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Implicit iteration scheme for two phi-hemicontractive operators in arbitrary Banach spaces

Guiwen Lv[1*], Arif Rafiq[2] and Zhiqun Xue[1]

*Correspondence:
lvguiwenyy@126.com
[1]Department of Mathematics and Physics, Shijiazhuang Tiedao University, Shijiazhuang, 050043, P.R. China
Full list of author information is available at the end of the article

Abstract
The purpose of this paper is to characterize conditions for the convergence of the implicit Ishikawa iterative scheme with errors in the sense of Agarwal *et al.* (J. Math. Anal. Appl. 272:435-447, 2002) to a common fixed point of two ϕ-hemicontractive mappings in a nonempty convex subset of an arbitrary Banach space.
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1 Introduction and preliminaries

Let K be a nonempty subset of an arbitrary Banach space E and E^* be its dual space. The symbols T and $F(T)$ stand for the self-map of K and the set of fixed points of T. We denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$J(x) = \{f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2\}.$$

Definition 1.1 [1–4] (i) T is said to be strongly pseudocontractive if there exists a constant $t > 1$ such that for each $x, y \in K$, there exists $j(x-y) \in J(x-y)$ satisfying

$$\langle Tx - Ty, j(x-y) \rangle \le \frac{1}{t}\|x-y\|^2.$$

(ii) T is said to be strictly hemicontractive if $F(T) \neq \emptyset$ and if there exists a constant $t > 1$ such that for each $x \in K$ and $q \in F(T)$, there exists $j(x-q) \in J(x-q)$ satisfying

$$\langle Tx - Tq, j(x-q) \rangle \le \frac{1}{t}\|x-q\|^2.$$

(iii) T is said to be ϕ-strongly pseudocontractive if there exists a strictly increasing function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ such that for each $x, y \in K$, there exists $j(x-y) \in J(x-y)$ satisfying

$$\langle Tx - Ty, j(x-y) \rangle \le \|x-y\|^2 - \phi(\|x-y\|).$$

(iv) T is said to be ϕ-hemicontractive if $F(T) \neq \emptyset$ and if there exists a strictly increasing function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ such that for each $x \in K$ and $q \in F(T)$, there exists $j(x-q) \in J(x-q)$ satisfying

$$\langle Tx - Tq, j(x-q) \rangle \le \|x-q\|^2 - \phi(\|x-q\|).$$

Clearly, each strictly hemicontractive operator is ϕ-hemicontractive.

Chidume [1] established that the Mann iteration sequence converges strongly to the unique fixed point of T in the case that T is a Lipschitz strongly pseudo-contractive mapping from a bounded, closed, convex subset of L_p (or l_p) into itself. Afterwards, several authors have generalized this result of Chidume in various directions [4–12].

In 2001, Xu and Ori [13] introduced the following implicit iteration process for a finite family of nonexpansive mappings $\{T_i : i \in I\}$ (here $I = \{1, 2, \dots, N\}$), with $\{\alpha_n\}$ a real sequence in $(0, 1)$, and an initial point $x_0 \in K$:

$$\begin{aligned}
&x_1 = (1-\alpha_1)x_0 + \alpha_1 T_1 x_1,\\
&x_2 = (1-\alpha_2)x_1 + \alpha_2 T_2 x_2,\\
&\vdots\\
&x_N = (1-\alpha_N)x_{N-1} + \alpha_N T_N x_N,\\
&x_{N+1} = (1-\alpha_{N+1})x_N + \alpha_{N+1} T_{N+1} x_{N+1},\\
&\vdots
\end{aligned}$$

which can be written in the following compact form:

$$x_n = (1-\alpha_n)x_{n-1} + \alpha_n T_n x_n \quad \text{for all } n \ge 1, \tag{XO}$$

where $T_n = T_{n(\mathrm{mod}\,N)}$ (here the mod$N$ function takes values in I). Xu and Ori [13] proved that the process converges weakly to a common fixed point of a finite family in a Hilbert space. They remarked further that it is yet unclear what assumptions on the mappings and the parameters $\{\alpha_n\}$ are sufficient to guarantee the strong convergence of the sequence $\{x_n\}$.

In [14], Osilike proved the following results.

