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# Oscillatory behaviour of a higher-order dynamic equation

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**Abstract**

In this paper we are concerned with the oscillation of solutions of a certain more general higher-order nonlinear neutral-type functional dynamic equation with oscillating coefficients. We obtain some sufficient criteria for oscillatory behaviour of its solutions.

**MSC:** 34N05

**Keywords:** time scale; higher-order nonlinear neutral dynamic equation; oscillating coefficient

## 1 Introduction

The calculus on time scales has been introduced in order to unify the theories of continuous and discrete processes and in order to extend those theories to a more general class of the so-called dynamic equations. In recent years there has been much research activity concerning the oscillation and non-oscillation of solutions of neutral dynamic equations on time scales.

In this paper we consider the higher-order nonlinear dynamic equation

$$\left[y(t)+P(t)y\big(\tau(t)\big)\right]^{\Delta^n}+\sum_{i=1}^{m}Q_i(t)f_i\Big(y\big(\phi_i(t)\big)\Big)=0, \tag{1.1}$$

where $n \geq 2$, $P(t), Q_i(t) \in C_{rd}\,[t_0,\infty)_{\mathbb{T}}$ for $i = 1,2,\dots,m$; $P(t)$ is an oscillating function ($P(t) : \mathbb{T} \to \mathbb{R}$), $Q_i(t)$ are positive real-valued functions for $i = 1,2,\dots,m$; $\phi_i(t) \in C_{rd}\,[t_0,\infty)_{\mathbb{T}}$, $\phi_i^{\Delta}(t) > 0$, the variable delays $\tau, \phi_i : [t_0,\infty)_{\mathbb{T}} \to \mathbb{T}$ with $\tau(t), \phi_i(t) < t$ for all $t \in [t_0,\infty)_{\mathbb{T}}$, $\phi_i(t) \to \infty$ as $t \to \infty$ for $i = 1,2,\dots,m$; $\tau(t) \to \infty$ as $t \to \infty$; $f_i(u) \in C(\mathbb{R},\mathbb{R})$ are nondecreasing functions, $uf_i(u) > 0$ for $u \neq 0$ and $i = 1,2,\dots,m$.

The purpose of the paper is to study oscillatory behaviour of solutions of equation (1.1). For the sake of convenience, the function $z(t)$ is defined by

$$z(t)=y(t)+P(t)y\big(\tau(t)\big). \tag{1.2}$$

## 2 Basic definitions and some auxiliary lemmas

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. For $t \in \mathbb{T}$, we define the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$$

while the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\}.$$

If $\sigma(t) > t$, we say that $t$ is *right-scattered*, while if $\rho(t) < t$, we say that $t$ is *left-scattered*. Also, if $\sigma(t) = t$, then $t$ is called *right-dense*, and if $\rho(t) = t$, then $t$ is called *left-dense*. The *graininess function* $\mu : \mathbb{T} \to [0, \infty)$ is defined by

$$\mu(t) := \sigma(t) - t.$$

We introduce the set $\mathbb{T}^\kappa$ which is derived from the time scale $\mathbb{T}$ as follows. If $\mathbb{T}$ has left-scattered maximum $m$, then $\mathbb{T}^\kappa = \mathbb{T} - \{m\}$, otherwise $\mathbb{T}^\kappa = \mathbb{T}$.

**Definition 1** [1] The function $f : \mathbb{T} \to \mathbb{R}$ is called *rd-continuous* provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$.

**Theorem 1** [1] *Assume that* $\nu : \mathbb{T} \to \mathbb{R}$ *is strictly increasing and* $\widetilde{\mathbb{T}} := \nu(\mathbb{T})$ *is a time scale. Let* $w : \widetilde{\mathbb{T}} \to \mathbb{R}$. *If* $\nu^\Delta(t)$ *and* $w^{\widetilde{\Delta}}(\nu(t))$ *exist for* $t \in \mathbb{T}^\kappa$, *then*

$$(w \circ \nu)^\Delta = \left(w^{\widetilde{\Delta}} \circ \nu\right)\nu^\Delta,$$

*where we denote the derivative on* $\widetilde{\mathbb{T}}$ *by* $\widetilde{\Delta}$.

