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Mappings of type generalized de La Vallée Poussin's mean

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Abstract
In the present paper, we study the operator ideals generated by the approximation numbers and generalized de La Vallée Poussin's mean defined in (Şimşek *et al.* in J. Comput. Anal. Appl. 12(4):768-779, 2010). Our results coincide with those in (Faried and Bakery in J. Inequal. Appl. 2013, doi:10.1186/1029-242X-2013-186) for the generalized Cesáro sequence space.

Keywords: approximation numbers; operator ideal; generalized de La Vallée Poussin's mean sequence space

1 Introduction

By $L(X,Y)$ we denote the space of all bounded linear operators from a normed space X into a normed space Y. The set of nonnegative integers is denoted by $\mathbb{N}=\{0,1,2,\dots\}$ and the real numbers by $\mathbb{R}$. By ω we denote the space of all real sequences. A map which assigns to every operator $T\in L(X,Y)$ a unique sequence $(s_n(T))_{n=0}^{\infty}$ is called an s-function and the number $s_n(T)$ is called the nth s-numbers of T if the following conditions are satisfied:

(a) $\|T\|=s_0(T)\geq s_1(T)\geq\cdots\geq 0$ for all $T\in L(X,Y)$.
(b) $s_n(T_1+T_2)\leq s_n(T_1)+\|T_2\|$ for all $T_1,T_2\in L(X,Y)$.
(c) $s_n(RST)\leq\|R\|s_n(S)\|T\|$ for all $T\in L(X_0,X)$, $S\in L(X,Y)$ and $R\in L(Y,Y_0)$.
(d) $s_n(\lambda T)=|\lambda|s_n(T)$ for all $T\in L(X,Y)$, $\lambda\in\mathbb{R}$.
(e) $\operatorname{rank}(T)\leq n$ if $s_n(T)=0$ for all $T\in L(X,Y)$.
(f)

$$s_r(I_n)=\begin{cases}1 & \text{for } r<n,\\ 0 & \text{for } r\geq n,\end{cases}$$

where I_n is the identity operator on the Euclidean space ℓ_2^n.

As examples of s-numbers, we mention approximation numbers $\alpha_n(T)$, Gelfand numbers $c_n(T)$, Kolmogorov numbers $d_n(T)$ and Tichomirov numbers $d_n^*(T)$ defined by:

(I) $\alpha_n(T)=\inf\{\|T-A\|:A\in L(X,Y)\text{ and }\operatorname{rank}(A)\leq n\}$.
(II) $c_n(T)=a_n(J_YT)$, where J_Y is a metric injection (a metric injection is a one-to-one operator with closed range and with norm equal to one) from the space Y into a higher space $\ell^\infty(\Lambda)$ for a suitable index set Λ.
(III) $d_n(T)=\inf_{\dim Y\leq n}\sup_{\|x\|\leq 1}\inf_{y\in Y}\|Tx-y\|$.
(IV) $d_n^*(T)=d_n(J_YT)$.

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All these numbers satisfy the following condition:

(g) $s_{n+m}(T_1 + T_2) \leq s_n(T_1) + s_m(T_2)$ for all $T_1, T_2 \in L(X, Y)$.

An operator ideal U is a subclass of $L = \{L(X, Y); X$ and Y are Banach spaces$\}$ such that its components $\{U(X, Y); X$ and Y are Banach spaces$\}$ satisfy the following conditions:

(i) $I_K \in U$, where K denotes the 1-dimensional Banach space, where $U \subset L$.

(ii) If $T_1, T_2 \in U(X, Y)$, then $\lambda_1 T_1 + \lambda_2 T_2 \in U(X, Y)$ for any scalars λ_1, λ_2.

(iii) If $V \in L(X_0, X)$, $T \in U(X, Y)$ and $R \in L(Y, Y_0)$, then $RTV \in U(X_0, Y_0)$. See [1, 2] and [3].