Theorem 1.2 [14, Theorem 2] *Let E be a real Banach space and K be a nonempty closed convex subset of E. Let $\{T_i : i \in I\}$ be N strictly pseudocontractive self-mappings of K with $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $\{\alpha_n\}_{n=1}^\infty$ be a real sequence satisfying the conditions*:

(i) $0 < \alpha_n < 1$,

(ii) $\sum_{n=1}^\infty (1-\alpha_n) = \infty$,

(iii) $\sum_{n=1}^\infty (1-\alpha_n)^2 < \infty$.

For arbitrary $x_0 \in K$, define the sequence $\{x_n\}$ by the implicit iteration process (XO). *Then $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{T_i : i \in I\}$ if and only if* $\lim_{n\to\infty} \inf d(x_n, F) = 0$.

It is well known that $\alpha_n = 1 - \frac{1}{n^{\frac{1}{2}}}$, $\sum(1-\alpha_n)^2 = \infty$. Hence the results of Osilike [14] need to be improved.

The purpose of this paper is to characterize conditions for the convergence of the implicit Ishikawa iterative scheme with errors in the sense of Agarwal *et al.* [15] to a common fixed point of two ϕ-hemicontractive mappings in a nonempty convex subset of an arbitrary Banach space. Our studying results improve and generalize most results in recent literature [4, 5, 7–9, 12].

2 Main results

The following results are now well known.

Lemma 2.1 [16] *For all $x, y \in E$ and $j(x+y) \in J(x+y)$,*

$$\|x+y\|^2 \le \|x\|^2 + \langle y, j(x+y)\rangle.$$

Lemma 2.2 [17] *Let $\{\theta_n\}$ be a sequence of nonnegative real numbers, and let $\{\lambda_n\}$ be a real sequence satisfying*

$$0 \le \lambda_n \le 1, \quad \sum_{n=0}^{\infty} \lambda_n = \infty.$$

Suppose that there exists a strictly increasing function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$. If there exists a positive integer n_0 such that

$$\theta_{n+1}^2 \le \theta_n^2 - \lambda_n \phi(\theta_{n+1}) + \sigma_n + \gamma_n$$

for all $n \ge n_0$, with $\sigma_n \ge 0$, $\forall n \in \mathbb{N}$, $\sigma_n = 0(\lambda_n)$ and $\sum_{n=0}^{\infty} \gamma_n < \infty$, then $\lim_{n\to\infty} \theta_n = 0$.

Now we prove our main results.

Theorem 2.3 *Let K be a nonempty convex subset of an arbitrary Banach space E, and let $T, S : K \to K$ be two uniformly continuous with $F(T) \cap F(S) \neq \emptyset$ and ϕ-hemicontractive mappings. Suppose that $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ are bounded sequences in K and $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, $\{c_n\}_{n=1}^{\infty}$, $\{a'_n\}_{n=1}^{\infty}$, $\{b'_n\}_{n=1}^{\infty}$ and $\{c'_n\}_{n=1}^{\infty}$ are sequences in $[0,1]$ satisfying conditions*

(i) $a_n + b_n + c_n = a'_n + b'_n + c'_n = 1$,
(ii) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = \lim_{n\to\infty} b'_n = 0$,
(iii) $\sum_{n=1}^{\infty} c'_n < \infty$, *and*
(iv) $\sum_{n=1}^{\infty} b'_n = \infty$.

For any $x_0 \in K$, define the sequence $\{x_n\}_{n=1}^{\infty}$ inductively as follows:

$$\begin{aligned} y_n &= a_n x_{n-1} + b_n S x_n + c_n u_n, \\ x_n &= a'_n x_{n-1} + b'_n T y_n + c'_n v_n, \quad n \ge 1. \end{aligned} \tag{2.1}$$

Then the following conditions are equivalent:

(a) $\{x_n\}_{n=1}^{\infty}$ *converges strongly to the common fixed point* q *of* T *and* S,

(b) $\lim_{n\to\infty} Ty_n = q$,

(c) $\{Ty_n\}_{n=1}^{\infty}$ *is bounded.*

Proof Since T and S are ϕ-hemicontractive, then the common fixed point of $F(T)\cap F(S)$ is unique. Suppose that p and q are all common fixed points of T and S, then

$$\|p-q\|^2 = \langle p-q, j(p-q)\rangle = \langle Tp-Tq, j(p-q)\rangle \le \|p-q\|^2 - \phi\big(\|p-q\|\big) < \|p-q\|^2,$$

which is a contradiction. So, we denote the unique fixed point q.

