implies that $Tx_0 \in X_{i_0+1}$. Thus there exists x_1 in X_{i_0+1} such that $Tx_0 = x_1$. Similarly,

$Tx_n = x_{n+1}$, where $x_n \in X_{i_n}$. Hence, for $n \geq 0$, there exists $i_n \in \{1, 2, \dots, m\}$ such that $x_n \in X_{i_n}$ and $x_{n+1} \in X_{i_n+1}$. In case $x_{n_0} = x_{n_0+1}$ for some $n_0 = 0, 1, 2, \dots$, then it is clear that x_{n_0} is a fixed point of T. Now assume that $x_n \neq x_{n+1}$ for all n. Hence, we have $d(x_{n-1}, x_n) > 0$ for all n. Set $d_n := d(x_n, x_{n+1})$. We shall show that the sequence $\{d_n\}$ is non-increasing. Due to (2) with $x = x_{n-1}$ and $y = x_n$, we get

$$\begin{aligned}
&r^2 d(x_n, x_{n+1}) \\
&\quad = r^2 d(Tx_{n-1}, Tx_n) \leq \eta(x_{n-1})\eta(x_n) d(Tx_{n-1}, Tx_n) \\
&\quad \leq \xi(x_{n-1})\xi(x_n)\left[\frac{1}{2}\left[d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1})\right] - \psi\left(d(x_{n-1}, Tx_n), d(x_n, Tx_{n-1})\right)\right] \\
&\quad = \xi(x_{n-1})\xi(x_n)\left[\frac{1}{2} d(x_{n-1}, x_{n+1}) - \psi\left(d(x_{n-1}, x_{n+1}), 0\right)\right] \\
&\quad \leq r^2\left[\frac{1}{2} d(x_{n-1}, x_{n+1}) - \psi\left(d(x_{n-1}, x_{n+1}), 0\right)\right],
\end{aligned}$$

which implies

$$\begin{aligned}
d(x_n, x_{n+1}) &\leq \frac{1}{2} d(x_{n-1}, x_{n+1}) - \psi\left(d(x_{n-1}, x_{n+1}), 0\right) \\
&\leq \frac{1}{2} d(x_{n-1}, x_{n+1}) \\
&\leq \frac{1}{2}\left[d(x_{n-1}, x_n) + d(x_n, x_{n+1})\right],
\end{aligned} \tag{5}$$

and so $d_n \leq d_{n-1}$ for all $n \in \mathbb{N}$. Then there exist $d \geq 0$ such that $\lim_{n\to\infty} d_n = d$. Suppose, on the contrary, that $d > 0$. Also, taking limit as $n \to \infty$ in (5), we deduce

$$d \leq \frac{1}{2} \lim_{n\to\infty} d(x_{n-1}, x_{n+1}) \leq \frac{1}{2}(d + d) = d,$$

that is,

$$\lim_{n\to\infty} d(x_{n-1}, x_{n+1}) = 2d. \tag{6}$$

Taking limit as $n \to \infty$ in (5) and using (6), we get

$$d \leq \frac{1}{2}[2d] - \psi(2d, 0).$$

Consequently, we have $\psi(2d, 0) = 0$, which yields $d = 0$. Hence

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{7}$$

We shall show that $\{x_n\}$ is a Cauchy sequence. To reach this goal, first we prove the following claim:

(K) For every $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that if $r, q \geq n$ with $r - q \equiv 1(m)$, then $d(x_r, x_q) < \varepsilon$.

Suppose, to the contrary, that there exists $\varepsilon > 0$ such that for any $n \in \mathbb{N}$, we can find $r_n > q_n \geq n$ with $r_n - q_n \equiv 1(m)$ satisfying

$$d(x_{q_n}, x_{r_n}) \geq \varepsilon. \tag{8}$$

Now, we take $n > 2m$. Then, corresponding to $q_n \geq n$, one can choose r_n in such a way that it is the smallest integer with $r_n > q_n$ satisfying $r_n - q_n \equiv 1(m)$ and $d(x_{q_n}, x_{r_n}) \geq \varepsilon$. Therefore, $d(x_{q_n}, x_{r_n - m}) < \varepsilon$. By using the triangular inequality,

$$\varepsilon \leq d(x_{q_n}, x_{r_n}) \leq d(x_{q_n}, x_{r_n - m}) + \sum_{i=1}^{m} d(x_{r_n - i}, x_{r_n - i - 1}) < \varepsilon + \sum_{i=1}^{m} d(x_{r_n - i}, x_{r_n - i - 1}).$$

Letting $n \to \infty$ in the last inequality, keeping (7) in mind, we derive that

$$\lim_{n \to \infty} d(x_{q_n}, x_{r_n}) = \varepsilon. \tag{9}$$

Again,

$$\begin{aligned}
\varepsilon &\leq d(x_{q_n}, x_{r_n}) \\
&\leq d(x_{q_n}, x_{q_n+1}) + d(x_{q_n+1}, x_{r_n+1}) + d(x_{r_n+1}, x_{r_n}) \\
&\leq d(x_{q_n}, x_{q_n+1}) + d(x_{q_n+1}, x_{q_n}) + d(x_{q_n}, x_{r_n}) + d(x_{r_n}, x_{r_n+1}) + d(x_{r_n+1}, x_{r_n}).
\end{aligned}$$

Taking (7) and (9) into account, we get

$$\lim_{n \to \infty} d(x_{q_n+1}, x_{r_n+1}) = \varepsilon \tag{10}$$

as $n \to \infty$ in (9).

