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Riemann-Liouville fractional Hermite-Hadamard inequalities. Part II: for twice differentiable geometric-arithmetically *s*-convex functions

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Abstract
Motivated by the definition of geometric-arithmetically *s*-convex functions in (Shuang *et al.* in Analysis 33:197-208, 2013) and second-order fractional integral identities in (Zhang and Wang in J. Inequal. Appl. 2013:220, 2013; Wang *et al.* in Appl. Anal. 2012, doi:10.1080/00036811.2012.727986), we establish some interesting Riemann-Liouville fractional Hermite-Hadamard inequalities for twice differentiable geometric-arithmetically *s*-convex functions via beta function and incomplete beta function.

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1 Introduction

Fractional calculus, which is a generalization of classical differentiation and integration to arbitrary order, was born in 1695. In the past three hundred years, fractional calculus developed not only in pure theoretical field but also in diverse fields ranging from physical sciences and engineering to biological sciences and economics [1–8].

The classical Hermite-Hadamard inequalities have attracted many researchers since 1893. Researchers investigated Hermite-Hadamard inequalities involving fractional integrals according to the associated fractional integral equalities and different types of convex functions. For instance, one can refer to [9–11] for convex functions and to [12] for nondecreasing functions, to [13–15] for m-convex functions and to [16] for (s,m)-convex functions, to [17] for functions satisfying s-e-condition, to [18] for (α,m)-logarithmically convex functions and see the references therein.

In [19], Shuang *et al.* introduced a new concept of geometric-arithmetically s-convex functions and presented interesting Hermite-Hadamard type inequalities for integer integrals of such functions. In [20], the authors used the definition of geometric-arithmetically s-convex functions in [19] and applied first-order fractional integral identities in [9, 10, 14] to establish some interesting Riemann-Liouville fractional Hermite-Hadamard inequalities for once differentiable geometric-arithmetically s-convex functions.

However, fractional Hermite-Hadamard inequalities for twice geometric-arithmetically s-convex functions have not been reported. In this work, we continue the development in [20]. Note that Wang *et al.* [13, 16] presented some elementary fractional integral equalities for twice differential functions. Motivated by [13, 16, 19], we study Riemann-Liouville fractional Hermite-Hadamard type inequalities for geometric-arithmetically s-convex functions by means of first-order fractional integral equalities.

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts.

Definition 2.1 (see [3]) Let $f \in L[a,b]$. The symbols $J_{a^+}^{\alpha} f$ and $J_{b^-}^{\alpha} f$ denote the left-sided and right-sided Riemann-Liouville fractional integrals of the order $\alpha \in R^+$ and are defined by

$$\left(J_{a^+}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\,dt \quad (0 \le a < x \le b)$$

and

$$\left(J_{b^-}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)\,dt \quad (0 \le a \le x < b),$$

respectively, here $\Gamma(\cdot)$ is the gamma function.

Definition 2.2 (see [19]) Let $f : I \subseteq R^+ \to R^+$ and $s \in (0,1]$. A function $f(x)$ is said to be geometric-arithmetically s-convex on I if for every $x, y \in I$ and $t \in [0,1]$, we have

$$f\left(x^t y^{1-t}\right) \le t^s\left(f(x)\right) + (1-t)^s f(y).$$

Definition 2.3 (see [21]) The incomplete beta function is defined as follows:

$$B_x(a,b) = \int_0^x t^{a-1}(1-t)^{b-1}\,dt,$$

where $x \in [0,1]$, $a, b > 0$.

The following inequality will be used in the sequel.

Lemma 2.4 (see [18]) *For* $t \in [0,1]$, *we have*

$$(1-t)^n \le 2^{1-n} - t^n \quad \text{for } n \in [0,1],$$
$$(1-t)^n \ge 2^{1-n} - t^n \quad \text{for } n \in [1,\infty).$$

The following elementally inequality was used in the proof directly in [19]. Here, we revisit this inequality from the point of our view and give a proof in [20].

Lemma 2.5 (see [20]) *For* $t \in [0,1]$, $x, y > 0$, *we have*

$$tx + (1-t)y \ge y^{1-t}x^t.$$

We introduce the following integral identities.

Lemma 2.6 (see [16]) *Let $f : [a,b] \to R$ be a twice differentiable mapping on (a,b) with $a < b$. If $f'' \in L[a,b]$, then the following equality for fractional integrals holds*:

$$\begin{aligned}&\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\big[J^\alpha_{a^+}f(b) + J^\alpha_{b^-}f(a)\big]\\&\quad = \frac{(b-a)^2}{2}\int_0^1 \frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1} f''\big(ta+(1-t)b\big)\,dt.\end{aligned}$$

Lemma 2.7 (see [13]) *Let $f : [a,b] \to R$ be a twice differentiable mapping on (a,b) with $a < b$. If $f'' \in L[a,b]$, then*

$$\begin{aligned}&\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\big[J^\alpha_{a^+}f(b) + J^\alpha_{b^-}f(a)\big] - f\left(\frac{a+b}{2}\right)\\&\quad = \frac{(b-a)^2}{2}\int_0^1 m(t) f''\big(ta+(1-t)b\big)\,dt,\end{aligned}$$

where

$$m(t) = \begin{cases} t - \frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1}, & t \in [0,\frac{1}{2}),\\ 1-t - \frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1}, & t \in [\frac{1}{2},1).\end{cases}$$

