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Estimating the polygamma functions for negative integers

Ahmed Salem[1] and Adem Kılıçman[2*]

*Correspondence:
akilicman@putra.upm.edu.my
[2]Department of Mathematics and Institute of Mathematical Research, Universiti Putra Malaysia (UPM), Serdang, Selangor 43400, Malaysia
Full list of author information is available at the end of the article

Abstract

The polygamma functions $\psi^{(r)}(x)$ are defined for all $x > 0$ and $r \in \mathbb{N}$. In this paper, the concepts of neutrix and neutrix limit are applied to generalize and redefine the polygamma functions $\psi^{(r)}(x)$ for all $x \in \mathbb{R}$ and $r \in \mathbb{N}$. Also, further results are given.

MSC: 33B15; 46F10

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1 Introduction

Neutrices are additive groups of negligible functions that do not contain any constants except zero. Their calculus was developed by van der Corput [1] and Hadamard in connection with asymptotic series and divergent integrals. Recently, the concepts of neutrix and neutrix limit have been used widely in many applications in mathematics, physics and statistics.

For example, Jack Ng and van Dam applied the neutrix calculus, in conjunction with the Hadamard integral developed by van der Corput, to the quantum field theories, in particular, to obtain finite results for the coefficients in the perturbation series. They also applied the neutrix calculus to quantum field theory and obtained finite renormalization in the loop calculations, see [2] and [3].

Further, many special functions have been generalized and redefined by using the neutrices. Fisher used the neutrices and the neutrix limit to define gamma, beta and incomplete gamma functions [4–8].

Ozcag *et al.* [9, 10] applied the neutrix limit to extend the definition of incomplete beta function and its partial derivatives for negative integers. Also, the digamma function was generalized for negative integers by Jolevska-Tuneska *et al.* [11]. Salem [12, 13] applied the neutrix limit to redefine the q-gamma and the incomplete q-gamma functions and their derivatives. In continuation, in this paper, we apply the concepts of neutrix and neutrix limit to generalize and redefine the polygamma functions.

A neutrix N is defined as a commutative additive group of functions $f(\xi)$ defined on a domain N' with values in an additive group N'', where further if for some f in N, $f(\xi) = \gamma$ for all $\xi \in N'$, then $\gamma = 0$. The functions in N are called negligible functions.

Let N be a set contained in a topological space with a limit point a which does not belong to N. If $f(\xi)$ is a function defined on N' with values in N'' and it is possible to find a constant c such that $f(\xi) - c \in N$, then c is called the neutrix limit of f as ξ tends to a, and we write $N\text{-}\lim_{\xi\to a} f(\xi) = c$.

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Note that if a neutrix limit c exists, then it is unique, since if $f(x) - c$ and $f(x) - c'$ are in N, then the constant function $c - c'$ is also in N and so $c = c'$.

Also note that if N is a neutrix containing the set of all functions which converge to zero in the normal sense as x tends to y, then

$$\lim_{x\to y} f(x) = c \quad \Rightarrow \quad N\text{-}\lim_{x\to y} f(x) = c.$$

In this paper, we let N be the neutrix having the domain $N' = \{\epsilon : 0 < \epsilon < \infty\}$ and the range N'' of real numbers, with negligible functions being finite linear sums of the functions

$$\epsilon^{\lambda} \ln^{r-1} \epsilon, \qquad \ln^r \epsilon \quad (\lambda < 0, r \in \mathbb{N})$$

and all functions $o(\epsilon)$ which converge to zero in the normal sense as ϵ tends to zero [1].

2 Gamma and digamma functions

The gamma function is defined as a locally summable function on the real line by [14]

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-x}\, dx, \quad x > 0. \tag{2.1}$$

In the classical sense, $\Gamma(x)$ function is not defined for the negative integer thus still is an open problem to give satisfactory definition. However, by using the neutrix limit, it was shown in [6] that gamma function (2.1) is defined as follows:

$$\Gamma(x) = N\text{-}\lim_{\epsilon\to 0} \int_{\epsilon}^{\infty} t^{x-1} e^{-t}\, dt \tag{2.2}$$

for $x \neq 0, -1, -2, \ldots$, and this function is also defined by the neutrix limit

$$\begin{aligned}\Gamma(-m) &= N\text{-}\lim_{\epsilon\to 0} \int_{\epsilon}^{\infty} t^{-m-1} e^{-t}\, dt \\ &= \int_1^{\infty} t^{-m-1} e^{-t}\, dt + \int_0^1 t^{-m-1} \left[e^{-t} - \sum_{i=0}^{m} \frac{(-1)^i}{i!} t^i \right] dt - \sum_{i=0}^{m-1} \frac{(-1)^i}{i!(m-i)}\end{aligned} \tag{2.3}$$