Also we have the following inequalities:

$$d(x_{q_n}, x_{r_n+1}) \leq d(x_{q_n}, x_{r_n}) + d(x_{r_n}, x_{r_n+1}) \tag{11}$$

and

$$d(x_{q_n}, x_{r_n}) \leq d(x_{q_n}, x_{r_n+1}) + d(x_{r_n}, x_{r_n+1}). \tag{12}$$

Letting $n \to \infty$ in (11) and (12), we derive that

$$\lim_{n \to \infty} d(x_{q_n}, x_{r_n+1}) = \varepsilon. \tag{13}$$

Again, we have

$$d(x_{r_n}, x_{q_n+1}) \leq d(x_{r_n}, x_{r_n+1}) + d(x_{r_n+1}, x_{q_n+1}) \tag{14}$$

and

$$d(x_{r_n+1}, x_{q_n+1}) \leq d(x_{r_n+1}, x_{r_n}) + d(x_{r_n}, x_{q_n+1}). \tag{15}$$

Letting $n \to \infty$ in (14) and (15), we conclude that

$$\lim_{n\to\infty} d(x_{r_n}, x_{q_n+1}) = \varepsilon. \tag{16}$$

Since x_{q_n} and x_{r_n} lie in different adjacently labeled sets X_i and X_{i+1} for certain $1 \leq i \leq m$, using the fact that T is a cyclic weak r-(η,ξ)-C-contractive mapping of the first kind, we have

$$\begin{aligned}
&r^2 d(x_{r_n+1}, x_{q_n+1}) \\
&\quad = r^2 d(Tx_{r_n}, Tx_{q_n}) \leq \eta(x_{r_n})\eta(x_{q_n}) d(Tx_{r_n}, Tx_{q_n}) \\
&\quad \leq \xi(x_{r_n})\xi(x_{q_n}) \left[\frac{1}{2}\big[d(x_{r_n}, Tx_{q_n}) + d(x_{q_n}, Tx_{r_n})\big] - \psi\big(d(x_{r_n}, Tx_{q_n}), d(x_{q_n}, Tx_{r_n})\big)\right] \\
&\quad \leq r^2 \left[\frac{1}{2}\big[d(x_{r_n}, x_{q_n+1}) + d(x_{q_n}, x_{r_n+1})\big] - \psi\big(d(x_{r_n}, x_{q_n+1}), d(x_{q_n}, x_{r_n+1})\big)\right],
\end{aligned} \tag{17}$$

which implies

$$d(x_{r_n+1}, x_{q_n+1}) \leq \frac{1}{2}\big[d(x_{r_n}, x_{q_n+1}) + d(x_{q_n}, x_{r_n+1})\big] - \psi\big(d(x_{r_n}, x_{q_n+1}), d(x_{q_n}, x_{r_n+1})\big).$$

Letting $n \to \infty$ in the inequality above and keeping the expressions (7), (9), (10), (13), (16) in mind, we conclude that

$$\varepsilon \leq \varepsilon - \psi(\varepsilon, \varepsilon).$$

Thus, we have $\psi(\varepsilon,\varepsilon) = 0$, which yields that $\varepsilon = 0$. Hence, (K) is satisfied.

We shall show that the sequence $\{x_n\}$ is Cauchy. Fix $\varepsilon > 0$. By the claim, we find $n_0 \in \mathbb{N}$ such that if $r, q \geq n_0$ with $r - q \equiv 1(m)$, then

$$d(x_r, x_q) \leq \frac{\varepsilon}{2}. \tag{18}$$

Since $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$, we also find $n_1 \in \mathbb{N}$ such that

$$d(x_n, x_{n+1}) \leq \frac{\varepsilon}{2m} \tag{19}$$

for any $n \geq n_1$. Suppose that $r, s \geq \max\{n_0, n_1\}$ and $s > r$. Then, there exists $k \in \{1, 2, \ldots, m\}$ such that $s - r \equiv k(m)$. Therefore, $s - r + \varphi \equiv 1(m)$ for $\varphi = m - k + 1$. So, we have, for $j \in \{1, \ldots, m\}$, $s + j - r \equiv 1(m)$

$$d(x_r, x_s) \leq d(x_r, x_{s+j}) + d(x_{s+j}, x_{s+j-1}) + \cdots + d(x_{s+1}, x_s).$$

By (18) and (19) and from the last inequality, we get

$$d(x_r, x_s) \leq \frac{\varepsilon}{2} + j \times \frac{\varepsilon}{2m} \leq \frac{\varepsilon}{2} + m \times \frac{\varepsilon}{2m} = \varepsilon.$$