Lemma 2.8 (see [13]) *Let $f : [a,b] \to R$ be a twice differentiable mapping on (a,b) with $a < b$. If $f'' \in L[a,b]$, $r > 0$, then*

$$\begin{aligned}&\frac{f(a)+f(b)}{r(r+1)} + \frac{2}{r+1} f\left(\frac{a+b}{2}\right) - \frac{\Gamma(\alpha+1)}{r(b-a)^\alpha}\big[J^\alpha_{a^+}f(b) + J^\alpha_{b^-}f(a)\big]\\&\quad = (b-a)^2\int_0^1 k(t) f''\big(ta+(1-t)b\big)\,dt,\end{aligned}$$

where

$$k(t) = \begin{cases} \frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{r(\alpha+1)} - \frac{t}{r+1}, & t \in [0,\frac{1}{2}),\\ \frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{r(\alpha+1)} - \frac{1-t}{r+1}, & t \in [\frac{1}{2},1).\end{cases}$$

3 The first main results

By using Lemma 2.6, we can obtain the main results in this section.

Theorem 3.1 *Let $f : [0,b] \to R$ be a differentiable mapping. If $|f''|$ is measurable and $|f''|$ is decreasing and geometric-arithmetically s-convex on $[0,b]$ for some fixed $\alpha \in (0,\infty)$, $s \in (0,1]$, $0 \le a < b$, then the following inequality for fractional integrals holds*:

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\big[J^\alpha_{a^+}f(b) + J^\alpha_{b^-}f(a)\big]\right|\\&\quad \le \frac{(b-a)^2(|f''(a)|+|f''(b)|)}{2(\alpha+1)}\left(\frac{1}{s+1} - \frac{1}{\alpha+s+2}\right).\end{aligned}$$

Proof By using Definition 2.2, Lemma 2.5 and Lemma 2.6, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f(a)\right]\right| \\
&\quad\le \frac{(b-a)^2}{2}\int_0^1\left|\frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1}\right|\left|f''\bigl(ta+(1-t)b\bigr)\right|dt \\
&\quad\le \frac{(b-a)^2}{2(\alpha+1)}\int_0^1\bigl(1-(1-t)^{\alpha+1}-t^{\alpha+1}\bigr)\left|f''\bigl(ta+(1-t)b\bigr)\right|dt \\
&\quad\le \frac{(b-a)^2}{2(\alpha+1)}\int_0^1\bigl(1-(1-t)^{\alpha+1}-t^{\alpha+1}\bigr)\left|f''\bigl(a^tb^{1-t}\bigr)\right|dt \\
&\quad\le \frac{(b-a)^2}{2(\alpha+1)}\int_0^1\bigl(1-(1-t)^{\alpha+1}-t^{\alpha+1}\bigr)\bigl[t^s\left|f''(a)\right|+(1-t)^s\left|f''(b)\right|\bigr]dt \\
&\quad\le -\frac{(b-a)^2|f''(a)|}{2(\alpha+1)}\int_0^1 t^{\alpha+s+1}\,dt+\frac{(b-a)^2|f''(a)|}{2(\alpha+1)}\int_0^1 t^s\,dt \\
&\qquad -\frac{(b-a)^2|f''(a)|}{2(\alpha+1)}\int_0^1 t^s(1-t)^{\alpha+1}\,dt \\
&\qquad -\frac{(b-a)^2|f''(b)|}{2(\alpha+1)}\int_0^1 (1-t)^{\alpha+s+1}\,dt-\frac{(b-a)^2|f''(b)|}{2(\alpha+1)}\int_0^1 t^{\alpha+1}(1-t)^s\,dt \\
&\qquad +\frac{(b-a)^2|f''(b)|}{2(\alpha+1)}\int_0^1 (1-t)^s\,dt \\
&\quad\le -\frac{(b-a)^2|f''(a)|}{2(\alpha+1)}\frac{1}{\alpha+s+2}+\frac{(b-a)^2|f''(a)|}{2(\alpha+1)}\frac{1}{s+1} \\
&\qquad -\frac{(b-a)^2|f''(a)|}{2(\alpha+1)}B(s+1,\alpha+2) \\
&\qquad -\frac{(b-a)^2|f''(b)|}{2(\alpha+1)}\frac{1}{\alpha+s+2}+\frac{(b-a)^2|f''(b)|}{2(\alpha+1)}\frac{1}{s+1} \\
&\qquad -\frac{(b-a)^2|f''(b)|}{2(\alpha+1)}B(s+1,\alpha+2) \\
&\quad\le \frac{(b-a)^2(|f''(a)|+|f''(b)|)}{2(\alpha+1)}\left(\frac{1}{s+1}-\frac{1}{\alpha+s+2}\right).
\end{aligned}$$

The proof is done. □

Theorem 3.2 *Let $f:[0,b]\to R$ be a differentiable mapping and $1<q<\infty$. If $|f''|^q$ is measurable and $|f''|^q$ is decreasing and geometric-arithmetically s-convex on $[0,b]$ for some fixed $\alpha\in(0,\infty)$, $s\in(0,1]$, $0\le a<b$, then the following inequality for fractional integrals holds*:

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f(a)\right]\right| \\
&\quad\le \frac{(b-a)^2\max\{1-2^{1-\alpha},2^{1-\alpha}-1\}}{2(\alpha+1)}\left(\frac{|f''(a)|^q+|f''(b)|^q}{s+1}\right)^{\frac{1}{q}}.
\end{aligned}$$

Proof To achieve our aim, we divide our proof into two cases.