For a sequence $p = (p_n)$ of positive real numbers with $p_n \geq 1$, for all $n \in \mathbb{N}$, the generalized Cesáro sequence space is defined by

$$\mathrm{Ces}(p) = \left\{x = (x_k) \in \omega : \rho(\lambda x) < \infty \text{ for some } \lambda > 0\right\},$$
$$\text{where } \rho(x) = \sum_{n=0}^{\infty} \left(\frac{1}{n+1} \sum_{k=0}^{n} |x_k| \right)^{p_n}.$$

The space $\mathrm{Ces}(p)$ is a Banach space with the norm $\|x\| = \inf\{\lambda > 0 : \rho(\frac{x}{\lambda}) \leq 1\}$.

If $p = (p_n)$ is bounded, we can simply write $\mathrm{Ces}(p) = \{x \in \omega : \sum_{n=0}^{\infty} (\frac{1}{n+1} \sum_{k=0}^{n} |x_k|)^{p_n} < \infty\}$. Also, some geometric properties of $\mathrm{Ces}(p)$ are studied in [4–6] and [7].

Let $\Lambda = (\lambda_n)$ be a nondecreasing sequence of positive real numbers tending to infinity, and let $\lambda_0 = 1$ and $\lambda_{n+1} \leq \lambda_n + 1$.

De La Vallée Poussin's means of a sequence $x = (x_k)$ are defined as follows:

$$t_n(x) = \frac{1}{\lambda_n} \sum_{j \in I_n} |x_j|, \quad \text{where } I_n = [n - \lambda_n + 1, n], \text{ for } k \in \mathbb{N}.$$

The generalized de La Vallée Poussin's mean sequence space was defined in [8].

$$V(\lambda, p) = \left\{x \in \omega : \rho(\lambda x) < \infty \text{ for some } \lambda > 0\right\}, \quad \text{where } \rho(x) = \sum_{n=0}^{\infty} \left(\frac{1}{\lambda_n} \sum_{k \in I_n} |x_k| \right)^{p_n}.$$

The space $V(\lambda, p)$ is a Banach space with the norm

$$\|x\| = \inf\left\{\lambda > 0 : \rho\left(\frac{x}{\lambda}\right) \leq 1\right\}.$$

If $p = (p_n)$ is bounded, we can simply write

$$V(\lambda, p) = \left\{x \in \omega : \sum_{n=0}^{\infty} \left(\frac{1}{\lambda_n} \sum_{k \in I_n} |x_k| \right)^{p_n} < \infty\right\}.$$

Also, some geometric properties of $V(\lambda, p)$ are studied in [9, 10] and [11].

Throughout this paper, the sequence (p_n) is a bounded sequence of positive real numbers with

(b1) the sequence (p_n) of positive real numbers is increasing and bounded with $\limsup p_n < \infty$ and $\liminf p_n > 1$,

(b2) the sequence (λ_n) is a nondecreasing sequence of positive real numbers tending to infinity, $\lambda_0 = 1$ and $\lambda_{n+1} \leq \lambda_n + 1$ with $\sum_{n=0}^{\infty} (\frac{1}{\lambda_n})^{p_n} < \infty$.

Also we define $e_i = (0, 0, \ldots, 1, 0, 0, \ldots)$, where 1 appears at the ith place for all $i \in \mathbb{N}$.

Different classes of paranormed sequence spaces have been introduced and their different properties have been investigated. See [12–15] and [16].

For any bounded sequence of positive numbers (p_n), we have the following well-known inequality $|a_n + b_n|^{p_n} \leq 2^{h-1}(|a_n|^{p_n} + |b_n|^{p_n})$, $h = \sup_n p_n$, and $p_n \geq 1$ for all $n \in \mathbb{N}$. See [17].

2 Preliminary and notation

Definition 2.1 A class of linear sequence spaces E is called a special space of sequences (sss) having the following conditions:

(1) E is a linear space and $e_n \in E$ for each $n \in \mathbb{N}$.
(2) If $x \in \omega$, $y \in E$ and $|x_n| \leq |y_n|$ for all $n \in \mathbb{N}$, then $x \in E$ '*i.e.*, E is solid'.
(3) If $(x_n)_{n=0}^{\infty} \in E$, then $(x_{[\frac{n}{2}]})_{n=0}^{\infty} = (x_0, x_0, x_1, x_1, x_2, x_2, \ldots) \in E$, where $[\frac{n}{2}]$ denotes the integral part of $\frac{n}{2}$.