This proves that $\{x_n\}$ is a Cauchy sequence. Since Y is closed in (X,d), then (Y,d) is also complete, there exists $x \in Y = \bigcup_{i=1}^m X_i$ such that $\lim_{n\to\infty} x_n = x$ in (Y,d). In what follows, we prove that x is a fixed point of T. In fact, since $\lim_{n\to\infty} x_n = x$ and, as $Y = \bigcup_{i=1}^m X_i$ is a cyclic representation of Y with respect to T, the sequence $\{x_n\}$ has infinite terms in each X_i for $i \in \{1, 2, \dots, m\}$. Suppose that $x \in X_i$, $Tx \in X_{i+1}$ and we take a subsequence x_{n_k} of $\{x_n\}$ with $x_{n_k} \in X_{i-1}$. Now from (iii) we have $\eta(x) \ge r$ and $\xi(x) \le r$. By using the contractive condition, we can obtain

$$\begin{aligned} r^2 d(Tx, Tx_{n_k}) &\le \eta(x)\eta(x_{n_k}) d(Tx, Tx_{n_k}) \\ &\le \xi(x)\xi(x_{n_k}) \left[\frac{1}{2}\left[d(x, Tx_{n_k}) + d(x_{n_k}, Tx)\right] - \psi\left(d(x, Tx_{n_k}), d(x_{n_k}, Tx)\right) \right] \\ &\le r^2 \left[\frac{1}{2}\left[d(x, Tx_{n_k}) + d(x_{n_k}, Tx)\right] - \psi\left(d(x, Tx_{n_k}), d(x_{n_k}, Tx)\right) \right], \end{aligned} \tag{20}$$

which implies

$$d(Tx, x_{n_k+1}) \le \frac{1}{2}\left[d(x, x_{n_k+1}) + d(x_{n_k}, Tx)\right] - \psi\left(d(x, x_{n_k+1}), d(x_{n_k}, Tx)\right).$$

Passing to the limit as $k \to \infty$ in the last inequality, we get

$$\begin{aligned} d(x, Tx) &\le \frac{1}{2} d(x, Tx) - \psi\left(0, d(x, Tx)\right) \\ &\le \frac{1}{2} d(x, Tx), \end{aligned}$$

which implies $d(x, Tx) = 0$, *i.e.*, $x = Tx$. Finally, to prove the uniqueness of the fixed point, suppose that $x, y \in Fix(T)_Y$ such that $\eta(x) \ge r$, $\eta(y) \ge r$, $\xi(x) \le r$, $\xi(y) \le r$, where $x \ne y$. The cyclic character of T and the fact that $x, y \in X$ are fixed points of T imply that $x, y \in \bigcap_{i=1}^m X_i$. Suppose that $x \ne y$. That is, $d(x, y) > 0$. Using the contractive condition, we obtain

$$\begin{aligned} r^2 d(Tx, Ty) &\le \eta(x)\eta(y) d(Tx, Ty) \\ &\le \xi(x)\xi(y) \left[\frac{1}{2}\left[d(x, Ty) + d(y, Tx)\right] - \psi\left(d(x, Ty), d(y, Tx)\right) \right] \\ &\le r^2 \left[\frac{1}{2}\left[d(x, Ty) + d(y, Tx)\right] - \psi\left(d(x, Ty), d(y, Tx)\right) \right], \end{aligned}$$

which implies

$$d(x, y) \le d(x, y) - \psi\left(d(x, y), d(x, y)\right).$$

Then $\psi(d(x, y), d(x, y)) = 0$ and so $d(x, y) = 0$, *i.e.*, $x = y$, which is a contradiction. This finishes the proof. □

Example 2.2 Let $X = \mathbb{R}$ with the metric $d(x,y) = |x-y|$ for all $x,y \in X$. Suppose $A_1 = (-\infty, 0]$ and $A_2 = [0,\infty)$ and $Y = \bigcup_{i=1}^{2} A_i$. Define $T : Y \to Y$ and $\eta, \xi : Y \to [0,\infty)$ by

$$Tx = \begin{cases} \frac{x+9}{\sqrt{x^8+1}} & \text{if } x \in (-\infty, -10], \\ \frac{\sin^2 x}{x^4} & \text{if } x \in [-10,-5), \\ -3x & \text{if } x \in [-5,-1), \\ 0 & \text{if } x \in [-1,1], \\ -5\ln x & \text{if } x \in (1,5), \\ \frac{4-x}{(3-x)(2-x)} & \text{if } x \in [5,10), \\ \sqrt[3]{9-x} & \text{if } x \in [10,\infty), \end{cases}$$

$$\eta(x) = \begin{cases} 4 & \text{if } x \in [-1,1], \\ 0 & \text{otherwise}, \end{cases}$$

$$\xi(x) = \begin{cases} 4 & \text{if } x \in [-1,1], \\ 10 & \text{otherwise}. \end{cases}$$

Also, define $\psi : [0,\infty)^2 \to [0,\infty)$ by $\psi(t,s) = \frac{1}{4}(t+s)$. Clearly, $TA_1 \subseteq A_2$, $TA_2 \subseteq A_1$ and $\eta(0) \geq 4$ and $\xi(0) \leq 4$. Let $\eta(x) \geq 4$, then $x \in [-1,1]$. On the other hand, $Tw \in [-1,1]$ for all $w \in [-1,1]$, *i.e.*, $\eta(Tx) \geq 1$. Similarly, $\xi(x) \leq 4$ implies $\xi(Tx) \leq 4$. Therefore, T is an r-(η,ξ)-admissible mapping. Let $\{x_n\}$ be a sequence in X such that $\eta(x_n) \geq 1$, $\xi(x_n) \leq 1$ and $x_n \to x$ as $n \to \infty$. Then $x_n \in [-1,1]$. So, $x \in [-1,1]$, *i.e.*, $\eta(x) \geq 1$ and $\xi(x) \leq 1$.