**Definition 2** [1] Let $f : \mathbb{T} \to \mathbb{R}$ be a function. If there exists a function $F : \mathbb{T} \to \mathbb{R}$ such that $F^\Delta(t) = f(t)$ for all $t \in \mathbb{T}^\kappa$, then $F$ is said to be an antiderivative of $f$. We define the Cauchy integral by

$$\int_a^b f(\tau)\Delta(\tau) = F(b) - F(a) \quad \text{for } a, b \in \mathbb{T}.$$

**Theorem 2** [2] *Let* $u$ *and* $v$ *be continuous functions on* $[a, b]$ *that are* $\Delta$*-differentiable on* $[a, b)$. *If* $u^\Delta$ *and* $v^\Delta$ *are integrable from* $a$ *to* $b$, *then*

$$\int_a^b u^\Delta(t)v(t)\Delta(t) + \int_a^b u^\sigma(t)v^\Delta(t)\Delta(t) = u(b)v(b) - u(a)v(a).$$

Let $\widetilde{\mathbb{T}} = \mathbb{T} \cup \{\sup \mathbb{T}\} \cup \{\inf \mathbb{T}\}$. If $\infty \in \widetilde{\mathbb{T}}$, we call $\infty$ left-dense, and $-\infty$ is called right-dense provided $-\infty \in \widetilde{\mathbb{T}}$. For any left-dense $t_0 \in \widetilde{\mathbb{T}}$ and any $\varepsilon > 0$, the set

$$L_\varepsilon(t_0) = \{t \in \mathbb{T} : 0 < t_0 - t < \varepsilon\}$$

is nonempty, and so is $L_\varepsilon(\infty) = \{t \in \mathbb{T} : t > \frac{1}{\varepsilon}\}$ if $\infty \in \widetilde{\mathbb{T}}$.

**Lemma 1** [3] *Let* $n \in \mathbb{N}$ *and* $f$ *be* $n$*-times differentiable on* $\mathbb{T}$. *Assume* $\infty \in \widetilde{\mathbb{T}}$. *Suppose there exists* $\varepsilon > 0$ *such that*

$$f(t) > 0, \quad \operatorname{sgn}\left(f^{\Delta^n}(t)\right) \equiv s \in \{-1, +1\} \quad \textit{for all } t \in L_\varepsilon(\infty)$$

*and $f^{\Delta^n}(t) \neq 0$ on $L_\delta(\infty)$ for any $\delta > 0$. Then there exists $p \in [0,n] \cap \mathbb{N}_0$ such that $n+p$ is even for $s = 1$ and odd for $s = -1$ with*

$$\begin{cases} (-1)^{p+j} f^{\Delta^j}(t) > 0 \quad \text{for all } t \in L_\varepsilon(\infty), j \in [p, n-1] \cap \mathbb{N}_0, \\ f^{\Delta^j}(t) > 0 \quad \text{for all } t \in L_{\delta_j}(\infty) \ (\text{with } \delta_j \in (0,\varepsilon)), j \in [1, p-1] \cap \mathbb{N}_0. \end{cases}$$

**Lemma 2** [3] *Let $f$ be $n$-times differentiable on $\mathbb{T}^{\kappa^n}$, $t \in \mathbb{T}$, and $\alpha \in \mathbb{T}^{\kappa^n}$. Then with the functions $h_k$ defined as $h_n(t,s) = (-1)^n g_n(s,t)$,*

$$h_0(r,s) \equiv 1 \quad \text{and} \quad h_{k+1}(r,s) = \int_s^r h_k(\tau, s)\Delta s \quad \text{for } k \in \mathbb{N}_0,$$

*we have*

$$f(t) = \sum_{k=0}^{n-1} h_k(t,\alpha) f^{\Delta^k}(\alpha) + \int_\alpha^{\rho^{n-1}(t)} h_{n-1}\bigl(t, \sigma(\tau)\bigr) f^{\Delta^n}(\tau)\Delta\tau.$$

**Lemma 3** [3] *Let $f$ be $n$-times differentiable on $\mathbb{T}^{\kappa^n}$ and $m \in \mathbb{N}$ with $m < n$. Then we have, for all $\alpha \in \mathbb{T}^{\kappa^{n-1+m}}$ and $t \in \mathbb{T}^{\kappa^m}$,*

$$f^{\Delta^m}(t) = \sum_{k=0}^{n-m-1} h_k(t,\alpha) f^{\Delta^{k+m}}(\alpha) + \int_\alpha^{\rho^{n-m-1}(t)} h_{n-m-1}\bigl(t, \sigma(\tau)\bigr) f^{\Delta^n}(\tau)\Delta\tau.$$

**Lemma 4** [3] *Suppose $f$ is $n$-times differentiable and $g_k$, $0 \le k \le n-1$, are differentiable at $t \in \mathbb{T}^{\kappa^n}$ with*

$$g_{k+1}^\Delta(t) = g_k\bigl(\sigma(t)\bigr) \quad \text{for all } 0 \le k \le n-2.$$