Example 2.2 ℓ_p is a special space of sequences for $0 < p < \infty$.

Example 2.3 ces_p is a special space of sequences for $1 < p < \infty$.

Definition 2.4 $U_E^{\mathrm{app}} := \{U_E^{\mathrm{app}}(X,Y); X, Y \text{ are Banach spaces}\}$, where $U_E^{\mathrm{app}}(X,Y) := \{T \in L(X,Y) : (\alpha_n(T))_{n=0}^{\infty} \in E\}$.

Theorem 2.5 *U_E^{app} is an operator ideal if E is a special space of sequences* (*sss*).

Proof See [18]. □

We give here the sufficient conditions on the generalized de La Vallée Poussin's mean such that the class of all bounded linear operators between any arbitrary Banach spaces with nth approximation numbers of the bounded linear operators in the generalized de La Vallée Poussin's mean form an operator ideal.

3 Main results

Theorem 3.1 *$U_{V(\lambda,p)}^{\mathrm{app}}$ is an operator ideal, if conditions* (b1) *and* (b2) *are satisfied.*

Proof (1-i) Let $x, y \in V(\lambda, p)$ since

$$\sum_{n=0}^{\infty}\left(\frac{1}{\lambda_n}\sum_{k\in I_n}|x_k + y_k|\right)^{p_n} \leq 2^{h-1}\left(\sum_{n=0}^{\infty}\left(\frac{1}{\lambda_n}\sum_{k\in I_n}|x_k|\right)^{p_n} + \sum_{n=0}^{\infty}\left(\frac{1}{\lambda_n}\sum_{k\in I_n}|y_k|\right)^{p_n}\right),$$

$h = \sup_n p_n$, then $x + y \in V(\lambda, p)$.

(1-ii) Let $\lambda \in \mathbb{R}$, $x \in V(\lambda, p)$, then

$$\sum_{n=0}^{\infty}\left(\frac{1}{\lambda_n}\sum_{k\in I_n}|\lambda x_k|\right)^{p_n} \leq \sup_n |\lambda|^{p_n}\sum_{n=0}^{\infty}\left(\frac{1}{\lambda_n}\sum_{k\in I_n}|x_k|\right)^{p_n} < \infty,$$

we get $\lambda x \in V(\lambda, p)$, from (1-i) and (1-ii), $V(\lambda, p)$ is a linear space.

To show that $e_m \in V(\lambda, p)$ for each $m \in \mathbb{N}$, since $\sum_{n=0}^{\infty}(\frac{1}{\lambda_n})^{p_n} < \infty$. Thus we get

$$\rho(e_m) = \sum_{n=m}^{\infty}\left(\frac{1}{\lambda_n}\sum_{k\in I_n}\left|e_m(k)\right|\right)^{p_n} = \sum_{n=m}^{\infty}\left(\frac{1}{\lambda_n}\right)^{p_n} < \infty.$$

Hence $e_m \in V(\lambda, p)$.

(2) Let $|x_n| \le |y_n|$ for each $n \in \mathbb{N}$, then $\sum_{n=0}^{\infty}(\frac{1}{\lambda_n}\sum_{k\in I_n}|x_k|)^{p_n} \le \sum_{n=0}^{\infty}(\frac{1}{\lambda_n}\sum_{k\in I_n}|y_k|)^{p_n}$ since $y \in V(\lambda, p)$. Thus $x \in V(\lambda, p)$.