Let $x \in A_1$ and $y \in A_2$. Now, if $x \notin [-1,0]$ or $y \notin [0,1]$, then $\eta(x)\eta(y) = 0$. Also, if $x \in [-1,0]$ and $y \in [0,1]$, then $d(Tx,Ty) = 0$. That is, $\eta(x)\eta(y)d(Tx,Ty) = 0$ for all $x \in A_1$ and all $y \in A_2$. Hence,

$$\eta(x)\eta(y)d(Tx,Ty) = 0 \leq \xi(x)\xi(y)\left[\frac{1}{2}\bigl[d(x,Ty) + d(y,Tx)\bigr] - \psi\bigl(d(x,Ty), d(y,Tx)\bigr)\right]$$

for all $x \in A_1$ and $y \in A_2$. Then T is a cyclic weak r-(η,ξ)-C-contractive mapping of the first kind. Therefore all the conditions of Theorem 2.1 hold and T has a fixed point in $A_1 \cap A_2$. Here, $x = 0$ is a fixed point of T.

Theorem 2.3 *Let (X,d) be a complete metric space, $m \in \mathbb{N}$, $X_1, X_2, \ldots, X_m$ be nonempty closed subsets of (X,p) and $Y = \bigcup_{i=1}^{m} X_i$. Suppose that $T : Y \to Y$ is a cyclic weak r-(η,ξ)-C-contractive mapping of the second kind such that*

(i) *T is r-(η,ξ)-admissible*;
(ii) *there exists $x_0 \in Y$ such that $\eta(x_0) \geq r$ and $\xi(x_0) \leq r$*;
(iii) *if $\{x_n\}$ is a sequence in Y such that $\eta(x_n) \geq r$ and $\xi(x_n) \leq r$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$, then $\eta(x) \geq r$ and $\xi(x) \leq r$.*

Then T has a fixed point $x \in \bigcap_{i=1}^{n} X_i$. Moreover, if $\eta(x) \geq r$, $\eta(y) \geq r$, $\xi(x) \leq r$, $\xi(y) \leq r$ for all $x,y \in Fix(T)_Y$, then T has a unique fixed point.

Proof By a similar method as in the proof of Theorem 2.1, we have

$$x_{n+1} = Tx_n, \qquad \eta(x_n) \geq r \quad \text{and} \quad \xi(x_n) \leq r \quad \text{for all } n \in \mathbb{N}. \tag{21}$$

We shall show that the sequence $\{d_n := d(x_n, x_{n+1})\}$ is non-increasing. Due to (3) with $x = x_{n-1}$ and $y = x_n$, we get

$$\begin{aligned}
(r^2+r)^{d(x_n,x_{n+1})} &= (r^2+r)^{d(Tx_{n-1},Tx_n)} \leq (\eta(x_{n-1})\eta(x_n)+r)^{d(Tx_{n-1},Tx_n)} \\
&\leq (\xi(x_{n-1})\xi(x_n)+r)^{[\frac{1}{2}[d(x_{n-1},Tx_n)+d(x_n,Tx_{n-1})]-\psi(d(x_{n-1},Tx_n),d(x_n,Tx_{n-1}))]} \\
&= (\xi(x_{n-1})\xi(x_n)+r)^{[\frac{1}{2}d(x_{n-1},x_{n+1})-\psi(d(x_{n-1},x_{n+1}),0)]} \\
&\leq (r^2+r)^{[\frac{1}{2}d(x_{n-1},x_{n+1})-\psi(d(x_{n-1},x_{n+1}),0)]},
\end{aligned}$$

which implies

$$\begin{aligned}
d(x_n,x_{n+1}) &\leq \frac{1}{2}d(x_{n-1},x_{n+1}) - \psi(d(x_{n-1},x_{n+1}),0) \\
&\leq \frac{1}{2}d(x_{n-1},x_{n+1}) \\
&\leq \frac{1}{2}[d(x_{n-1},x_n)+d(x_n,x_{n+1})], \qquad (22)
\end{aligned}$$

and so $d_n \leq d_{n-1}$ for all $n \in \mathbb{N}$. Then there exists $d \geq 0$ such that $\lim_{n\to\infty} d_n = d$. We shall show that $d = 0$ by the method of *reductio ad absurdum*. Suppose that $d > 0$. By letting $n \to \infty$ in (22), we deduce

$$d \leq \frac{1}{2}\lim_{n\to\infty} d(x_{n-1},x_{n+1}) \leq \frac{1}{2}(d+d) = d,$$

that is,

$$\lim_{n\to\infty} d(x_{n-1},x_{n+1}) = 2d. \qquad (23)$$

Taking limit as $n \to \infty$ in (22) and using (23), we get

$$d \leq \frac{1}{2}[2d] - \psi(2d,0).$$

Thus, we have $\psi(2d,0) = 0$ and hence $d = 0$, which is a contradiction. Consequently, we have

$$\lim_{n\to\infty} d_n = \lim_{n\to\infty} d(x_n,x_{n+1}) = 0. \qquad (24)$$

We shall show that $\{x_n\}$ is a Cauchy sequence. To reach this goal, first we prove the following claim:

(K) For every $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that if $r, q \geq n$ with $r - q \equiv 1(m)$, then $d(x_r, x_q) < \varepsilon$.