Case 1: $\alpha \in (0,1)$. By using Definition 2.2, Lemma 2.4, Lemma 2.5, Hölder's inequality and Lemma 2.6, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)\right]\right| \\
&\quad\le \frac{(b-a)^2}{2}\int_0^1 \left|\frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1}\right|\left|f''\bigl(ta+(1-t)b\bigr)\right|dt \\
&\quad\le \frac{(b-a)^2}{2(\alpha+1)}\left(\int_0^1 \left|1-(1-t)^\alpha - t^\alpha\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f''\bigl(ta+(1-t)b\bigr)\right|^q dt\right)^{\frac{1}{q}} \\
&\quad\le \frac{(b-a)^2}{2(\alpha+1)}\left(\int_0^1 \left|1-(1-t)^\alpha - t^\alpha\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f''\bigl(a^t b^{1-t}\bigr)\right|^q dt\right)^{\frac{1}{q}} \\
&\quad\le \frac{(b-a)^2}{2(\alpha+1)}\left(\int_0^1 \left|1-(1-t)^\alpha - t^\alpha\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 \left[t^s\left|f''(a)\right|^q + (1-t)^s\left|f''(b)\right|^q\right] dt\right)^{\frac{1}{q}} \\
&\quad\le \frac{(b-a)^2}{2(\alpha+1)}\left(\frac{|f''(a)|^q + |f''(b)|^q}{s+1}\right)^{\frac{1}{q}}\left(\int_0^1 \left[(1-t)^\alpha + t^\alpha - 1\right]^p dt\right)^{\frac{1}{p}} \\
&\quad\le \frac{(b-a)^2}{2(\alpha+1)}\left(\frac{|f''(a)|^q + |f''(b)|^q}{s+1}\right)^{\frac{1}{q}}\left(\int_0^1 \left[2^{1-\alpha} - 1\right]^p dt\right)^{\frac{1}{p}} \\
&\quad\le \frac{(b-a)^2(2^{1-\alpha}-1)}{2(\alpha+1)}\left(\frac{|f''(a)|^q + |f''(b)|^q}{s+1}\right)^{\frac{1}{q}},
\end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Case 2: $\alpha \in [1,\infty)$. By using Definition 2.2, Lemma 2.4, Lemma 2.5, Hölder's inequality and Lemma 2.6, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)\right]\right| \\
&\quad\le \frac{(b-a)^2}{2(\alpha+1)}\left(\frac{|f''(a)|^q + |f''(b)|^q}{s+1}\right)^{\frac{1}{q}}\left(\int_0^1 \left[1-(1-t)^\alpha - t^\alpha\right]^p dt\right)^{\frac{1}{p}} \\
&\quad\le \frac{(b-a)^2(1-2^{1-\alpha})}{2(\alpha+1)}\left(\frac{|f''(a)|^q + |f''(b)|^q}{s+1}\right)^{\frac{1}{q}}.
\end{aligned}$$

The proof is done. □

4 The second main results

By using Lemma 2.7, we can obtain the main results in this section.

Theorem 4.1 *Let $f : [0,b] \to R$ be a differentiable mapping. If $|f''|$ is measurable and $|f''|$ is decreasing geometric-arithmetically s-convex functions on $[0,b]$ for some fixed $\alpha \in (0,\infty)$, $s \in (0,1]$, $0 \le a < b$, then the following inequality for fractional integrals holds*:

$$\begin{aligned}
&\left|\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)\right] - f\left(\frac{a+b}{2}\right)\right| \\
&\quad\le \frac{(b-a)^2|f''(a)|}{2(\alpha+1)}\left[\frac{\alpha - \alpha 2^{-s-1} - 2^{-s-1}}{1+s} - \frac{\alpha+1}{2+s}\right.
\end{aligned}$$

$$+ 2B(s+1, \alpha+2) + \frac{1}{\alpha+s+2}\bigg]$$
$$+ \frac{(b-a)^2|f''(b)|}{2(\alpha+1)}\left[\frac{\alpha 2^{-s-1} + 2^{-s-1} - 1}{1+s} + \frac{1}{\alpha+s+2} + 2B(\alpha+2, s+1)\right],$$