(3) Let $(x_n) \in V(\lambda, p)$, then we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}\left(\frac{1}{\lambda_n}\sum_{k\in I_n}|x_{[\frac{k}{2}]}|\right)^{p_n} \\
&\quad= \sum_{n=0}^{\infty}\left(\frac{1}{\lambda_{2n}}\sum_{k\in I_{2n}}|x_{[\frac{k}{2}]}|\right)^{p_{2n}} + \sum_{n=0}^{\infty}\left(\frac{1}{\lambda_{2n+1}}\sum_{k\in I_{2n+1}}|x_{[\frac{k}{2}]}|\right)^{p_{2n+1}} \\
&\quad= \sum_{n=0}^{\infty}\left(\frac{1}{\lambda_{2n}}\left(\left(\sum_{k\in I_n}2|x_k|\right)+|x_n|\right)\right)^{p_n} + \sum_{n=0}^{\infty}\left(\frac{1}{\lambda_{2n+1}}\left(\sum_{k\in I_n}2|x_k|\right)\right)^{p_n} \\
&\quad\le 2^{h-1}\left(\sum_{n=0}^{\infty}\left(\frac{1}{\lambda_n}\left(2\sum_{k\in I_n}|x_k|\right)\right)^{p_n} + \sum_{n=0}^{\infty}\left(\frac{1}{\lambda_n}\sum_{k\in I_n}|x_k|\right)^{p_n}\right) \\
&\qquad + 2^h\sum_{n=0}^{\infty}\left(\frac{1}{\lambda_n}\sum_{k\in I_n}|x_k|\right)^{p_n} \\
&\quad\le 2^{h-1}(2^h+1)\sum_{n=0}^{\infty}\left(\frac{1}{\lambda_n}\sum_{k\in I_n}|x_k|\right)^{p_n} + 2^h\sum_{n=0}^{\infty}\left(\frac{1}{\lambda_n}\sum_{k\in I_n}|x_k|\right)^{p_n} \\
&\quad\le \left(2^{2h-1}+2^{h-1}+2^h\right)\sum_{n=0}^{\infty}\left(\frac{1}{\lambda_n}\sum_{k\in I_n}|x_k|\right)^{p_n} < \infty.
\end{aligned}$$

Hence $(x_{[\frac{n}{2}]})_{n=0}^{\infty} \in V(\lambda, p)$. Hence from Theorem 2.5 it follows that $U_{V(\lambda,p)}^{\text{app}}$ is an operator ideal. □

Corollary 3.2 $U_{\text{ces}(p)}^{\text{app}}$ *is an operator ideal if* (p_n) *is an increasing sequence of positive real numbers*, $\lim_{n\to\infty}\sup p_n < \infty$ *and* $\lim_{n\to\infty}\inf p_n > 1$.

Corollary 3.3 $U_{\text{ces}_p}^{\text{app}}$ *is an operator ideal if* $1 < p < \infty$.

Theorem 3.4 *The linear space* $F(X, Y)$ *is dense in* $U_{V(\lambda,p)}^{\text{app}}(X, Y)$ *if conditions* (b1) *and* (b2) *are satisfied.*

Proof First we prove that every finite mapping $T \in F(X, Y)$ belongs to $U_{V(\lambda,p)}^{\text{app}}(X, Y)$. Since $e_m \in V(\lambda, p)$ for each $m \in \mathbb{N}$ and $V(\lambda, p)$ is a linear space, then for every finite mapping $T \in F(X, Y)$, *i.e.*, the sequence $(\alpha_n(T))_{n=0}^{\infty}$ contains only finitely many numbers different from zero. Now we prove that $U_{V(\lambda,p)}^{\text{app}}(X, Y) \subseteq \overline{F(X, Y)}$. Since letting $T \in U_{V(\lambda,p)}^{\text{app}}(X, Y)$ we get $(\alpha_n(T))_{n=0}^{\infty} \in V(\lambda, p)$, and since $\rho((\alpha_n(T))_{n=0}^{\infty}) < \infty$, let $\varepsilon \in]0, 1[$, then there exists a natural number $s > 0$ such that $\rho((\alpha_n(T))_{n=s}^{\infty}) < \frac{\varepsilon}{2^{h+2}\delta c}$ for some $c \ge 1$, where $\delta = \max\{1, \sum_{n=s}^{\infty}(\frac{1}{\lambda_n})^{p_n}\}$.