*Then we have*

$$\left[\sum_{k=0}^{n-1} (-1)^k f^{\Delta^k} g_k\right]^\Delta = f g_0^\Delta + (-1)^{n-1} f^{\Delta^n} g_{n-1}^\sigma.$$

## 3 Main results

**Lemma 5** *Let $f$ be $n$-times differentiable on $\mathbb{T}^{\kappa^n}$. If $f^\Delta > 0$, then for every $\lambda$, $0 < \lambda < 1$, we have*

$$f(t) \ge \lambda (-1)^{n-1} g_{n-1}\bigl(\sigma(T^*), t\bigr) f^{\Delta^{n-1}}(t). \tag{3.1}$$

*Proof* Let $p$, $0 \le p \le n-1$, be the integer assigned to the function $f$ as in Lemma 1. Because of $f^\Delta > 0$, we always have $p > 0$. Furthermore, let $T^* \ge T$ be assigned to $f$ by Lemma 1. Then, by using the Taylor formula on time scales, for every $\rho^{n-1}(t) \ge T^*$, we obtain

$$f(t) \ge \int_{T^*}^{\rho^{n-1}(t)} (-1)^{n-1} g_{n-1}\bigl(\sigma(\tau), t\bigr) f^{\Delta^n}(\tau)\Delta\tau. \tag{3.2}$$

By using Theorem 2 and (3.2), we have

$$f(t) \ge (-1)^{n-1} g_{n-1}\bigl(\sigma(t), t\bigr) f^{\Delta^{n-1}}(t) - \int_{T^*}^{\rho^{n-1}(t)} (-1)^{n-1} g_{n-1}\bigl(\sigma(\tau), t\bigr) f^{\Delta^{n-1}}(\tau)\Delta\tau.$$

Since $f$ is $n$-times differentiable on $\mathbb{T}^{\kappa^n}$ and $m \in \mathbb{N}$ with $m < n$, we have with $n$ and $f$ substituted by $n - m$ and $f^{\Delta^m}$, respectively

$$f^{\Delta^m}(t) \geq \int_{T^*}^{\rho^{n-m-1}(t)} (-1)^{n-m-1} g_{n-m-1}\bigl(\sigma(\tau), t\bigr) f^{\Delta^n}(\tau)\Delta\tau.$$

Also, for every $\rho^{n-1}(t)$, $s$ with $\rho^{n-1}(t) \geq T^*$ and $T^* \leq s \leq t$, we have

$$f^{\Delta^m}(s) \geq (-1)^{n-m-1} g_{n-m-1}\bigl(\sigma\bigl(T^*\bigr), t\bigr) f^{\Delta^n}(t).$$

This is obvious for $m = n - 1$ and, when $m < n - 1$, it can be derived by applying the Taylor formula. Thus, for all $t \geq T^*$, we get

$$f(t) \geq (-1)^{n-1} g_{n-1}\bigl(\sigma\bigl(T^*\bigr), t\bigr) f^{\Delta^{n-1}}(t)$$

and therefore the proof of the lemma can be immediately completed. □

The result of Lemma 5 is an extension of studies in [4] and [5]. In order that the reader sees how the results in [4] (1.8.14) and [5] (Lemma 2) follow from (3.1), it is at this point only necessary to know that in the case $\mathbb{T} = \mathbb{Z}$, we have $\rho(t) = t - 1$, $\sigma(t) = t + 1$ and

$$g_{n-1}\bigl(\sigma\bigl(T^*\bigr), t\bigr) = \frac{(t - T^* - 1)^{(n-1)}}{(n-1)!},$$

then we get the inequality in [4]

$$u(t) \geq \frac{1}{(n-1)!}(n - n_1)^{(n-1)} \Delta^{n-1} u\bigl(2^{n-m-1} n\bigr);$$

and in the case $\mathbb{T} = \mathbb{R}$, we have $\rho(t) = \sigma(t) = t$ and

$$g_{n-1}\bigl(\sigma\bigl(T^*\bigr), t\bigr) = \frac{(t - T^*)^{(n-1)}}{(n-1)!},$$

then we get the inequality in [5]

$$u(t) \geq \frac{\vartheta}{(n-1)!}(t)^{n-1} u^{n-1}(t).$$

For the cases $\mathbb{T} = \mathbb{Z}$ and $\mathbb{T} = \mathbb{R}$, some sufficient criterias for oscillatory behaviour of the solutions of the equation (1.1) were obtained by Bolat and Akın in [6] and [7], respectively. Furthermore, there might be other time scales that we cannot appreciate at this time due to our current lack of 'real-world' examples.