Suppose that $\lim_{n\to\infty} x_n = q$. Then (ii) and the uniform continuity of T and S yield that

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} [a_n x_{n-1} + b_n S x_n + c_n u_n] = q,$$

which implies that $\lim_{n\to\infty} Ty_n = q$. Therefore $\{Ty_n\}_{n=1}^{\infty}$ is bounded.

Put

$$M_1 = \|x_0 - q\| + \sup_{n\ge1} \|Ty_n - q\| + \sup_{n\ge1}\|u_n - q\| + \sup_{n\ge1}\|v_n - q\|.$$

Obviously, $M_1 < \infty$. It is clear that $\|x_0 - p\| \le M_1$. Let $\|x_{n-1} - p\| \le M_1$. Next we prove that $\|x_n - p\| \le M_1$.

Consider

$$\begin{aligned}
\|x_n - q\| &= \big\|a'_n x_{n-1} + b'_n Ty_n + c'_n v_n - q\big\| \\
&= \big\|a'_n(x_{n-1}-q) + b'_n(Ty_n - q) + c'_n(v_n - q)\big\| \\
&\le \big(1-b'_n\big)\|x_{n-1}-q\| + b'_n\|Ty_n - q\| + c'_n\|v_n - q\| \\
&\le \big(1-b'_n\big)M_1 + b'_n\|Ty_n - q\| + c'_n\|v_n - q\| \\
&= \big(1-b'_n\big)\Big[\|x_0 - q\| + \sup_{n\ge1}\|Ty_n - q\| + \sup_{n\ge1}\|u_n - q\| + \sup_{n\ge1}\|v_n - q\|\Big] \\
&\quad + b'_n\|Ty_n - q\| + c'_n\|v_n - q\| \\
&\le \|x_0 - q\| + \Big(\big(1-b'_n\big)\sup_{n\ge1}\|Ty_n - q\| + b'_n\|Ty_n - q\|\Big) \\
&\quad + \sup_{n\ge1}\|u_n - q\| + \Big(\big(1-b'_n\big)\sup_{n\ge1}\|v_n - q\| + b'_n\|v_n - q\|\Big) \\
&\le \|x_0 - q\| + \Big(\big(1-b'_n\big)\sup_{n\ge1}\|Ty_n - q\| + b'_n\sup_{n\ge1}\|Ty_n - q\|\Big) \\
&\quad + \sup_{n\ge1}\|u_n - q\| + \Big(\big(1-b'_n\big)\sup_{n\ge1}\|v_n - q\| + b'_n\sup_{n\ge1}\|v_n - q\|\Big) \\
&= \|x_0 - q\| + \sup_{n\ge1}\|Ty_n - q\| + \sup_{n\ge1}\|u_n - q\| + \sup_{n\ge1}\|v_n - q\| \\
&= M_1.
\end{aligned}$$

So, from the above discussion, we can conclude that the sequence $\{x_n - q\}_{n\ge0}$ is bounded. Since S is uniformly continuous, so $\{\|Sx_n - q\|\}_{n=1}^{\infty}$ is also bounded. Thus there is a constant

$M_2 > 0$ satisfying

$$M_2 = \sup_{n\geq 1}\|x_n - q\| + \sup_{n\geq 1}\|Sx_n - q\| + \sup_{n\geq 1}\|Ty_n - q\| + \sup_{n\geq 1}\|u_n - q\| + \sup_{n\geq 1}\|v_n - q\|. \tag{2.2}$$

Denote $M = M_1 + M_2$. Obviously, $M < \infty$. Let $w_n = \|Ty_n - Tx_n\|$ for each $n \geq 1$. The uniform continuity of T ensures that

$$\lim_{n\to\infty} w_n = 0. \tag{2.3}$$

Because

$$\begin{aligned}
\|y_n - x_n\| &= \left\|b_n(Sx_n - x_{n-1}) + b'_n(x_{n-1} - Ty_n) + c_n(u_n - x_{n-1}) - c'_n(v_n - x_{n-1})\right\| \\
&\leq b_n\|Sx_n - x_{n-1}\| + b'_n\|x_{n-1} - Ty_n\| + c_n\|u_n - x_{n-1}\| + c'_n\|v_n - x_{n-1}\| \\
&\leq 2M_2\left(b_n + c_n + b'_n + c'_n\right) \\
&\to 0
\end{aligned}$$

as $n \to \infty$.