Suppose, to the contrary, that there exists $\varepsilon > 0$ such that for any $n \in \mathbb{N}$ we can find $r_n > q_n \geq n$ with $r_n - q_n \equiv 1(m)$ satisfying

$$d(x_{q_n}, x_{r_n}) \geq \varepsilon. \qquad (25)$$

Following the related lines in Theorem 2.1, we deduce

$$\lim_{n\to\infty} d(x_{q_n}, x_{r_n}) = \varepsilon, \tag{26}$$

$$\lim_{n\to\infty} d(x_{q_n+1}, x_{r_n+1}) = \varepsilon, \tag{27}$$

$$\lim_{n\to\infty} d(x_{q_n}, x_{r_n+1}) = \varepsilon \tag{28}$$

and

$$\lim_{n\to\infty} d(x_{r_n}, x_{q_n+1}) = \varepsilon. \tag{29}$$

Since x_{q_n} and x_{r_n} lie in different adjacently labeled sets X_i and X_{i+1} for certain $1 \le i \le m$, using the fact that a cyclic weak r-(η,ξ)-C-contractive mapping of the second kind, we have

$$\begin{aligned}
(r^2+r)^{d(x_{r_n+1},x_{q_n+1})} &= (r^2+r)^{d(Tx_{r_n},Tx_{q_n})} \le \bigl(\eta(x_{r_n})\eta(x_{q_n})+r\bigr)^{d(Tx_{r_n},Tx_{q_n})} \\
&\le \bigl(\xi(x_{r_n})\xi(x_{q_n})+r\bigr)^{[\frac{1}{2}[d(x_{r_n},Tx_{q_n})+d(x_{q_n},Tx_{r_n})]-\psi(d(x_{r_n},Tx_{q_n}),d(x_{q_n},Tx_{r_n}))]} \\
&\le (r^2+r)^{[\frac{1}{2}[d(x_{r_n},x_{q_n+1})+d(x_{q_n},x_{r_n+1})]-\psi(d(x_{r_n},x_{q_n+1}),d(x_{q_n},Tx_{r_n+1}))]},
\end{aligned}$$

which implies

$$d(x_{r_n+1}, x_{q_n+1}) \le \frac{1}{2}\bigl[d(x_{r_n}, x_{q_n+1}) + d(x_{q_n}, x_{r_n+1})\bigr] - \psi\bigl(d(x_{r_n}, x_{q_n+1}), d(x_{q_n}, Tx_{r_n+1})\bigr).$$

Letting $n \to \infty$ in the inequality above and by applying (24) (26), (27), (28), (29), we deduce that

$$\varepsilon \le \varepsilon - \psi(\varepsilon, \varepsilon).$$

Consequently, we have $\psi(\varepsilon,\varepsilon) = 0$, and hence $\varepsilon = 0$. As a result, we conclude that (K) is satisfied. We assert that the sequence $\{x_n\}$ is Cauchy. Fix $\varepsilon > 0$. By the claim, we find $n_0 \in \mathbb{N}$ such that if $r, q \ge n_0$ with $r - q \equiv 1(m)$, then

$$d(x_r, x_q) \le \frac{\varepsilon}{2}. \tag{30}$$

Since $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$, we also find $n_1 \in \mathbb{N}$ such that

$$d(x_n, x_{n+1}) \le \frac{\varepsilon}{2m} \tag{31}$$

for any $n \ge n_1$. Suppose that $r, s \ge \max\{n_0, n_1\}$ and $s > r$. Then there exists $k \in \{1, 2, \ldots, m\}$ such that $s - r \equiv k(m)$. Therefore, $s - r + \varphi \equiv 1(m)$ for $\varphi = m - k + 1$. So, we have, for $j \in \{1, \ldots, m\}$, $s + j - r \equiv 1(m)$,

$$d(x_r, x_s) \le d(x_r, x_{s+j}) + d(x_{s+j}, x_{s+j-1}) + \cdots + d(x_{s+1}, x_s).$$

By (30) and (31) and from the last inequality, we get

$$\begin{aligned} d(x_r, x_s) &\le \frac{\varepsilon}{2} + j \times \frac{\varepsilon}{2m} \\ &\le \frac{\varepsilon}{2} + m \times \frac{\varepsilon}{2m} = \varepsilon. \end{aligned}$$

This proves that $\{x_n\}$ is a Cauchy sequence. Since Y is closed in (X, d), then (Y, d) is also complete, there exists $x \in Y = \bigcup_{i=1}^m X_i$ such that $\lim_{n\to\infty} x_n = x$ in (Y, d). In what follows, we prove that x is a fixed point of T. In fact, since $\lim_{n\to\infty} x_n = x$ and, as $Y = \bigcup_{i=1}^m X_i$ is a cyclic representation of Y with respect to T, the sequence $\{x_n\}$ has infinite terms in each X_i for $i \in \{1, 2, \ldots, m\}$. Suppose that $x \in X_i$, $Tx \in X_{i+1}$ and we take a subsequence x_{n_k} of $\{x_n\}$ with $x_{n_k} \in X_{i-1}$. Now from (iii) we have $\eta(x) \ge r$ and $\xi(x) \le r$. By using the contractive condition, we can obtain

$$\begin{aligned} \left(r^2 + r\right)^{d(Tx, Tx_{n_k})} &\le \left(\eta(x)\eta(x_{n_k}) + r\right)^{d(Tx, Tx_{n_k})} \\ &\le \left(\xi(x)\xi(x_{n_k}) + r\right)^{[\frac{1}{2}[d(x, Tx_{n_k}) + d(x_{n_k}, Tx)] - \psi(d(x, Tx_{n_k}), d(x_{n_k}, Tx))]} \\ &\le \left(r^2 + r\right)^{[\frac{1}{2}[d(x, Tx_{n_k}) + d(x_{n_k}, Tx)] - \psi(d(x, Tx_{n_k}), d(x_{n_k}, Tx))]}, \end{aligned} \tag{32}$$

which implies

$$d(Tx, x_{n_k+1}) \le \frac{1}{2}\left[d(x, x_{n_k+1}) + d(x_{n_k}, Tx)\right] - \psi\left(d(x, x_{n_k+1}), d(x_{n_k}, Tx)\right).$$