for $m \in \mathbb{N}$. The existence of the rth derivative of the gamma function was also proven in [6] and defined by the equation

$$\begin{aligned}\Gamma^{(r)}(0) &= N\text{-}\lim_{\epsilon\to 0} \int_{\epsilon}^{\infty} t^{-1} \ln^r t e^{-t}\, dt \\ &= \int_1^{\infty} t^{-1} \ln^r t e^{-t}\, dt + \int_0^1 t^{-1} \ln^r t \big[e^{-t} - 1 \big]\, dt,\end{aligned} \tag{2.4}$$

$$\begin{aligned}\Gamma^{(r)}(-m) &= N\text{-}\lim_{\epsilon\to 0} \int_{\epsilon}^{\infty} t^{-m-1} \ln^r t e^{-t}\, dt \\ &= \int_1^{\infty} t^{-m-1} \ln^r t e^{-t}\, dt + \int_0^1 t^{-m-1} \ln^r t \left[e^{-t} - \sum_{i=0}^{m} \frac{(-1)^i}{i!} t^i \right] dt \\ &\quad - \sum_{i=0}^{m-1} \frac{(-1)^i}{i!} r!(m-i)^{-r-1}\end{aligned} \tag{2.5}$$

for $r \in \mathbb{N}_0$ and $m \in \mathbb{N}$, see also [15]. Further, it was proven that

$$\Gamma(-r) = \frac{(-1)^r}{r!}\phi(r) - \frac{(-1)^r}{r!}\gamma \tag{2.6}$$

for $r = 1, 2, \ldots$, where

$$\phi(r) = \sum_{i=1}^{r} \frac{1}{i},$$

thus we can extend the definition to the whole real line where

$$\Gamma(0) = \Gamma'(1) = -\gamma,$$

where γ denotes Euler's constant, see [16].

The logarithmic derivative of the gamma function is known as the *psi* or *digamma* function $\psi(x)$, that is, it is given by

$$\psi(x) = \frac{d}{dx}\bigl(\ln\Gamma(x)\bigr) = \frac{\Gamma'(x)}{\Gamma(x)}. \tag{2.7}$$

The digamma function is also defined as a locally summable function on the real line by

$$\psi(x) = -\gamma + \int_0^\infty \frac{e^{-t} - e^{-xt}}{1 - e^{-t}}\,dt, \quad x > 0, \tag{2.8}$$

and this can be read as

$$\psi(x) = -\gamma + \int_0^1 \frac{1 - t^{x-1}}{1 - t}\,dt, \quad x > 0, \tag{2.9}$$

where γ is also Euler's constant.

We note that the digamma function $\psi(x)$ is redefined by [11] to be

$$\psi(x) = -\gamma + N\text{-}\lim_{\epsilon \to 0} \int_\epsilon^1 \frac{1 - t^{x-1}}{1 - t}\,dt, \quad x \in \mathbb{R}, \tag{2.10}$$

and it has values for negative integers as

$$\psi(x) = -\gamma + H_n, \quad n \in \mathbb{N}_0, \tag{2.11}$$

where H_n denotes the Harmonic number defined as

$$H_n = \sum_{k=1}^{n} \frac{1}{k}, \quad n \in \mathbb{N} \text{ with } H_0 = 0.$$

It was also proven in [11] that

$$\psi'(-n) = \sum_{k=1}^{\infty} \frac{1}{(n-k)^2}, \quad n \in \mathbb{N}, \tag{2.12}$$

which is correct for all $n \neq k$; however, if $n = k$, then $\psi'(-n)$ is undefined, and we will give the exact formula for (2.12) in the following section.

3 Polygamma functions

The polygamma functions $\psi^{(r)}(x)$ are defined as the rth derivative of the digamma function, that is,

$$\psi^{(r)}(x) = -\int_0^1 \frac{t^{x-1}\ln^r t}{1-t}\,dt, \quad x > 0, r \in \mathbb{N}. \tag{3.1}$$

In the present section, we seek to redefine the polygamma functions $\psi^{(r)}(x)$ for all $x \in \mathbb{R}$ and $r \in \mathbb{N}$.

Lemma 3.1 *The neutrix limit, as ϵ tends to zero of the integral*

$$\int_\epsilon^1 t^{\alpha-1}\ln^r t\,dt, \tag{3.2}$$

exists for all values of $\alpha \in \mathbb{R}$ and $r \in \mathbb{N}$ and

$$N\text{-}\lim_{\epsilon\to 0}\int_\epsilon^1 t^{\alpha-1}\ln^r t\,dt = \begin{cases} \frac{(-1)^r r!}{\alpha^{r+1}}, & \alpha \neq 0, \\ 0, & \alpha = 0. \end{cases} \tag{3.3}$$