**Theorem 3** *Assume that n is odd and*

(C1) $\lim_{t\to\infty} P(t) = 0$,
(C2) $\int_{t_0}^{\infty} s^{n-1} \sum_{i=1}^{m} Q_i(s)\Delta s = \infty$.

*Then every bounded solution of equation* (1.1) *is either oscillatory or tends to zero as* $t \to \infty$.

*Proof* Assume that equation (1.1) has a bounded non-oscillatory solution $y(t)$. Without loss of generality, assume that $y(t)$ is eventually positive (the proof is similar when $y(t)$ is eventually negative). That is, $y(t) > 0$, $y(\tau(t)) > 0$ and $y(\phi_i(t)) > 0$ for $t \geq t_1 \geq t_0$ and $i = 1, 2, \ldots, m$. Assume further that $y(t)$ does not tend to zero as $t \to \infty$. By (1.1), (1.2), we have for $t \geq t_1$

$$z^{\Delta^n}(t) = -\sum_{i=1}^{m} Q_i(t) f_i\bigl(y\bigl(\phi_i(t)\bigr)\bigr) < 0. \tag{3.3}$$

It follows that $z^{\Delta^j}(t)$ ($j \in [0, n-1] \cap \mathbb{N}_0$) is strictly monotone and eventually of constant sign. Since $P(t)$ is an oscillatory function, there exists a $t_2 \geq t_1$ such that if $t \geq t_2$, then $z(t) > 0$. Since $y(t)$ is bounded, by virtue of (C1) and (1.2), there is a $t_3 \geq t_2$ such that $z(t)$ is also bounded for $t \geq t_3$. Because $n$ is odd and $z(t)$ is bounded, by Lemma 1, when $p = 0$ (otherwise $z(t)$ is not bounded), there exists $t_4 \geq t_3$ such that for $t \geq t_4$ we have $(-1)^j z^{\Delta^j}(t) > 0$, $j \in [0, n-1] \cap \mathbb{N}_0$.

In particular, since $z^{\Delta}(t) < 0$ for $t \geq t_4$, $z(t)$ is decreasing. Since $z(t)$ is bounded, we write $\lim_{t\to\infty} z(t) = L$ ($-\infty < L < \infty$). Assume that $0 \leq L < \infty$. Let $L > 0$. Then there exists a constant $c > 0$ and a $t_5 \geq t_4$ such that $z(t) > c > 0$ for $t \geq t_5$. Since $y(t)$ is bounded, $\lim_{t\to\infty} P(t) y(\tau(t)) = 0$ by (C1). Therefore, there exists a constant $c_1 > 0$ and a $t_6 \geq t_5$ such that $y(t) = z(t) - P(t) y(\tau(t)) > c_1 > 0$ for $t \geq t_6$. So that we can find a $t_7$ with $t_7 \geq t_6$ such that $y(\phi_i(t)) > c_1 > 0$ for $t \geq t_7$. From (3.3) we have

$$z^{\Delta^n}(t) = -\sum_{i=1}^{m} Q_i(t) f_i(c_1) < 0 \tag{3.4}$$

for $t \geq t_7$. If we multiply (3.4) by $t^{n-1}$ and integrate it from $t_7$ to $t$, we obtain

$$F(t) - F(t_7) \leq -f(c_1) \int_{t_7}^{t} \sum_{i=1}^{m} Q_i(s) s^{n-1} \Delta s, \tag{3.5}$$

where

$$F(t) = \sum_{i=1}^{n-1} (-1)^{i+1} \bigl(t^{n-1}\bigr)^{\Delta^i} z^{\Delta^{n-i}} \bigl(\sigma^i(t)\bigr)$$

and

$$\sigma^i(t) = \sigma\bigl(\sigma^{i-1}(t)\bigr).$$

Since $(-1)^k z^{\Delta^k}(t) > 0$ for $k = 0, 1, 2, \ldots, n-1$ and $t \geq t_4$, we have $F(t) > 0$ for $t \geq t_7$. From (3.5) we have

$$-F(t_7) \leq -f(c_1) \int_{t_7}^{t} \sum_{i=1}^{m} Q_i(s) s^{n-1} \Delta s.$$

By (C2) we obtain

$$-F(t_7) \leq -f(c_1) \int_{t_7}^{t} \sum_{i=1}^{m} Q_i(s) s^{n-1} \Delta s = -\infty$$

as $t \to \infty$. This is a contradiction. So, $L > 0$ is impossible. Therefore, $L = 0$ is the only possible case. That is, $\lim_{t\to\infty} z(t) = 0$. Since $y(t)$ is bounded, by (C1) we obtain

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} z(t) - \lim_{t\to\infty} P(t)y(t) = 0$$

from (1.2).