where

$$\int_0^1 t^s (1-t)^{\alpha+1}\,dt = B(s+1, \alpha+2).$$

Proof By using Definition 2.2, Lemma 2.5 and Lemma 2.7, we have

$$\begin{aligned}
&\left|\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)\right] - f\left(\frac{a+b}{2}\right)\right| \\
&\quad \le \frac{(b-a)^2}{2}\int_0^1 |m(t)|\left|f''\left(ta + (1-t)b\right)\right|dt \\
&\quad \le \frac{(b-a)^2}{2}\int_0^1 |m(t)|\left|f''\left(a^t b^{1-t}\right)\right|dt \\
&\quad \le \frac{(b-a)^2}{2}\int_0^1 |m(t)|\left[t^s\left|f''(a)\right| + (1-t)^s\left|f''(b)\right|\right]dt \\
&\quad \le \frac{(b-a)^2}{2}\int_{\frac{1}{2}}^1 \left|1 - t - \frac{1-(1-t)^{\alpha+1} - t^{\alpha+1}}{\alpha+1}\right|\left[t^s\left|f''(a)\right| + (1-t)^s\left|f''(b)\right|\right]dt \\
&\qquad + \frac{(b-a)^2}{2}\int_0^{\frac{1}{2}} \left|t - \frac{1-(1-t)^{\alpha+1} - t^{\alpha+1}}{\alpha+1}\right|\left[t^s\left|f''(a)\right| + (1-t)^s\left|f''(b)\right|\right]dt \\
&\quad \le \frac{(b-a)^2}{2(\alpha+1)}\int_{\frac{1}{2}}^1 \left|\alpha - t\alpha - t + (1-t)^{\alpha+1} + t^{\alpha+1}\right|\left[t^s\left|f''(a)\right| + (1-t)^s\left|f''(b)\right|\right]dt \\
&\qquad + \frac{(b-a)^2}{2(\alpha+1)}\int_0^{\frac{1}{2}} \left|t\alpha + t - 1 + (1-t)^{\alpha+1} + t^{\alpha+1}\right|\left[t^s\left|f''(a)\right| + (1-t)^s\left|f''(b)\right|\right]dt \\
&\quad \le \frac{(b-a)^2}{2(\alpha+1)}\left|f''(a)\right|\int_{\frac{1}{2}}^1 \left[\alpha t^s - (\alpha+1)t^{s+1} + t^s(1-t)^{\alpha+1} + t^{\alpha+s+1}\right]dt \\
&\qquad + \frac{(b-a)^2}{2(\alpha+1)}\left|f''(b)\right|\int_{\frac{1}{2}}^1 \left[\alpha(1-t)^s - (\alpha+1)t(1-t)^s + (1-t)^{\alpha+s+1} + t^{\alpha+1}(1-t)^s\right]dt \\
&\qquad + \frac{(b-a)^2}{2(\alpha+1)}\left|f''(a)\right|\int_0^{\frac{1}{2}} \left[-t^s + (\alpha+1)t^{s+1} + t^s(1-t)^{\alpha+1} + t^{\alpha+s+1}\right]dt \\
&\qquad + \frac{(b-a)^2}{2(\alpha+1)}\left|f''(b)\right|\int_0^{\frac{1}{2}} \left[-(1-t)^s + (\alpha+1)t(1-t)^s + (1-t)^{\alpha+s+1} + t^{\alpha+1}(1-t)^s\right]dt \\
&\quad \le \frac{(b-a)^2}{2(\alpha+1)}\left|f''(a)\right|\left[\alpha\frac{1-2^{-s-1}}{1+s} - (\alpha+1)\frac{1-2^{-s-2}}{2+s} + B(s+1,\alpha+2) + \frac{1-2^{-\alpha-s-2}}{\alpha+s+2}\right] \\
&\qquad + \frac{(b-a)^2}{2(\alpha+1)}\left|f''(b)\right|\left[\alpha\frac{2^{-s-1}}{s+1} - (\alpha+1)B(2,s+1) + \frac{2^{-\alpha-s-2}}{\alpha+s+2} + B(\alpha+2,s+1)\right] \\
&\qquad + \frac{(b-a)^2}{2(\alpha+1)}\left|f''(a)\right|\left[-\frac{2^{-s-1}}{s+1} + (\alpha+1)\frac{2^{-s-2}}{s+2} + B(\alpha+2,s+1) + \frac{2^{-\alpha-s-2}}{\alpha+s+2}\right] \\
&\qquad + \frac{(b-a)^2}{2(\alpha+1)}\left|f''(b)\right|\left[-\frac{1-2^{-s-1}}{1+s} + (\alpha+1)B(2,s+1) + \frac{1-2^{-\alpha-s-2}}{\alpha+s+2} + B(\alpha+2,s+1)\right]
\end{aligned}$$

$$\begin{aligned}&\leq \frac{(b-a)^2|f''(a)|}{2(\alpha+1)}\left[\frac{\alpha-\alpha 2^{-s-1}-2^{-s-1}}{1+s}-\frac{\alpha+1}{2+s}\right.\\&\quad\left.+2B(s+1,\alpha+2)+\frac{1}{\alpha+s+2}\right]\\&\quad+\frac{(b-a)^2|f''(b)|}{2(\alpha+1)}\left[\frac{\alpha 2^{-s-1}+2^{-s-1}-1}{1+s}+\frac{1}{\alpha+s+2}+2B(\alpha+2,s+1)\right].\end{aligned}$$

The proof is done. □

Theorem 4.2 *Let $f:[0,b]\to R$ be a differentiable mapping. $|f''|$ is measurable and $1<q<\infty$. If $|f''|^q$ is decreasing and geometric-arithmetically s-convex on $[0,b]$ for some fixed $\alpha\in(0,\infty)$, $s\in(0,1]$, $0\leq a<b$, then the following inequality for fractional integrals holds*:

$$\begin{aligned}&\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b)+J_{b-}^\alpha f(a)\right]-f\left(\frac{a+b}{2}\right)\\&\quad\leq \frac{(b-a)^2}{2(\alpha+1)}\left(\frac{|f''(a)|^q+|f''(b)|^q}{s+1}\right)^{\frac{1}{q}}\left(\frac{(\alpha+1)2^{-p-1}+(\alpha+0.5)^{p+1}-\alpha^{p+1}}{p+1}\right)^{\frac{1}{p}},\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof By using Definition 2.2, Lemma 2.5, Hölder's inequality and Lemma 2.7, we have