By virtue of Lemma 2.1 and (2.1), we infer that

$$\begin{aligned}
\|x_n - q\|^2 &= \left\|a'_n x_{n-1} + b'_n Ty_n + c'_n v_n - q\right\|^2 \\
&= \left\|a'_n(x_{n-1} - q) + b'_n(Ty_n - q) + c'_n(v_n - q)\right\|^2 \\
&\leq \left(1 - b'_n\right)^2\|x_{n-1} - q\|^2 + 2b'_n\langle Ty_n - q, j(x_n - q)\rangle \\
&\quad + 2c'_n\langle v_n - q, j(x_n - q)\rangle \\
&\leq \left(1 - b'_n\right)^2\|x_{n-1} - q\|^2 + 2b'_n\langle Ty_n - Tx_n, j(x_n - q)\rangle \\
&\quad + 2b'_n\langle Tx_n - q, j(x_n - q)\rangle + 2c'_n\|v_n - q\|\|x_n - q\| \\
&\leq \left(1 - b'_n\right)^2\|x_{n-1} - q\|^2 + 2b'_n\|Ty_n - Tx_n\|\|x_n - q\| \\
&\quad + 2b'_n\|x_n - q\|^2 - 2b'_n\phi\left(\|x_n - q\|\right) + 2M^2c'_n \\
&\leq \left(1 - b'_n\right)^2\|x_{n-1} - q\|^2 + 2Mb'_n w_n + 2b'_n\|x_n - q\|^2 \\
&\quad - 2b'_n\phi\left(\|x_n - q\|\right) + 2M^2c'_n.
\end{aligned} \tag{2.4}$$

Consider

$$\begin{aligned}
\|x_n - q\|^2 &= \left\|a'_n x_{n-1} + b'_n Ty_n + c'_n v_n - q\right\|^2 \\
&= \left\|a'_n(x_{n-1} - q) + b'_n(Ty_n - q) + c'_n(v_n - q)\right\|^2 \\
&\leq a'_n\|x_{n-1} - q\|^2 + b'_n\|Ty_n - q\|^2 + c'_n\|v_n - q\|^2 \\
&\leq \|x_{n-1} - q\|^2 + M^2\left(b'_n + c'_n\right),
\end{aligned} \tag{2.5}$$

where the first inequality holds by the convexity of $\|\cdot\|^2$.

Substituting (2.5) in (2.4), we get

$$\begin{aligned}
&\|x_n - q\|^2 \\
&\quad \le \big[(1-b'_n)^2 + 2b'_n\big]\|x_{n-1} - q\|^2 + 2Mb'_n\big(w_n + M(b'_n + c'_n)\big) + 2M^2c'_n \\
&\qquad - 2b'_n\phi\big(\|x_n - q\|\big) \\
&\quad = \big(1 + b'^2_n\big)\|x_{n-1} - q\|^2 + 2Mb'_n\big(w_n + M(b'_n + c'_n)\big) + 2M^2c'_n \\
&\qquad - 2b'_n\phi\big(\|x_n - q\|\big) \\
&\quad \le \|x_{n-1} - q\|^2 + Mb'_n\big(3Mb'_n + 2w_n + 2Mb'_n\big) + 2M^2c'_n \\
&\qquad - 2b'_n\phi\big(\|x_n - q\|\big) \\
&\quad = \|x_{n-1} - q\|^2 + b'_nl_n + 2M^2c'_n - 2b'_n\phi\big(\|x_n - q\|\big),
\end{aligned} \tag{2.6}$$

where

$$l_n = M\big(3Mb'_n + 2w_n + 2Mb'_n\big) \to 0 \tag{2.7}$$

as $n \to \infty$.

Denote

$$\begin{aligned}
&\theta_n = \|x_n - q\|, \\
&\lambda_n = 2b'_n, \\
&\sigma_n = b'_nl_n, \\
&\gamma_n = 2M^2c'_n.
\end{aligned}$$

Condition (i) assures the existence of a rank $n_0 \in \mathbb{N}$ such that $\lambda_n = 2b'_n \le 1$ for all $n \ge n_0$. Now, with the help of conditions (ii), (iii), (2.3), (2.7) and Lemma 2.2, we obtain from (2.6) that

$$\lim_{n\to\infty} \|x_n - q\| = 0,$$

completing the proof. □

Using the method of proof in Theorem 2.3, we have the following result.