Passing to the limit as $k \to \infty$ in the last inequality, we get

$$d(x, Tx) \le \frac{1}{2} d(x, Tx) - \psi\left(0, d(x, Tx)\right) \le \frac{1}{2} d(x, Tx),$$

which implies $d(x, Tx) = 0$, *i.e.*, $x = Tx$. Finally, to prove the uniqueness of the fixed point, suppose that $x, y \in Fix(T)_Y$ such that $\eta(x) \ge r$, $\eta(y) \ge r$, $\xi(x) \le r$, $\xi(y) \le r$, where $x \ne y$. The cyclic character of T and the fact that $x, y \in X$ are fixed points of T imply that $x, y \in \bigcap_{i=1}^m X_i$. Suppose that $x \ne y$. That is, $d(x, y) > 0$. Using the contractive condition, we obtain

$$\begin{aligned} \left(r^2 + r\right)^{d(Tx, Ty)} &\le \left(\eta(x)\eta(y) + r\right)^{d(Tx, Ty)} \\ &\le \left(\xi(x)\xi(y) + r\right)^{[\frac{1}{2}[d(x, Ty) + d(y, Tx)] - \psi(d(x, Ty), d(y, Tx))]} \\ &\le \left(r^2 + r\right)^{[\frac{1}{2}[d(x, Ty) + d(y, Tx)] - \psi(d(x, Ty), d(y, Tx))]}, \end{aligned}$$

which implies

$$d(x, y) \le d(x, y) - \psi\left(d(x, y), d(x, y)\right).$$

Hence, we obtain $\psi(d(x, y), d(x, y)) = 0$, which implies $d(x, y) = 0$, that is, $x = y$ a contradiction. □

3 Existence of solutions of an integral equation

For $T > 0$, we denote by $X = C([0,T])$ the set of real continuous functions on $[0,T]$. We endow X with the metric

$$d_\infty(u,v) = \|u - v\|_\infty \quad \text{for all } u,v \in X.$$

It is evident that (X, d_∞) is a complete metric space.

Consider the integral equation

$$u(t) = \int_0^T G(t,s) f\bigl(s, u(s)\bigr)\, ds \quad \text{for all } t \in [0,T], \tag{33}$$

(1) $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ and $G : [0,T] \times [0,T] \to [0,\infty)$ are continuous functions.

(2) Let $(\alpha,\beta) \in X^2$, $(\alpha_0,\beta_0) \in \mathbb{R}^2$ such that

$$\alpha_0 \le \alpha(t) \le \beta(t) \le \beta_0 \quad \text{for all } t \in [0,T]. \tag{34}$$

Assume that for all $t \in [0,T]$, we have

$$\alpha(t) \le \int_0^T G(t,s) f\bigl(s, \beta(s)\bigr)\, ds \tag{35}$$

and

$$\beta(t) \ge \int_0^T G(t,s) f\bigl(s, \alpha(s)\bigr)\, ds. \tag{36}$$

Let for all $s \in [0,T]$, $f(s,\cdot)$ be a decreasing function, that is,

$$x,y \in \mathbb{R}, \quad x \ge y \quad \Longrightarrow \quad f(s,x) \le f(s,y). \tag{37}$$

Let $Z := \{u \in X : u \le \beta\} \cup \{u \in X : u \ge \alpha\}$. There exist $0 \le r < 1$ and $\theta, \pi : Z \to \mathbb{R}$ such that if $\theta(x) \ge 0$ and $\theta(y) \ge 0$ with ($x \le \beta_0$ and $y \ge \alpha_0$) or ($x \ge \alpha_0$ and $y \le \beta_0$), then for every $s \in [0,T]$, we have

$$\bigl|f\bigl(s,x(s)\bigr) - f\bigl(s,y(s)\bigr)\bigr| \le \frac{r|\pi(y)|}{2}\bigl(\bigl|x(s) - Ty(s)\bigr| + \bigl|y(s) - Tx(s)\bigr|\bigr). \tag{38}$$

(3) Assume that

$$\left\| \int_0^T \bigl|\pi(y)\bigr| G(t,s)\, ds \right\|_\infty \le 1 \tag{39}$$

for all $x \in Z$, where $\theta(x) \ge 0$. Suppose that

$$\theta(x) \ge 0 \quad \Longrightarrow \quad \theta(Tx) \ge 0 \quad \text{for } x \in \{u \in X : u \le \beta\} \cup \{u \in X : u \ge \alpha\}. \tag{40}$$

(4) If $\{x_n\}$ is a sequence in $\{u \in X : u \le \beta\} \cup \{u \in X : u \ge \alpha\}$ such that $\theta(x_n) \ge 0$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$, then $\theta(x) \ge 0$.

(5) There exists $x_0 \in \{u \in X : u \le \beta\} \cup \{u \in X : u \ge \alpha\}$ such that $\theta(x_0) \ge 0$.