Now let us consider the case of $y(t) < 0$ for $t \geq t_1$. By (1.1) and (1.2),

$$z^{\Delta^n}(t) = -\sum_{i=1}^{m} Q_i(t) f_i\bigl(y\bigl(\phi_i(t)\bigr)\bigr) > 0$$

for $t \geq t_1$. That is, $z^{\Delta^n} > 0$. It follows that $z^{\Delta^j}(t)$ ($j \in [0, n-1] \cap \mathbb{N}_0$) is strictly monotone and eventually of constant sign. Since $P(t)$ is an oscillatory function, there exists a $t_2 \geq t_1$ such that if $t \geq t_2$, then $z(t) < 0$. Since $y(t)$ is bounded, by (C1) and (1.2), there is a $t_3 \geq t_2$ such that $z(t)$ is also bounded for $t \geq t_3$. Assume that $x(t) = -z(t)$. Then $x^{\Delta^n}(t) = -z^{\Delta^n}(t)$. Therefore, $x(t) > 0$ and $x^{\Delta^n}(t) < 0$ for $t \geq t_3$. Hence, we observe that $x(t)$ is bounded. Since $n$ is odd, by Lemma 1, there exists a $t_4 \geq t_3$ and $p = 0$ (otherwise $x(t)$ is not bounded) such that $(-1)^j x^{\Delta^j}(t) > 0$, $j \in [0, n-1] \cap \mathbb{N}_0$ and $t \geq t_4$. That is, $(-1)^j z^{\Delta^j}(t) < 0$, $j \in [0, n-1] \cap \mathbb{N}_0$ and $t \geq t_4$. In particular, for $t \geq t_4$ we have $z^{\Delta}(t) > 0$. Therefore, $z(t)$ is increasing. So, we can assume that $\lim_{t\to\infty} z(t) = L$ ($-\infty < L \leq 0$). As in the proof of $y(t) > 0$, we may prove that $L = 0$. As for the rest, it is similar to the case of $y(t) > 0$. That is, $\lim_{t\to\infty} y(t) = 0$. This contradicts our assumption. Hence the proof is completed. □

**Theorem 4** *Assume that n is even and* (C1) *holds. If the following condition is satisfied*:

(C3) *There is a function $\varphi(t)$ such that $\varphi(t) \in C^1_{rd}[t_0, \infty)_{\mathbb{T}}$. Moreover,*

$$\lim_{t\to\infty} \sup \int_{t_0}^{t} \varphi(s) \sum_{i=1}^{m} Q_i(s) \Delta s = \infty$$

*and*

$$\lim_{t\to\infty} \sup \int_{t_{10}}^{t} \frac{[\varphi^{\Delta}(s)]^2}{\varphi(s) g^{\sigma}_{n-2}(\sigma(\phi_i(s)), \phi_i(s))} \Delta s < \infty$$

*for $\varphi(t)$ and $i = 1, 2, \ldots, m$. Then every bounded solution of equation* (1.1) *is oscillatory.*

*Proof* Assume that equation (1.1) has a bounded non-oscillatory solution $y(t)$. Without loss of generality, assume that $y(t)$ is eventually positive (the proof is similar when $y(t)$ is eventually negative). That is, $y(t) > 0$, $y(\tau(t)) > 0$ and $y(\phi_i(t)) > 0$ for $t \geq t_1 \geq t_0$. By (1.1) and (1.2), we have (3.3) for $t \geq t_1$. Then $z^{\Delta^n}(t) < 0$. It follows that $z^{\Delta^j}(t)$ ($j \in [0, n-1] \cap \mathbb{N}_0$) is strictly monotone and eventually of constant sign. Since $P(t)$ is an oscillatory function, there exists a $t_2 \geq t_1$ such that for $t \geq t_2$, we have $z(t) > 0$. Since $y(t)$ is bounded, by (C1) and (1.2), there is a $t_3 \geq t_2$, such that $z(t)$ is also bounded for $t \geq t_3$. Because $n$ is even, by Lemma 1 when $p = 1$ (otherwise $z(t)$ is not bounded), there exists $t_4 \geq t_3$ such that for $t \geq t_4$ we have

$$(-1)^{j+1} z^{\Delta^j}(t) > 0, \quad j \in [0, n-1] \cap \mathbb{N}_0. \tag{3.6}$$