Corollary 2.4 *Let* E, K, T, S, $\{u_n\}_{n=1}^{\infty}$, $\{v_n\}_{n=1}^{\infty}$, $\{x_n\}_{n=1}^{\infty}$ *and* $\{y_n\}_{n=1}^{\infty}$ *be as in Theorem* 2.3. *Suppose that* $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, $\{c_n\}_{n=1}^{\infty}$, $\{a'_n\}_{n=1}^{\infty}$, $\{b'_n\}_{n=1}^{\infty}$ *and* $\{c'_n\}_{n=1}^{\infty}$ *are sequences in* $[0,1]$ *satisfying conditions* (i), (ii), (iv) *and* $c'_n = o(b'_n)$. *Then the conclusions of Theorem* 2.3 *hold*.

Proof From the condition $c'_n = o(b'_n)$, set $c'_n = b'_nt_n$, where $\lim_{n\to\infty} t_n = 0$. Substituting (2.5) in (2.4), we get

$$\begin{aligned}
&\|x_n - q\|^2 \\
&\quad \le \big[(1-b'_n)^2 + 2b'_n\big]\|x_{n-1} - q\|^2 + 2Mb'_n\big(w_n + M(b'_n + c'_n + t_n)\big)
\end{aligned}$$

$$\begin{aligned}
&\quad - 2b'_n\phi(\|x_n - q\|)\|x_n - q\| \\
&= (1 + b'^2_n)\|x_{n-1} - q\|^2 + 2Mb'_n(w_n + M(b'_n + c'_n + t_n)) \\
&\quad - 2b'_n\phi(\|x_n - q\|) \\
&\le \|x_{n-1} - q\|^2 + Mb'_n(2w_n + M(3b'_n + 2c'_n + 2t_n)) \\
&\quad - 2b'_n\phi(\|x_n - q\|) \\
&= \|x_{n-1} - q\|^2 - 2b'_n\phi(\|x_n - q\|)\|x_n - q\| + b'_n l'_n,
\end{aligned} \tag{2.8}$$

where

$$l'_n = M(2w_n + M(3b'_n + 2c'_n + 2t_n)) \to 0 \tag{2.9}$$

as $n \to \infty$.

It follows from Lemma 2.2 that $\lim_{n\to\infty} \|x_n - q\| = 0$. □

Corollary 2.5 *Let K be a nonempty convex subset of an arbitrary Banach space E, and let $T : K \to K$ be a uniformly continuous and ϕ-hemicontractive mapping. Suppose that $\{u_n\}_{n=1}^\infty$ and $\{v_n\}_{n=1}^\infty$ are bounded sequences in K and $\{a_n\}_{n=1}^\infty$, $\{b_n\}_{n=1}^\infty$, $\{c_n\}_{n=1}^\infty$, $\{a'_n\}_{n=1}^\infty$, $\{b'_n\}_{n=1}^\infty$ and $\{c'_n\}_{n=1}^\infty$ are sequences in $[0,1]$ satisfying conditions*

(i) $a_n + b_n + c_n = a'_n + b'_n + c'_n = 1$,

(ii) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = \lim_{n\to\infty} b'_n = 0$,

(iii) $c'_n = o(b'_n)$, *and*

(iv) $\sum_{n=1}^\infty b'_n = \infty$.

For any $x_0 \in K$, define the sequence $\{x_n\}_{n=1}^\infty$ inductively as follows:

$$\begin{aligned}
y_n &= a_n x_{n-1} + b_n T x_n + c_n u_n, \\
x_n &= a'_n x_{n-1} + b'_n T y_n + c'_n v_n, \quad n \ge 1.
\end{aligned}$$

Then the following conditions are equivalent:

(a) *$\{x_n\}_{n=1}^\infty$ converges strongly to the unique fixed point q of T,*