Theorem 3.1 *Under assumptions* (1)-(5), *integral equation* (33) *has a solution in* $\{u \in C([0,T]) : \alpha(t) \leq u(t) \leq \beta(t) \text{ for all } t \in [0,T]\}$.

Proof Define the closed subsets of X, A_1 and A_2 by

$$A_1 = \{u \in X : u \leq \beta\}$$

and

$$A_2 = \{u \in X : u \geq \alpha\}.$$

Also define the mapping $T : X \to X$ by

$$Tu(t) = \int_0^T G(t,s)f\bigl(s,u(s)\bigr)\,ds \quad \text{for all } t \in [0,T].$$

Let us prove that

$$T(A_1) \subseteq A_2 \quad \text{and} \quad T(A_2) \subseteq A_1. \tag{41}$$

Suppose $u \in A_1$, that is,

$$u(s) \leq \beta(s) \quad \text{for all } s \in [0,T].$$

Applying condition (37), since $G(t,s) \geq 0$ for all $t,s \in [0,T]$, we obtain that

$$G(t,s)f\bigl(s,u(s)\bigr) \geq G(t,s)f\bigl(s,\beta(s)\bigr) \quad \text{for all } t,s \in [0,T].$$

The above inequality with condition (35) imply that

$$\int_0^T G(t,s)f\bigl(s,u(s)\bigr)\,ds \geq \int_0^T G(t,s)f\bigl(s,\beta(s)\bigr)\,ds \geq \alpha(t)$$

for all $t \in [0,T]$. Then we have $Tu \in A_2$.

Similarly, let $u \in A_2$, that is,

$$u(s) \geq \alpha(s) \quad \text{for all } s \in [0,T].$$

Using condition (37), since $G(t,s) \geq 0$ for all $t,s \in [0,T]$, we obtain that

$$G(t,s)f\bigl(s,u(s)\bigr) \leq G(t,s)f\bigl(s,\alpha(s)\bigr) \quad \text{for all } t,s \in [0,T].$$

The above inequality with condition (36) imply that

$$\int_0^T G(t,s)f\bigl(s,u(s)\bigr)\,ds \leq \int_0^T G(t,s)f\bigl(s,\alpha(s)\bigr)\,ds \leq \beta(t)$$

for all $t \in [0,T]$. Then we have $Tu \in A_1$. Also, we deduce that (41) holds.

Now, let $(u,v) \in A_1 \times A_2$, that is, for all $t \in [0,T]$,

$$u(t) \leq \beta(t), \qquad v(t) \geq \alpha(t).$$

This implies from condition (34) that for all $t \in [0,T]$,

$$u(t) \leq \beta_0, \qquad v(t) \geq \alpha_0.$$

Now, by conditions (39) and (38), we have, for all $s \in [0,T]$,

$$\begin{aligned}
\left|Tu(t) - Tv(t)\right| &= \left|\int_0^T G(t,s)\left[f\left(s,u(s)\right) - f\left(s,v(s)\right)\right] ds\right| \\
&\leq \int_0^T G(t,s)\left|f\left(s,u(s)\right) - f\left(s,v(s)\right)\right| ds \\
&\leq \int_0^T G(t,s)\frac{r|\pi(y)|}{2}\left(\left|u(s) - Tv(s)\right| + \left|v(s) - Tu(s)\right|\right) ds \\
&\leq \frac{r}{2}\left(\|u - Tv\|_\infty + \|v - Tu\|_\infty\right)\left\|\int_0^T \left|\pi(v)\right|G(t,s)\, ds\right\|_\infty \\
&\leq \frac{r}{2}\left(\|u - Tv\|_\infty + \|v - Tu\|_\infty\right),
\end{aligned}$$

which implies

$$\|Tu - Tv\|_\infty \leq \frac{r}{2}\left(\|u - Tv\|_\infty + \|v - Tu\|_\infty\right).$$

Define $\eta, \xi : Z \to [0,\infty)$ by $\eta(u) = \left\{\begin{smallmatrix} 1, & \theta(u) \geq 0, \\ 0, & \text{otherwise} \end{smallmatrix}\right.$ and $\xi(u) = 1$. Further, $\psi(t_1,t_2) = \frac{(1-r)}{2}(t_1 + t_2)$.

Hence,

$$\eta(u)\eta(v)d_\infty(Tu,Tv) \leq \frac{r}{2}\left(d_\infty(u,Tv) + d_\infty(v,Tu)\right)$$

for all $(u,v) \in A_1 \times A_2$. By a similar method, we can show that the above inequality holds if $(u,v) \in A_2 \times A_1$. Now, all the conditions of Theorem 2.1 hold and T has a fixed point z^* in

$$A_1 \cap A_2 = \left\{u \in C\left([0,T]\right) : \alpha \leq u(t) \leq \beta \text{ for all } t \in [0,T]\right\}.$$