Proof Let

$$I_r(a,b) = \int_a^b t^{\alpha-1}\ln^r t\,dt, \quad r \in \mathbb{N}.$$

It is easy to see by integration by parts that

$$I_r(a,b) = \frac{b^\alpha \ln^r b - a^\alpha \ln^r a}{\alpha} - \frac{r}{\alpha} I_{r-1}(a,b) \quad \text{and} \quad I_0(a,b) = \frac{b^\alpha - a^\alpha}{\alpha}.$$

The induction yields

$$I_r(a,b) = \frac{(-1)^r r!}{\alpha^{r+1}}\left(b^\alpha - a^\alpha\right) - \sum_{k=1}^r \frac{(-1)^r r!}{(r-k+1)!\alpha^k}\left(b^\alpha \ln^{r-k+1} b - a^\alpha \ln^{r-k+1} a\right).$$

Hence, we get

$$I_r(\epsilon,1) = \frac{(-1)^r r!}{\alpha^{r+1}}\left(1-\epsilon^\alpha\right) + \sum_{k=1}^r \frac{(-1)^r r!}{(r-k+1)!\alpha^k}\epsilon^\alpha \ln^{r-k+1}\epsilon.$$

Notice that the sum consists of (linear sum of $\epsilon^\lambda \ln^k \epsilon$) negligible functions which can be neglected when taking the neutrix limit as $\epsilon \to 0$. Taking the neutrix limit as $\epsilon \to 0$ yields the desired results. □

Theorem 3.2 *Let $n, r \in \mathbb{N}$ and $x \in \mathbb{R}$. Then the neutrix limit, as ϵ tends to zero of the integral*

$$\int_\epsilon^1 \frac{t^{x-1}\ln^r t}{1-t}\,dt, \tag{3.4}$$

exists for all values of $-n < x < -n+1$.

Proof Using the geometric sequence sum rule gives

$$\begin{aligned}\int_\epsilon^1 \frac{t^{x-1}\ln^r t}{1-t}\,dt &= \int_\epsilon^1 \frac{t^{x-1}\ln^r t(1-t^n)}{1-t}\,dt + \int_\epsilon^1 \frac{t^{x+n-1}\ln^r t}{1-t}\,dt \\ &= \sum_{k=0}^{n-1}\int_\epsilon^1 t^{x+k-1}\ln^r t\,dt + \int_\epsilon^1 \frac{t^{x+n-1}\ln^r t}{1-t}\,dt.\end{aligned}$$

Taking the neutrix limit as $\epsilon \to 0$ yields

$$N\text{-}\lim_{\epsilon\to 0}\int_\epsilon^1 \frac{t^{x-1}\ln^r t}{1-t}\,dt = \sum_{k=0}^{n-1}\left(N\text{-}\lim_{\epsilon\to 0}\int_\epsilon^1 t^{x+k-1}\ln^r t\,dt\right) + \int_0^1 \frac{t^{x+n-1}\ln^r t}{1-t}\,dt.$$

Lemma 3.1 indicates that the neutrix limit of the integrals

$$N\text{-}\lim_{\epsilon\to 0}\int_\epsilon^1 t^{x+k-1}\ln^r t\,dt$$

exist for all $-n < x < -n+1$ and thus integral (3.4) also exists. □

Theorem 3.3 *The neutrix limit, as ϵ tends to zero of the integral*

$$\int_\epsilon^1 \frac{t^{-1}\ln^r t}{1-t}\,dt, \tag{3.5}$$

exists for all $r \in \mathbb{N}$ and

$$N\text{-}\lim_{\epsilon\to 0}\int_\epsilon^1 \frac{t^{-1}\ln^r t}{1-t}\,dt = -\psi^{(r)}(1). \tag{3.6}$$

Proof We have

$$\int_\epsilon^1 \frac{t^{-1}\ln^r t}{1-t}\,dt = \int_\epsilon^1 t^{-1}\ln^r t\,dt + \int_\epsilon^1 \frac{\ln^r t}{1-t}\,dt.$$

In view of Lemma 3.1, the neutrix limit of the first integral exists and equals zero, and so we get

$$N\text{-}\lim_{\epsilon\to 0}\int_\epsilon^1 \frac{t^{-1}\ln^r t}{1-t}\,dt = \int_0^1 \frac{\ln^r t}{1-t}\,dt = -\psi^{(r)}(1).$$ □

Theorem 3.4 *The neutrix limit, as ϵ tends to zero of the integral*

$$\int_\epsilon^1 \frac{t^{-n-1}\ln^r t}{1-t}\,dt, \tag{3.7}$$

exists for all $n, r \in \mathbb{N}$ and

$$N\text{-}\lim_{\epsilon\to 0}\int_\epsilon^1 \frac{t^{-n-1}\ln^r t}{1-t}dt = -\psi^{(r)}(1) - \sum_{k=1}^{n}\frac{r!}{k^{r+1}}. \tag{3.8}$$