$$\begin{aligned}&\left|\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b)+J_{b-}^\alpha f(a)\right]-f\left(\frac{a+b}{2}\right)\right|\\&\quad\leq \frac{(b-a)^2}{2}\int_0^1\left|m(t)\right|\left|f''\big(ta+(1-t)b\big)\right|dt\\&\quad\leq \frac{(b-a)^2}{2}\left(\int_0^1\left|m(t)\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f''\big(ta+(1-t)b\big)\right|^q dt\right)^{\frac{1}{q}}\\&\quad\leq \frac{(b-a)^2}{2}\left(\int_0^1\left|m(t)\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f''\big(a^tb^{1-t}\big)\right|^q dt\right)^{\frac{1}{q}}\\&\quad\leq \frac{(b-a)^2}{2}\left(\int_0^1\left|m(t)\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left[t^s\left|f''(a)\right|^q+(1-t)^s\left|f''(b)\right|^q\right]dt\right)^{\frac{1}{q}}\\&\quad\leq \frac{(b-a)^2}{2}\left(\frac{|f''(a)|^q+|f''(b)|^q}{s+1}\right)^{\frac{1}{q}}\left(\int_0^1\left|m(t)\right|^p dt\right)^{\frac{1}{p}}\\&\quad\leq \frac{(b-a)^2}{2}\left(\frac{|f''(a)|^q+|f''(b)|^q}{s+1}\right)^{\frac{1}{q}}\left(\int_0^{\frac{1}{2}}\left|t-\frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1}\right|^p dt\right.\\&\quad\quad\left.+\int_{\frac{1}{2}}^1\left|1-t-\frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1}\right|^p dt\right)^{\frac{1}{p}}\\&\quad\leq \frac{(b-a)^2}{2(\alpha+1)}\left(\frac{|f''(a)|^q+|f''(b)|^q}{s+1}\right)^{\frac{1}{q}}\left(\int_0^{\frac{1}{2}}\left|\alpha t+t-1+(1-t)^{\alpha+1}+t^{\alpha+1}\right|^p dt\right.\\&\quad\quad\left.+\int_{\frac{1}{2}}^1\left|\alpha-t+(1-t)^{\alpha+1}+t^{\alpha+1}\right|^p dt\right)^{\frac{1}{p}}\end{aligned}$$

$$\le \frac{(b-a)^2}{2(\alpha+1)}\left(\frac{|f''(a)|^q+|f''(b)|^q}{s+1}\right)^{\frac{1}{q}}\left((\alpha+1)\int_0^{\frac{1}{2}} t^p\,dt+\int_{\frac{1}{2}}^1(\alpha-t+1)^p\,dt\right)^{\frac{1}{p}}$$

$$\le \frac{(b-a)^2}{2(\alpha+1)}\left(\frac{|f''(a)|^q+|f''(b)|^q}{s+1}\right)^{\frac{1}{q}}\left(\frac{(\alpha+1)2^{-p-1}+(\alpha+0.5)^{p+1}-\alpha^{p+1}}{p+1}\right)^{\frac{1}{p}}.$$

The proof is done. □

5 The third main results

By using Lemma 2.8, we can obtain the main results in this section.

Theorem 5.1 *Let $f:[0,b]\to R$ be a differentiable mapping. If $|f''|$ is measurable and $|f''|$ is decreasing and geometric-arithmetically s-convex on $[0,b]$ for some fixed $\alpha\in(0,\infty)$, $s\in(0,1]$, $0\le a<b$, then the following inequality for fractional integrals holds*:

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{r(r+1)}+\frac{2}{r+1}f\left(\frac{a+b}{2}\right)-\frac{\Gamma(\alpha+1)}{r(b-a)^\alpha}\left[J_{a^+}^\alpha f(b)+J_{b^-}^\alpha f(a)\right]\right|\\
&\quad\le\frac{(b-a)^2}{r(r+1)(\alpha+1)}\max\left\{\left[r+1-(r+1)2^{-\alpha}\right]\left[\frac{2^{-s-1}|f''(a)|}{s+1}+\frac{(1-2^{-s-1})|f''(b)|}{s+1}\right]\right.\\
&\qquad-r(\alpha+1)\left[\frac{2^{-s-2}|f''(a)|}{s+2}+B_{0.5}(2,s+1)\left|f''(b)\right|\right],\\
&\qquad\left.r(\alpha+1)\left[\frac{2^{-s-2}|f''(a)|}{s+2}+B_{0.5}(2,s+1)\left|f''(b)\right|\right]\right\}\\
&\qquad+\frac{(b-a)^2}{r(r+1)(\alpha+1)}\max\left\{\left[r+1-(r+1)2^{-\alpha}-r(\alpha+1)\right]\right.\\
&\qquad\times\left[\frac{(1-2^{-s-1})|f''(a)|}{s+1}-\frac{2^{-s-1}|f''(b)|}{s+1}\right]\\
&\qquad+r(\alpha+1)\left[\frac{(1-2^{-s-2})|f''(a)|}{s+2}+B_{0.5}(s+1,2)\left|f''(b)\right|\right],\\
&\qquad r(\alpha+1)\left[\frac{(1-2^{-s-1})|f''(a)|-2^{-s-1}|f''(b)|}{s+1}\right.\\
&\qquad\left.\left.-\frac{(1-2^{-s-2})|f''(a)|}{s+2}-B_{0.5}(s+1,2)\left|f''(b)\right|\right]\right\}.
\end{aligned}$$