In particular, since $z^{\Delta}(t) > 0$ for $t \geq t_4$, $z(t)$ is increasing. Since $y(t)$ is bounded,

$$\lim_{t\to\infty} P(t) y\bigl(\tau(t)\bigr) = 0$$

by (C1). Let $\delta > 1$; *i.e.*, there exists a $t_5 \geq t_4$ such that by (1.2)

$$y(t) = z(t) - P(t) y\bigl(\tau(t)\bigr) > \frac{1}{\delta} z(t) > 0$$

for $t \geq t_5$. We may find a $t_6 \geq t_5$ such that for $t \geq t_6$ and $i = 1, 2, \ldots, m$,

$$y\bigl(\phi_i(t)\bigr) > \frac{1}{\delta} z\bigl(\phi_i(t)\bigr) > 0. \tag{3.7}$$

From (3.3), (3.7) and the properties of $f$, we have

$$\begin{aligned} z^{\Delta^n}(t) &\leq -\sum_{i=1}^{m} Q_i(t) f_i\left(\frac{1}{\delta} z\bigl(\phi_i(t)\bigr)\right) \\ &= -\sum_{i=1}^{m} Q_i(t) \frac{f_i(\frac{1}{\delta} z(\phi_i(t)))}{z(\phi_i(t))} z\bigl(\phi_i(t)\bigr) \end{aligned} \tag{3.8}$$

for $t \geq t_6$. Since $z(t) > 0$ is bounded and increasing, $\lim_{t\to\infty} z(t) = L$ $(0 < L < \infty)$. By the continuity of $f$, we have

$$\lim_{t\to\infty} \frac{f_i(\frac{1}{\delta} z(\phi_i(t)))}{z(\phi_i(t))} = \frac{f_i(\frac{L}{\delta})}{L} > 0.$$

Then there is a $t_7 \geq t_6$ such that for $t \geq t_7$, $i = 1, 2, \ldots, m$, we have

$$\frac{f_i(\frac{1}{\delta} z(\phi_i(t)))}{z(\phi_i(t))} \geq \frac{f_i(\frac{L}{\delta})}{2L} = \alpha > 0. \tag{3.9}$$

By (3.8), (3.9),

$$z^{\Delta^n}(t) \leq -\alpha \sum_{i=1}^{m} Q_i(t) z\bigl(\phi_i(t)\bigr) \quad \text{for } t \geq t_7. \tag{3.10}$$

Set

$$w(t) = \frac{z^{\Delta^{n-1}}(t)}{z(\frac{1}{\delta}\phi_i(t))}. \tag{3.11}$$

We know from (3.6) that there is a $t_8 \geq t_7$ such that for a sufficiently large $t \geq t_8$, $w(t) > 0$.

Therefore, $\Delta$-derivating (3.11) we obtain

$$\begin{aligned} w^{\Delta}(t) &= \frac{z^{\Delta^n}(t)}{z(\delta^{-1}\phi_i(t))} + z^{\Delta^{n-1}}(\sigma(t))\left(\frac{1}{z(\delta^{-1}\phi_i(t))}\right)^{\Delta} \\ &= \frac{z^{\Delta^n}(t)}{z(\delta^{-1}\phi_i(t))} - \frac{\delta^{-1}\phi_i^{\Delta}(t)z^{\Delta^{n-1}}(\sigma(t))z^{\Delta}(\delta^{-1}\phi_i(t))}{z(\delta^{-1}\phi_i(t))z(\delta^{-1}\phi_i(\sigma(t)))} \qquad (3.12) \\ &\leq \frac{z^{\Delta^n}(t)}{z(\delta^{-1}\phi_i(t))} - \frac{\delta^{-1}\phi_i^{\Delta}(t)z^{\Delta^{n-1}}(\sigma(t))z^{\Delta}(\delta^{-1}\phi_i(t))}{z^2(\delta^{-1}\phi_i(\sigma(t)))} \\ &= \frac{z^{\Delta^n}(t)}{z(\delta^{-1}\phi_i(t))} - \delta^{-1}w^{\sigma}(t)\frac{\phi_i^{\Delta}(t)z^{\Delta}(\delta^{-1}\phi_i(t))}{z(\delta^{-1}\phi_i(\sigma(t)))}. \qquad (3.13) \end{aligned}$$

We know from (3.6) that there is a $t \geq t_9$ such that $z^{\Delta}(t) > 0$ and $z^{\Delta^{n-1}}(t) > 0$ for an even $n$. Since $z(t) > 0$ is increasing $z(\delta^{-1}\phi_i(\sigma(t))) \geq z(\delta^{-1}\phi_i(t))$ for $i = 1, 2, \ldots, m$. Therefore, by Lemma 5,

$$z\big(\delta^{-1}\phi_i(t)\big) \geq \lambda(-1)^{n-1}g_{n-1}\big(\sigma\big(\phi_i(t)\big), \phi_i(t)\big)z^{\Delta^{n-1}}\big(\phi_i(t)\big). \qquad (3.14)$$