Since $\alpha_n(T)$ is decreasing for each $n \in \mathbb{N}$, we get

$$\begin{aligned}\sum_{n=s+1}^{2s}\left(\frac{1}{\lambda_n}\sum_{k\in I_n}\alpha_{2s}(T)\right)^{p_n} &\le \sum_{n=s+1}^{2s}\left(\frac{1}{\lambda_n}\sum_{k\in I_n}\alpha_n(T)\right)^{p_n} \\ &\le \sum_{n=s}^{\infty}\left(\frac{1}{\lambda_n}\sum_{k\in I_n}\alpha_k(T)\right)^{p_n} < \frac{\varepsilon}{2^{h+2}\delta c},\end{aligned} \tag{1}$$

then there exists $A \in F_{2s}(X,Y)$, $\operatorname{rank}(A) \le 2s$ with

$$\sum_{n=2s+1}^{3s}\left(\frac{1}{\lambda_n}\sum_{k\in I_n}\|T-A\|\right)^{p_n} \le \sum_{n=s+1}^{2s}\left(\frac{1}{\lambda_n}\sum_{k\in I_n}\|T-A\|\right)^{p_n} < \frac{\varepsilon}{2^{h+2}\delta c}, \tag{2}$$

and since (p_n) is a bounded sequence of positive real numbers, so we can take

$$\sup_{n=s}^{\infty}\left(\sum_{k\in I_s}\|T-A\|\right)^{p_n} < \frac{\varepsilon}{2^h\delta}, \tag{3}$$

also $\alpha_n(T) = \inf\{\|T-A\| : A \in L(X,Y) \text{ and } \operatorname{rank}(A) \le n\}$. Then there exists a natural number $N > 0$, A_N with $\operatorname{rank}(A_N) \le N$ and $\|T-A_N\| \le 2\alpha_N(T)$. Since $\alpha_n(T) \overset{n\to\infty}{\longrightarrow} 0$, then

$$\|T-A_N\| \overset{N\to\infty}{\longrightarrow} 0, \quad \text{so we can take } \sum_{n=0}^{s}\left(\frac{1}{\lambda_n}\sum_{k\in I_n}\|T-A\|\right)^{p_n} < \frac{\varepsilon}{2^{h+3}\delta c}. \tag{4}$$

Since (p_n) is an increasing sequence, by using (1), (2), (3) and (4), we get

$$\begin{aligned} d(T,A) &= \rho\big(\alpha_n(T-A)\big)_{n=0}^{\infty} \\ &= \sum_{n=0}^{3s-1}\left(\frac{1}{\lambda_n}\sum_{k\in I_n}\alpha_k(T-A)\right)^{p_n} + \sum_{n=3s}^{\infty}\left(\frac{1}{\lambda_n}\sum_{k\in I_n}\alpha_k(T-A)\right)^{p_n} \\ &\le \sum_{n=0}^{3s}\left(\frac{1}{\lambda_n}\sum_{k\in I_n}\|T-A\|\right)^{p_n} + \sum_{n=s}^{\infty}\left(\frac{1}{\lambda_n}\sum_{k\in I_{n+2s}}\alpha_k(T-A)\right)^{p_{n+2s}} \\ &\le 3\sum_{n=0}^{s}\left(\frac{1}{\lambda_n}\sum_{k\in I_n}\|T-A\|\right)^{p_n} \\ &\quad + \sum_{n=s}^{\infty}\left(\frac{1}{\lambda_n}\sum_{k\in I_{2s-1}}\alpha_k(T-A) + \frac{1}{\lambda_n}\sum_{k\in I_{n+2s}\setminus I_{2s-1}}\alpha_k(T-A)\right)^{p_n} \\ &\le 3\sum_{n=0}^{s}\left(\frac{1}{\lambda_n}\sum_{k\in I_n}\|T-A\|\right)^{p_n} \\ &\quad + 2^{h-1}\left(\sum_{n=s}^{\infty}\left(\frac{1}{\lambda_n}\sum_{k\in I_{2s-1}}\alpha_k(T-A)\right)^{p_n} + \sum_{n=s}^{\infty}\left(\frac{1}{\lambda_n}\sum_{k\in I_{n+2s}\setminus I_{2s-1}}\alpha_k(T-A)\right)^{p_n}\right) \\ &\le 3\sum_{n=0}^{s}\left(\frac{1}{\lambda_n}\sum_{k\in I_n}\|T-A\|\right)^{p_n} \end{aligned}$$