Proof By using Definition 2.2, Definition 2.3, Lemma 2.5 and Lemma 2.8, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{r(r+1)}+\frac{2}{r+1}f\left(\frac{a+b}{2}\right)-\frac{\Gamma(\alpha+1)}{r(b-a)^\alpha}\left[J_{a^+}^\alpha f(b)+J_{b^-}^\alpha f(a)\right]\right|\\
&\quad\le(b-a)^2\int_0^1\left|k(t)\right|\left|f''\big(ta+(1-t)b\big)\right|dt\\
&\quad\le(b-a)^2\int_0^1\left|k(t)\right|\left|f''\big(a^tb^{1-t}\big)\right|dt\\
&\quad\le(b-a)^2\int_0^1\left|k(t)\right|\left[t^s\left|f''(a)\right|+(1-t)^s\left|f''(b)\right|\right]dt\\
&\quad\le(b-a)^2\int_0^{\frac{1}{2}}\left|\frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{r(\alpha+1)}-\frac{t}{r+1}\right|\left[t^s\left|f''(a)\right|+(1-t)^s\left|f''(b)\right|\right]dt
\end{aligned}$$

$$
\begin{aligned}
&+ (b-a)^2 \int_{\frac{1}{2}}^{1} \left| \frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{r(\alpha+1)} - \frac{1-t}{r+1} \right| \left[t^s \left| f''(a) \right| + (1-t)^s \left| f''(b) \right| \right] dt \\
&\le \frac{(b-a)^2}{r(r+1)(\alpha+1)} \int_0^{\frac{1}{2}} \left| r+1-(r+1)\left(t^{\alpha+1}+(1-t)^{\alpha+1}\right) \right. \\
&\quad \left. - \operatorname{tr}(\alpha+1) \right| \left[t^s \left| f''(a) \right| + (1-t)^s \left| f''(b) \right| \right] dt \\
&\quad + \frac{(b-a)^2}{r(r+1)(\alpha+1)} \int_{\frac{1}{2}}^{1} \left| r+1+\operatorname{tr}(\alpha+1)-(r+1)\left(t^{\alpha+1}+(1-t)^{\alpha+1}\right) \right. \\
&\quad \left. - r(\alpha+1) \right| \left[t^s \left| f''(a) \right| + (1-t)^s \left| f''(b) \right| \right] dt \\
&\le \frac{(b-a)^2}{r(r+1)(\alpha+1)} \max\left\{ \left[r+1-(r+1)2^{-\alpha} \right] \left[\frac{2^{-s-1}|f''(a)|}{s+1} + \frac{(1-2^{-s-1})|f''(b)|}{s+1} \right] \right. \\
&\quad - r(\alpha+1) \left[\frac{2^{-s-2}|f''(a)|}{s+2} + B_{0.5}(2,s+1) \left| f''(b) \right| \right], \\
&\quad \left. r(\alpha+1) \left[\frac{2^{-s-2}|f''(a)|}{s+2} + B_{0.5}(2,s+1) \left| f''(b) \right| \right] \right\} \\
&\quad + \frac{(b-a)^2}{r(r+1)(\alpha+1)} \max\left\{ \left[r+1-(r+1)2^{-\alpha} - r(\alpha+1) \right] \right. \\
&\quad \times \left[\frac{(1-2^{-s-1})|f''(a)|}{s+1} - \frac{2^{-s-1}|f''(b)|}{s+1} \right] \\
&\quad + r(\alpha+1) \left[\frac{(1-2^{-s-2})|f''(a)|}{s+2} + B_{0.5}(s+1,2) \left| f''(b) \right| \right], \\
&\quad r(\alpha+1) \left[\frac{(1-2^{-s-1})|f''(a)| - 2^{-s-1}|f''(b)|}{s+1} - \frac{(1-2^{-s-2})|f''(a)|}{s+2} \right. \\
&\quad \left. \left. - B_{0.5}(s+1,2) \left| f''(b) \right| \right] \right\},
\end{aligned}
$$

where we have used the following inequalities:

$$
\begin{aligned}
&\int_0^{\frac{1}{2}} \left[r+1-(r+1)\left(t^{\alpha+1}+(1-t)^{\alpha+1}\right) - \operatorname{tr}(\alpha+1) \right] \left[t^s \left| f''(a) \right| + (1-t)^s \left| f''(b) \right| \right] dt \\
&\le \left[r+1-(r+1)2^{-\alpha} \right] \int_0^{\frac{1}{2}} \left[t^s \left| f''(a) \right| + (1-t)^s \left| f''(b) \right| \right] dt \\
&\quad - r(\alpha+1) \int_0^{\frac{1}{2}} \left[t^{s+1} \left| f''(a) \right| + t(1-t)^s \left| f''(b) \right| \right] dt \\
&\le \left[r+1-(r+1)2^{-\alpha} \right] \left[\frac{2^{-s-1}|f''(a)|}{s+1} + \frac{(1-2^{-s-1})|f''(b)|}{s+1} \right] \\
&\quad - r(\alpha+1) \left[\frac{2^{-s-2}|f''(a)|}{s+2} + B_{0.5}(2,s+1) \left| f''(b) \right| \right],
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^{\frac{1}{2}} \left[-r-1+(r+1)\left(t^{\alpha+1}+(1-t)^{\alpha+1}\right) + \operatorname{tr}(\alpha+1) \right] \left[t^s \left| f''(a) \right| + (1-t)^s \left| f''(b) \right| \right] dt \\
&\le \left[-r-1+(r+1) \right] \int_0^{\frac{1}{2}} \left[t^s \left| f''(a) \right| + (1-t)^s \left| f''(b) \right| \right] dt
\end{aligned}
$$