Then by $\Delta$-derivating (3.14) and using $g_{n-1}^{\Delta}(\sigma(t), t) = g_{n-2}^{\sigma}(\sigma(t), t)$, we get

$$\begin{aligned} \big[z\big(\delta^{-1}\phi_i(t)\big)\big]^{\Delta} &\geq \lambda(-1)^{n-2}g_{n-1}^{\Delta}\big(\sigma\big(\phi_i(t)\big), \phi_i(t)\big)z^{\Delta^{n-1}}\big(\phi_i(t)\big) \\ &\geq \lambda(-1)^{n-2}g_{n-2}^{\sigma}\big(\sigma\big(\phi_i(t)\big), \phi_i(t)\big)z^{\Delta^{n-1}}\big(\phi_i(t)\big) \end{aligned}$$

by Lemma 2

$$z^{\Delta}\big(\delta^{-1}\phi_i(t)\big)\delta^{-1}\phi_i^{\Delta}(t) \geq \lambda(-1)^{n-2}g_{n-2}^{\sigma}\big(\sigma\big(\phi_i(t)\big), \phi_i(t)\big)z^{\Delta^{n-1}}\big(\phi_i(t)\big).$$

Since $\phi_i(t) \leq t$, we obtain

$$z^{\Delta}\big(\delta^{-1}\phi_i(t)\big) \geq \frac{\delta\lambda(-1)^{n-2}g_{n-2}^{\sigma}(\sigma(\phi_i(t)), \phi_i(t))z^{\Delta^{n-1}}(t)}{\phi_i^{\Delta}(t)}. \qquad (3.15)$$

Hence by (3.10), (3.14) and (3.15), we have

$$\begin{aligned} w^{\Delta}(t) &\leq \frac{-\alpha\sum_{i=1}^{m}Q_i(t)z(\phi_i(t))}{z(\delta^{-1}\phi_i(t))} \\ &\quad - \delta^{-1}w^{\sigma}(t)\frac{\delta\lambda(-1)^{n-2}g_{n-2}^{\sigma}(\sigma(\phi_i(t)), \phi_i(t))z^{\Delta^{n-1}}(t)}{\phi_i^{\Delta}(t)}\frac{\phi_i^{\Delta}(t)}{z(\delta^{-1}\phi_i(\sigma(t)))} \\ &\leq \frac{-\alpha\sum_{i=1}^{m}Q_i(t)z(\phi_i(t))}{z(\delta^{-1}\phi_i(t))} \\ &\quad - \delta^{-1}w^{\sigma}(t)\frac{\delta\lambda(-1)^{n-2}g_{n-2}^{\sigma}(\sigma(\phi_i(t)), \phi_i(t))\phi_i^{\Delta}(t)}{\phi_i^{\Delta}(t)}\frac{z^{\Delta^{n-1}}(\sigma(t))}{z(\delta^{-1}\phi_i(\sigma(t)))} \\ &\leq -\alpha\sum_{i=1}^{m}Q_i(t) - \lambda(-1)^{n-2}g_{n-2}^{\sigma}\big(\sigma\big(\phi_i(t)\big), \phi_i(t)\big)\big(w^{\sigma}(t)\big)^2, \end{aligned}$$

and then

$$\alpha \sum_{i=1}^{m} Q_i(t) \le -w^{\Delta}(t) - \lambda(-1)^{n-2} w^2(t) g_{n-2}^{\sigma}\bigl(\sigma\bigl(\phi_i(t)\bigr), \phi_i(t)\bigr) \tag{3.16}$$

for $t \ge t_{10}$. If we multiply (3.16) by $\varphi(t)$ and integrate it from $t_{10}$ to $t$, we obtain by Theorem 2