That is, $z^* \in A_1 \cap A_2$ is the solution to (33). □

Example 3.2 In this example, we denote by $X = C([0,1])$ the set of real continuous functions on $[0,1]$. We endow X with the metric

$$d_\infty(u,v) = \|u - v\|_\infty \quad \text{for all } u,v \in X.$$

Consider the following continuous functions:

$$f(t,x) = \begin{cases} x^3 & \text{if } x \in (-\infty, 0), \\ 0 & \text{if } x \in [0,1], \\ x^2 - 1 & \text{if } x \in (1,4), \\ 15 & \text{if } x \in [4, \sqrt{e^{16}-1}], \\ x^2 + 16 - e^{16} & \text{if } x \in (\sqrt{e^{16}-1}, \infty) \end{cases} \quad \text{for all } t \in [0,1]$$

and

$$G(t,s) = \frac{t}{1+t} e^s \quad \text{for all } s,t \in [0,1] \times [0,1].$$

Let $\alpha(t) = 0$ and $\beta(t) = 1$. Then, for $(\alpha_0, \beta_0) = (0,1) \in \mathbb{R}^2$, we have

$$\alpha_0 \le \alpha(t) \le \beta(t) \le \beta_0;$$
$$\alpha(t) = 0 \le \int_0^1 G(t,s) f\bigl(s, \beta(s)\bigr)\, ds = 0$$

and

$$\beta(t) = 1 \ge \int_0^1 G(t,s) f\bigl(s, \alpha(s)\bigr)\, ds = 0.$$

Also, $Z := \{u \in X : u \le \beta\} \cup \{u \in X : u \ge \alpha\} = X$. Define $\theta, \pi : Z \to \mathbb{R}$ by

$$\theta\bigl(x(t)\bigr) = \begin{cases} 0 & \text{if } 0 \le x(t) \le 1 \text{ for all } t \in [0,1] \\ -1, & \text{otherwise} \end{cases} \quad \text{and} \quad \pi(x) = \frac{1}{e-1}.$$

Clearly, $\theta(0) \ge 0$. Also, if $\theta(x(t)) \ge 0$, then $0 \le x(t) \le 1$. On the other hand,

$$Tu(t) = \int_0^1 G(t,s) f\bigl(s, u(s)\bigr)\, ds = 0$$

for all $0 \le u(t) \le 1$. That is, $\theta(Tx(t)) \ge 0$. Hence, $\theta(x) \ge 0$ implies $\theta(Tx) \ge 0$.

Assume $\theta(x(s)) \ge 0$ and $\theta(y(s)) \ge 0$ with ($x \le \beta_0$ and $y \ge \alpha_0$) or ($x \ge \alpha_0$ and $y \le \beta_0$). Thus, $0 \le x(s) \le 1$ and $0 \le y(s) \le 1$, which implies $f(s, x(s)) = f(s, y(s)) = 0$. That is,

$$\bigl|f\bigl(s, x(s)\bigr) - f\bigl(s, y(s)\bigr)\bigr| = 0 \le \frac{r|\pi(y)|}{2} \bigl(\bigl|x(s) - Ty(s)\bigr| + \bigl|y(s) - Tx(s)\bigr|\bigr)$$

for all $s \in [0,1]$, where $0 \le r < 1$. Further,

$$\int_0^1 \bigl|\pi(y)\bigr| G(t,s)\, ds = \int_0^1 \frac{1}{e-1} \frac{t}{1+t} e^s\, ds = \frac{t}{1+t} \le 1,$$

and so

$$\biggl\| \int_0^T \bigl|\pi(y)\bigr| G(t,s)\, ds \biggr\|_\infty \le 1.$$

Assume that $\{x_n\}$ is a sequence in X such that $\theta(x_n) \geq 0$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$. Then $0 \leq x_n \leq 1$. So, $0 \leq x \leq 1$. That is, $\theta(x) \geq 0$.

Therefore, all of the conditions of Theorem 3.1 are satisfied. Then the integral equation

$$u(t) = \frac{t}{1+t} \int_0^1 e^s f\bigl(s, u(s)\bigr)\, ds$$

has a solution in $\{u \in C([0,1]) : 0 \leq u(t) \leq 1 \text{ for all } t \in [0,1]\}$. Here, $u(t) = 0$ is a solution.

But if we chose $x_0(t) = 0$ and $y_0(t) = \sqrt{e^{16} - 1}$, then $f(s, x_0(s)) = 0$ and $f(s, y_0(s)) = 15$. That is,

$$\bigl|f\bigl(s, x_0(s)\bigr) - f\bigl(s, y_0(s)\bigr)\bigr| = 15.$$

Also,

$$\sqrt{\ln\bigl(\bigl|x_0(s) - y_0(s)\bigr|^2 + 1\bigr)} = \sqrt{\ln\bigl(\bigl|0 - \sqrt{e^{16} - 1}\bigr|^2 + 1\bigr)} = \sqrt{\ln e^{16}} = 4,$$

and so

$$\bigl|f\bigl(s, x_0(s)\bigr) - f\bigl(s, y_0(s)\bigr)\bigr| = 15 > 4 = \sqrt{\ln\bigl(\bigl|x_0(s) - y_0(s)\bigr|^2 + 1\bigr)}.$$

That is, Theorem 3.1 of [6] cannot be applied to this example.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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Acknowledgements
The authors thank the anonymous referees for their remarkable comments, suggestions and ideas that helped to improve this paper. The third author (V Rakocević) is supported by Grant No. 174025 of the Ministry of Science, Technology and Development, Republic of Serbia.

Received: 14 May 2013 Accepted: 18 October 2013 Published: 11 Nov 2013

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10.1186/1029-242X-2013-529
Cite this article as: Gülyaz et al.: **Existence of a solution of integral equations via fixed point theorem.** *Journal of Inequalities and Applications* 2013, 2013:529