Proof Using the geometric sequence sum rule gives

$$\begin{aligned}\int_\epsilon^1 \frac{t^{-n-1}\ln^r t}{1-t}\,dt &= \int_\epsilon^1 \frac{t^{-n-1}(1-t^{n+1})\ln^r t}{1-t}\,dt + \int_\epsilon^1 \frac{\ln^r t}{1-t}\,dt \\ &= \sum_{k=0}^{n}\int_\epsilon^1 t^{k-n-1}\ln^r t\,dt + \int_\epsilon^1 \frac{\ln^r t}{1-t}\,dt \\ &= \sum_{k=0}^{n-1}\int_\epsilon^1 t^{k-n-1}\ln^r t\,dt + \int_\epsilon^1 t^{-1}\ln^r t\,dt + \int_\epsilon^1 \frac{\ln^r t}{1-t}\,dt \\ &= \sum_{k=1}^{n}\int_\epsilon^1 t^{-k-1}\ln^r t\,dt + \int_\epsilon^1 t^{-1}\ln^r t\,dt + \int_\epsilon^1 \frac{\ln^r t}{1-t}\,dt.\end{aligned}$$

Taking the neutrix limit as $\epsilon \to 0$ yields

$$\begin{aligned}N\text{-}\lim_{\epsilon\to 0}\int_\epsilon^1 \frac{t^{-n-1}\ln^r t}{1-t}\,dt &= \sum_{k=1}^{n}\left(N\text{-}\lim_{\epsilon\to 0}\int_\epsilon^1 t^{-k-1}\ln^r t\,dt\right) \\ &\quad + N\text{-}\lim_{\epsilon\to 0}\int_\epsilon^1 t^{-1}\ln^r t\,dt + \int_0^1 \frac{\ln^r t}{1-t}\,dt.\end{aligned}$$

The results obtained in Lemma 3.1 and definition (3.1) complete the proof. □

The above theorems lead us to introducing the following.

Definition 3.5 The polygamma function $\psi^{(r)}(x)$ can be redefined by

$$\psi^{(r)}(x) = -\left(N\text{-}\lim_{\epsilon\to 0}\int_\epsilon^1 \frac{t^{x-1}\ln^r t}{1-t}\,dt\right) \tag{3.9}$$

for all $x \in \mathbb{R}$ and $r \in \mathbb{N}$.

Corollary 3.6 *The polygamma function $\psi^{(r)}(x)$ has value at $x = 0$ as*

$$\psi^{(r)}(0) = \psi^{(r)}(1) \tag{3.10}$$

and for negative integers $x = -n$ as

$$\psi^{(r)}(-n) = \psi^{(r)}(1) + \sum_{k=1}^{n}\frac{r!}{k^{r+1}}, \quad n \in \mathbb{N}. \tag{3.11}$$

In particular, when $r = 1$, *we have*

$$\psi'(0) = \psi'(1) = \frac{\pi^2}{6} \simeq 1.64493 \tag{3.12}$$

and

$$\psi'(-n) = \psi'(1) + \sum_{k=1}^{n} \frac{1}{k^2} = \frac{\pi^2}{6} + \sum_{k=1}^{n} \frac{1}{k^2} \simeq 1.64493 + \sum_{k=1}^{n} \frac{1}{k^2}, \quad n \in \mathbb{N}. \tag{3.13}$$

Corollary 3.7 *If we let* $n \to \infty$ *to* (3.11), *then we get*

$$\psi^{(r)}(-\infty) = \psi^{(r)}(1) + r!\zeta(r+1), \quad r \in \mathbb{N}, \tag{3.14}$$

where $\zeta(s)$ *is the Riemann zeta function defined as* [14]

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}, \quad \Re(s) > 1.$$

Since $\psi^{(r)}(1) = (-1)^{r+1} r!\zeta(r+1)$, *then we get*

$$\psi^{(r)}(-\infty) = r!\zeta(r+1)\bigl(1 + (-1)^{r+1}\bigr) = \begin{cases} 2r!\zeta(r+1) & \textit{if } n \textit{ is odd}, \\ 0 & \textit{if } n \textit{ is even}, \end{cases} \tag{3.15}$$

and if $r = 1$, *we get*

$$\psi'(-\infty) = 2\zeta(2) = \frac{\pi^2}{3} \simeq 3.28987. \tag{3.16}$$

Remark 3.8 Formula (3.13) is the correct formula of (2.12).

Competing interests
The authors declare that they do not have competing interests in publishing this article.

Authors' contributions
Both authors contributed equally. All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, Faculty of Science, Al Jouf University, Sakaka, Al Jouf, Kingdom of Saudi Arabia.
[2]Department of Mathematics and Institute of Mathematical Research, Universiti Putra Malaysia (UPM), Serdang, Selangor 43400, Malaysia.

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