$$
\begin{aligned}
&+ 2^{h-1}\left(\sum_{n=s}^{\infty}\left(\frac{1}{\lambda_n}\sum_{k\in I_s}\|T-A\|\right)^{p_n} + \sum_{n=s}^{\infty}\left(\frac{1}{\lambda_n}\sum_{k\in I_n}\alpha_{k+2s}(T-A)\right)^{p_n}\right) \\
&\leq 3\sum_{n=0}^{s}\left(\frac{1}{\lambda_n}\sum_{k=0}^{n}\|T-A\|\right)^{p_n} \\
&+ 2^{h-1}\sup_{n=s}^{\infty}\left(\sum_{k\in I_s}\|T-A\|\right)^{p_n}\sum_{n=s}^{\infty}\left(\frac{1}{\lambda_n}\right)^{p_n} + 2^{h-1}\sum_{n=s}^{\infty}\left(\frac{1}{\lambda_n}\sum_{k\in I_n}\alpha_k(T)\right)^{p_n} < \varepsilon.
\end{aligned}
$$
□

Definition 3.5 A class of special space of sequences (sss) E_ρ is called a pre-modular special space of sequences if there exists a function $\rho : E \to [0,\infty[$ satisfying the following conditions:

(i) $\rho(x) \geq 0$ $\forall x \in E_\rho$ and $\rho(x) = 0 \Leftrightarrow x = \theta$, where θ is the zero element of E,

(ii) there exists a constant $l \geq 1$ such that $\rho(\lambda x) \leq l|\lambda|\rho(x)$ for all values of $x \in E$ and for any scalar λ,

(iii) for some numbers $k \geq 1$, we have the inequality $\rho(x + y) \leq k(\rho(x) + \rho(y))$ for all $x, y \in E$,

(iv) if $|x_n| \leq |y_n|$ for all $n \in \mathbb{N}$, then $\rho((x_n)) \leq \rho((y_n))$,

(v) for some numbers $k_0 \geq 1$, we have the inequality $\rho((x_n)) \leq \rho((x_{[\frac{n}{2}]})) \leq k_0\rho((x_n))$,

(vi) for each $x = (x(i))_{i=0}^{\infty} \in E$, there exists $s \in \mathbb{N}$ such that $\rho(x(i))_{i=s}^{\infty} < \infty$. This means the set of all finite sequences is ρ-dense in E,

(vii) for any $\lambda > 0$, there exists a constant $\zeta > 0$ such that $\rho(\lambda, 0, 0, 0, \dots) \geq \zeta\lambda\rho(1, 0, 0, 0, \dots)$.

It is clear from condition (ii) that ρ is continuous at θ. The function ρ defines a metrizable topology in E endowed with this topology which is denoted by E_ρ.

Example 3.6 ℓ_p is a pre-modular special space of sequences for $0 < p < \infty$, with $\rho(x) = \sum_{n=0}^{\infty}|x_n|^p$.

Example 3.7 ces_p is a pre-modular special space of sequences for $1 < p < \infty$, with $\rho(x) = \sum_{n=0}^{\infty}(\frac{1}{n+1}\sum_{k=0}^{n}|x_n|)^p$.

Theorem 3.8 *$V(\lambda, p)$ with $\rho(x) = \sum_{n=0}^{\infty}(\frac{1}{\lambda_n}\sum_{k\in I_n}|x_n|)^{p_n}$ is a pre-modular special space of sequences if conditions* (b1) *and* (b2) *are satisfied.*

Proof (i) Clearly, $\rho(x) \geq 0$ and $\rho(x) = 0 \Leftrightarrow x = \theta$.

(ii) Since (p_n) is bounded, then there exists a constant $l \geq 1$ such that $\rho(\lambda x) \leq l|\lambda|\rho(x)$ for all values of $x \in E$ and for any scalar λ.

(iii) For some numbers $k = \max(1, 2^{h-1}) \geq 1$, we have the inequality $\rho(x + y) \leq k(\rho(x) + \rho(y))$ for all $x, y \in V(\lambda, p)$.