$$
\begin{aligned}
&+ r(\alpha+1)\int_0^{\frac{1}{2}} \bigl[t^{s+1}\bigl|f''(a)\bigr| + t(1-t)^s\bigl|f''(b)\bigr|\bigr]\,dt \\
&\leq r(\alpha+1)\biggl[\frac{2^{-s-2}|f''(a)|}{s+2} + B_{0.5}(2,s+1)\bigl|f''(b)\bigr|\biggr],
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{\frac{1}{2}}^1 \bigl[r+1+\operatorname{tr}(\alpha+1)-(r+1)\bigl(t^{\alpha+1}+(1-t)^{\alpha+1}\bigr) \\
&\quad - r(\alpha+1)\bigr]\bigl[t^s\bigl|f''(a)\bigr| + (1-t)^s\bigl|f''(b)\bigr|\bigr]\,dt \\
&\leq \bigl[r+1-(r+1)2^{-\alpha}-r(\alpha+1)\bigr]\int_{\frac{1}{2}}^1 \bigl[t^s\bigl|f''(a)\bigr| + (1-t)^s\bigl|f''(b)\bigr|\bigr]\,dt \\
&\quad + r(\alpha+1)\int_{\frac{1}{2}}^1 \bigl[t^{s+1}\bigl|f''(a)\bigr| + t(1-t)^s\bigl|f''(b)\bigr|\bigr]\,dt \\
&\leq \bigl[r+1-(r+1)2^{-\alpha}-r(\alpha+1)\bigr]\biggl[\frac{(1-2^{-s-1})|f''(a)|}{s+1} - \frac{2^{-s-1}|f''(b)|}{s+1}\biggr] \\
&\quad + r(\alpha+1)\biggl[\frac{(1-2^{-s-2})|f''(a)|}{s+2} + B_{0.5}(s+1,2)\bigl|f''(b)\bigr|\biggr],
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{\frac{1}{2}}^1 \bigl[-r-1-\operatorname{tr}(\alpha+1)+(r+1)\bigl(t^{\alpha+1}+(1-t)^{\alpha+1}\bigr) \\
&\quad + r(\alpha+1)\bigr]\bigl[t^s\bigl|f''(a)\bigr| + (1-t)^s\bigl|f''(b)\bigr|\bigr]\,dt \\
&\leq \bigl[-r-1+(r+1)+r(\alpha+1)\bigr]\int_{\frac{1}{2}}^1 \bigl[t^s\bigl|f''(a)\bigr| + (1-t)^s\bigl|f''(b)\bigr|\bigr]\,dt \\
&\quad - r(\alpha+1)\int_{\frac{1}{2}}^1 \bigl[t^{s+1}\bigl|f''(a)\bigr| + t(1-t)^s\bigl|f''(b)\bigr|\bigr]\,dt \\
&\leq r(\alpha+1)\biggl[\frac{(1-2^{-s-1})|f''(a)|}{s+1} - \frac{2^{-s-1}|f''(b)|}{s+1}\biggr] \\
&\quad - r(\alpha+1)\biggl[\frac{(1-2^{-s-2})|f''(a)|}{s+2} + B_{0.5}(s+1,2)\bigl|f''(b)\bigr|\biggr] \\
&\leq r(\alpha+1)\biggl[\frac{(1-2^{-s-1})|f''(a)| - 2^{-s-1}|f''(b)|}{s+1} - \frac{(1-2^{-s-2})|f''(a)|}{s+2} \\
&\quad - B_{0.5}(s+1,2)\bigl|f''(b)\bigr|\biggr].
\end{aligned}
$$

The proof is done. □

Theorem 5.2 *Let $f : [0,b] \to R$ be a differentiable mapping. $|f''|$ is measurable and $1 < q < \infty$. If $|f''|^q$ is decreasing and geometric-arithmetically s-convex on $[0,b]$ for some fixed $\alpha \in (0,\infty)$, $s \in (0,1]$, $0 \leq a < b$, then the following inequality for fractional integrals* *holds*:

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{r(r+1)}+\frac{2}{r+1}f\left(\frac{a+b}{2}\right)-\frac{\Gamma(\alpha+1)}{r(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f(a)\right]\right|\\&\quad\le\frac{(b-a)^2}{[r(r+1)(\alpha+1)]^{1+p^{-1}}}\left(\frac{|f''(a)|^q+|f''(b)|^q}{s+1}\right)^{\frac{1}{q}}\\&\qquad\times\Big(\max\big\{\big[r+1-(r+1)2^{-\alpha}\big]^{p+1}-\big[1+0.5r(1-\alpha)-(1+r)2^{-\alpha}\big]^{p+1},\\&\qquad\big[r(\alpha+1)\big]^{p+1}2^{-p-1}\big\}\\&\qquad+\max\big\{\big[r+1-(r+1)2^{-\alpha}\big]^{p+1}-\big[1+0.5r(1-\alpha)+(1+r)2^{-\alpha}\big]^{p+1},\\&\qquad\big[0.5r(\alpha+1)\big]^{p+1}\big\}\Big)^{\frac{1}{p}},\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof By using Definition 2.2, Lemma 2.5, Hölder's inequality and Lemma 2.8, we have