(b) $\lim_{n\to\infty} Ty_n = q$,

(c) *$\{Ty_n\}_{n=1}^\infty$ is bounded.*

Corollary 2.6 *Let E, K, T, $\{u_n\}_{n=1}^\infty$, $\{v_n\}_{n=1}^\infty$, $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ be as in Corollary 2.5. Suppose that $\{a_n\}_{n=1}^\infty$, $\{b_n\}_{n=1}^\infty$, $\{c_n\}_{n=1}^\infty$, $\{a'_n\}_{n=1}^\infty$, $\{b'_n\}_{n=1}^\infty$ and $\{c'_n\}_{n=1}^\infty$ are sequences in $[0,1]$ satisfying conditions* (i), (ii), (iv) *and $\sum_{n=1}^\infty c'_n < \infty$. Then the conclusions of Corollary 2.5 hold.*

Corollary 2.7 *Let K be a nonempty convex subset of an arbitrary Banach space E, and let $T, S : K \to K$ be two uniformly continuous and ϕ-hemicontractive mappings. Suppose that $\{\alpha_n\}_{n=1}^\infty$, $\{\beta_n\}_{n=1}^\infty$ are any sequences in $[0,1]$ satisfying*

(i) $\lim_{n\to\infty} \beta_n = 0 = \lim_{n\to\infty} \alpha_n$,

(ii) $\sum_{n=1}^\infty \alpha_n = \infty$.

For any $x_0 \in K$, define the sequence $\{x_n\}_{n=1}^\infty$ inductively as follows:

$$\begin{aligned}
y_n &= (1 - \beta_n)x_{n-1} + \beta_n S x_n, \\
x_n &= (1 - \alpha_n)x_{n-1} + \alpha_n T y_n, \quad n \ge 1.
\end{aligned}$$

Then the following conditions are equivalent:

(a) $\{x_n\}_{n=1}^{\infty}$ *converges strongly to the common fixed point* q *of* T *and* S,

(b) $\lim_{n\to\infty} Ty_n = q$,

(c) $\{Ty_n\}_{n=1}^{\infty}$ *is bounded*.

Corollary 2.8 *Let* K *be a nonempty convex subset of an arbitrary Banach space* E, *and let* $T : K \to K$ *be a uniformly continuous and* ϕ-*hemicontractive mapping. Suppose that* $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ *are any sequences in* $[0,1]$ *satisfying*

(i) $\lim_{n\to\infty} \beta_n = 0 = \lim_{n\to\infty} \alpha_n$,

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$.

For any $x_0 \in K$, *define the sequence* $\{x_n\}_{n=1}^{\infty}$ *inductively as follows*:

$$\begin{aligned} y_n &= (1-\beta_n)x_{n-1} + \beta_n Tx_n, \\ x_n &= (1-\alpha_n)x_{n-1} + \alpha_n Ty_n, \quad n \geq 1. \end{aligned}$$

Then the following conditions are equivalent:

(a) $\{x_n\}_{n=1}^{\infty}$ *converges strongly to the unique fixed point* q *of* T,

(b) $\lim_{n\to\infty} Ty_n = q$,

(c) $\{Ty_n\}_{n=1}^{\infty}$ *is bounded*.

Corollary 2.9 *Let* K *be a nonempty convex subset of an arbitrary Banach space* E, *and let* $T : K \to K$ *be a uniformly continuous and* ϕ-*hemicontractive mapping. Suppose that* $\{\alpha_n\}_{n=1}^{\infty}$ *is any sequence in* $[0,1]$ *satisfying*

(i) $\lim_{n\to\infty} \alpha_n = 0$,

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$.

For any $x_0 \in K$, *define the sequence* $\{x_n\}_{n=1}^{\infty}$ *inductively as follows*:

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)Tx_n, \quad n \geq 1.$$

Then the following conditions are equivalent:

(a) $\{x_n\}_{n=1}^{\infty}$ *converges strongly to the unique fixed point* q *of* T,

(b) $\lim_{n\to\infty} Tx_n = q$,

(c) $\{Tx_n\}_{n=1}^{\infty}$ *is bounded*.

All of the above results are also valid for Lipschitz ϕ-hemicontractive mappings.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally in writing this paper, and read and approved the final manuscript.

Author details
[1]Department of Mathematics and Physics, Shijiazhuang Tiedao University, Shijiazhuang, 050043, P.R. China. [2]Hajvery University, 43-52 Industrial Area, Gulberg-III, Lahore, Pakistan.

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