(iv) Let $|x_n| \leq |y_n|$ for all $n \in \mathbb{N}$, then $\sum_{n=0}^{\infty}(\frac{1}{\lambda_n}\sum_{k\in I_n}|x_n|)^{p_n} \leq \sum_{n=0}^{\infty}(\frac{1}{\lambda_n}\sum_{k\in I_n}|y_n|)^{p_n}$.

(v) There exist some numbers $k_0 = 2^{h-1}(2^h + 1) + 2^h \geq 1$; by using (iv) we have the inequality $\rho((x_n)) \leq \rho((x_{[\frac{n}{2}]})) \leq k_0\rho((x_n))$.

(vi) It is clear that the set of all finite sequences is ρ-dense in $V(\lambda, p)$.

(vii) For any $\lambda > 0$, there exists a constant $0 < \zeta < \lambda^{p_0-1}$ such that $\rho(\lambda, 0, 0, 0, \dots) \geq \zeta\lambda\rho(1, 0, 0, 0, \dots)$. □

Theorem 3.9 *Let X be a normed space, Y be a Banach space, and let conditions* (b1) *and* (b2) *be satisfied, then $U^{\mathrm{app}}_{V_\rho(\lambda,p)}(X,Y)$ is complete.*

Proof Let (T_m) be a Cauchy sequence in $U^{\mathrm{app}}_{V_\rho(\lambda,p)}(X,Y)$. Since $V(\lambda,p)$ with $\rho(x) = \sum_{n=0}^{\infty}(\frac{1}{\lambda_n}\sum_{k\in I_n}|x_n|)^{p_n}$ is a pre-modular special space of sequences, then, by using condition (vii) and since $U^{\mathrm{app}}_{V_\rho(\lambda,p)}(X,Y) \subseteq L(X,Y)$, we have $\rho((\alpha_n(T_i - T_j))_{n=0}^{\infty}) \geq \rho(\alpha_0(T_i - T_j), 0, 0, 0, \ldots) = \rho(\|T_i - T_j\|, 0, 0, 0, \ldots) \geq \zeta\|T_i - T_j\|\rho(1, 0, 0, 0, \ldots)$, then (T_m) is also a Cauchy sequence in $L(X,Y)$. Since the space $L(X,Y)$ is a Banach space, then there exists $T \in L(X,Y)$ such that $\|T_m - T\| \overset{m\to\infty}{\longrightarrow} 0$ and since $(\alpha_n(T_m))_{n=0}^{\infty} \in E$ for all $m \in \mathbb{N}$, ρ is continuous at θ and using (iii), we have

$$\begin{aligned}\rho\big(\alpha_n(T)\big)_{n=0}^{\infty} &= \rho\big(\alpha_n(T - T_m + T_m)\big)_{n=0}^{\infty} \leq k\rho\big(\alpha_{[\frac{n}{2}]}(T_m - T)\big)_{n=0}^{\infty} + k\rho\big(\alpha_{[\frac{n}{2}]}(T_m)\big)_{n=0}^{\infty} \\ &\leq k\rho\big(\big(\|T_m - T\|\big)_{n=0}^{\infty}\big) + k\rho\big(\alpha_n(T_m)\big)_{n=0}^{\infty} < \varepsilon \quad \text{for some } k \geq 1.\end{aligned}$$

Hence $(\alpha_n(T))_{n=0}^{\infty} \in V_\rho(\lambda,p)$ as such $T \in U^{\mathrm{app}}_{V_\rho(\lambda,p)}(X,Y)$. □

Corollary 3.10 *Let X be a normed space, Y be a Banach space and (p_n) be an increasing sequence of positive real numbers with $\limsup p_n < \infty$ and $\liminf p_n > 1$, then $U^{\mathrm{app}}_{\mathrm{ces}_{(p)}}(X,Y)$ is complete.*

Corollary 3.11 *Let X be a normed space, Y be a Banach space and (p_n) be an increasing sequence of positive real numbers with $1 < p < \infty$, then $U^{\mathrm{app}}_{\mathrm{ces}_p}(X,Y)$ is complete.*

Competing interests
The author declares that he has no competing interests.

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