$$\begin{aligned}
\alpha \int_{t_{10}}^{t} \varphi(s) \sum_{i=1}^{m} Q_i(s)\Delta s &\le - \int_{t_{10}}^{t} \varphi(s) w^{\Delta}(s)\Delta s \\
&\quad - \int_{t_{10}}^{t} \lambda(-1)^{n-2} \varphi(s) w^2(s) g_{n-2}^{\sigma}\bigl(\sigma\bigl(\phi_i(s)\bigr), \phi_i(s)\bigr)\Delta s \\
&\le - \left[\varphi(t)w(t) - \varphi(t_{10})w(t_{10}) - \int_{t_{10}}^{t} \varphi^{\Delta}(s) w^{\sigma}(t)\Delta s\right] \\
&\quad - \int_{t_{10}}^{t} \lambda(-1)^{n-2} \varphi(s) w^2(s) g_{n-2}^{\sigma}\bigl(\sigma\bigl(\phi_i(s)\bigr), \phi_i(s)\bigr)\Delta s \\
&\le \varphi(t_{10})w(t_{10}) + \int_{t_{10}}^{t} \varphi^{\Delta}(s) w^{\sigma}(t)\Delta s \\
&\quad - \lambda \int_{t_{10}}^{t} \varphi(s) w^2(s) g_{n-2}^{\sigma}\bigl(\sigma\bigl(\phi_i(s)\bigr), \phi_i(s)\bigr)\Delta s \\
&\le \varphi(t_{10})w(t_{10}) - \lambda \int_{t_{10}}^{t} \varphi(s) g_{n-2}^{\sigma}\bigl(\sigma\bigl(\phi_i(s)\bigr), \phi_i(s)\bigr) \\
&\quad \times \left[w(s) - \frac{\varphi^{\Delta}(s)}{2\lambda\varphi(s) g_{n-2}^{\sigma}(\sigma(\phi_i(s)), \phi_i(s))}\right]^2 \Delta s \\
&\quad + \int_{t_{10}}^{t} \frac{[\varphi^{\Delta}(s)]^2}{4\lambda\varphi(s) g_{n-2}^{\sigma}(\sigma(\phi_i(s)), \phi_i(s))}\Delta s \\
&\le \varphi(t_{10})w(t_{10}) + \int_{t_{10}}^{t} \frac{[\varphi^{\Delta}(s)]^2}{4\lambda\varphi(s) g_{n-2}^{\sigma}(\sigma(\phi_i(s)), \phi_i(s))}\Delta s.
\end{aligned}$$

Therefore by (C3),

$$\begin{aligned}
\infty &= \alpha \limsup_{t\to\infty} \int_{t_{10}}^{t} \varphi(s) \sum_{i=1}^{m} Q_i(s)\Delta s \\
&\le \varphi(t_{10})w(t_{10}) + \frac{1}{4\lambda} \limsup_{t\to\infty} \int_{t_{10}}^{t} \frac{[\varphi^{\Delta}(s)]^2}{\varphi(s) g_{n-2}^{\sigma}(\sigma(\phi_i(s)), \phi_i(s))}\Delta s \\
&< \infty.
\end{aligned}$$

This is a contradiction.

Now let us consider the case of $y(t) < 0$ for $t \ge t_1$. By (1.1) and (1.2), we have

$$z^{\Delta^n}(t) = -\sum_{i=1}^{m} Q_i(t) f_i\bigl(y\bigl(\phi_i(t)\bigr)\bigr) > 0$$

for $t \geq t_1$. That is, $z^{\Delta^n} > 0$. It follows that $z^{\Delta^j}(t)$ ($j \in [0, n-1] \cap \mathbb{N}_0$) is strictly monotone and eventually of constant sign. Since $P(t)$ is an oscillatory function, there exists a $t_2 \geq t_1$ such that $z(t) < 0$ for $t \geq t_2$. Since $y(t)$ is bounded, by (C1) and (1.2), there is a $t_3 \geq t_2$ such that $z(t)$ is also bounded for $t \geq t_3$. Assume that $x(t) = -z(t)$. Then $x^{\Delta^n}(t) = -z^{\Delta^n}(t)$. Therefore, $x(t) > 0$ and $x^{\Delta^n}(t) < 0$ for $t \geq t_3$. Hence, we observe that $x(t)$ is bounded. Since $n$ is odd, by Lemma 1, there exists a $t_4 \geq t_3$ and $p = 1$ (otherwise $x(t)$ is not bounded) such that $(-1)^k x^{\Delta^k}(t) > 0$, $k \in [0, n-1] \cap \mathbb{N}_0$ and $t \geq t_4$. That is, $(-1)^k z^{\Delta^k}(t) < 0$, $k \in [0, n-1] \cap \mathbb{N}_0$ and $t \geq t_4$. In particular, for $t \geq t_4$ we have $z^{\Delta}(t) > 0$. Therefore, $z(t)$ is increasing. For the rest of the proof, we can proceed the proof similarly to the case of $y(t) > 0$. Hence, the proof is completed. □

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
DU carried out the time scale studies, participated in the sequence alignment, drafted the manuscript, and have given final approval of the version to be published. YB carried out the preliminaries of the manuscript and participated in the sequence alignment. Each author have participated sufficiently in the work to take public responsibility for appropriate portions of the content. Authors read and approved the final manuscript.

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