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{r(r+1)}+\frac{2}{r+1}f\left(\frac{a+b}{2}\right)-\frac{\Gamma(\alpha+1)}{r(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f(a)\right]\right|\\&\quad\le(b-a)^2\int_0^1\big|k(t)\big|\big|f''\big(ta+(1-t)b\big)\big|\,dt\\&\quad\le(b-a)^2\left(\int_0^1\big|k(t)\big|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1\big|f''\big(ta+(1-t)b\big)\big|^q\,dt\right)^{\frac{1}{q}}\\&\quad\le(b-a)^2\left(\int_0^1\big|k(t)\big|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1\big|f''\big(a^tb^{1-t}\big)\big|^q\,dt\right)^{\frac{1}{q}}\\&\quad\le(b-a)^2\left(\int_0^1\big|k(t)\big|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1\big[t^s\big|f''(a)\big|^q+(1-t)^s\big|f''(b)\big|^q\big]\,dt\right)^{\frac{1}{q}}\\&\quad\le(b-a)^2\left(\frac{|f''(a)|^q+|f''(b)|^q}{s+1}\right)^{\frac{1}{q}}\left(\int_0^1\big|k(t)\big|^p\,dt\right)^{\frac{1}{p}}\\&\quad\le(b-a)^2\left(\frac{|f''(a)|^q+|f''(b)|^q}{s+1}\right)^{\frac{1}{q}}\left(\int_0^{\frac{1}{2}}\left|\frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{r(\alpha+1)}-\frac{t}{r+1}\right|^p\,dt\right.\\&\qquad\left.+\int_{\frac{1}{2}}^1\left|\frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{r(\alpha+1)}-\frac{1-t}{r+1}\right|^p\,dt\right)^{\frac{1}{p}}\\&\quad\le\frac{(b-a)^2}{r(r+1)(\alpha+1)}\left(\frac{|f''(a)|^q+|f''(b)|^q}{s+1}\right)^{\frac{1}{q}}\left(\int_0^{\frac{1}{2}}\big|r+1-(r+1)\big(t^{\alpha+1}+(1-t)^{\alpha+1}\big)\right.\\&\qquad-\mathrm{tr}(\alpha+1)\big|^p\,dt\\&\qquad\left.+\int_{\frac{1}{2}}^1\big|r+1+\mathrm{tr}(\alpha+1)-(r+1)\big(t^{\alpha+1}+(1-t)^{\alpha+1}\big)-r(\alpha+1)\big|^p\,dt\right)^{\frac{1}{p}}\\&\quad\le\frac{(b-a)^2}{r(r+1)(\alpha+1)}\left(\frac{|f''(a)|^q+|f''(b)|^q}{s+1}\right)^{\frac{1}{q}}\\&\qquad\times\left(\max\left\{\frac{[r+1-(r+1)2^{-\alpha}]^{p+1}-[1+0.5r(1-\alpha)-(1+r)2^{-\alpha}]^{p+1}}{r(\alpha+1)(p+1)},\right.\right.\end{aligned}$$

$$
\begin{aligned}
&\frac{[r(\alpha+1)]^{p+1}2^{-p-1}}{r(\alpha+1)(p+1)}\bigg\} \\
&+\max\bigg\{\frac{[r+1-(r+1)2^{-\alpha}]^{p+1}-[1+0.5r(1-\alpha)+(1+r)2^{-\alpha}]^{p+1}}{r(\alpha+1)(p+1)}, \\
&\frac{[0.5r(\alpha+1)]^{p+1}}{r(\alpha+1)(p+1)}\bigg\}\bigg)^{\frac{1}{p}} \\
\le{}& \frac{(b-a)^2}{[r(r+1)(\alpha+1)]^{1+p^{-1}}}\left(\frac{|f''(a)|^q+|f''(b)|^q}{s+1}\right)^{\frac{1}{q}} \\
&\times\big(\max\big\{\big[r+1-(r+1)2^{-\alpha}\big]^{p+1}-\big[1+0.5r(1-\alpha)-(1+r)2^{-\alpha}\big]^{p+1}, \\
&\big[r(\alpha+1)\big]^{p+1}2^{-p-1}\big\} \\
&+\max\big\{\big[r+1-(r+1)2^{-\alpha}\big]^{p+1}-\big[1+0.5r(1-\alpha)+(1+r)2^{-\alpha}\big]^{p+1}, \\
&\big[0.5r(\alpha+1)\big]^{p+1}\big\}\big)^{\frac{1}{p}},
\end{aligned}
$$

where we have used the following inequalities:

$$
\begin{aligned}
&\int_0^{\frac{1}{2}}\big[r+1-(r+1)\big(t^{\alpha+1}+(1-t)^{\alpha+1}\big)-\mathrm{tr}(\alpha+1)\big]^p\,dt \\
&\quad\le\frac{[r+1-(r+1)2^{-\alpha}]^{p+1}-[1+0.5r(1-\alpha)-(1+r)2^{-\alpha}]^{p+1}}{r(\alpha+1)(p+1)},
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^{\frac{1}{2}}\big[-r-1+(r+1)\big(t^{\alpha+1}+(1-t)^{\alpha+1}\big)+\mathrm{tr}(\alpha+1)\big]^p\,dt \\
&\quad\le\frac{[r(\alpha+1)]^{p+1}2^{-p-1}}{r(\alpha+1)(p+1)},
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{\frac{1}{2}}^{1}\big[r+1+\mathrm{tr}(\alpha+1)-(r+1)\big(t^{\alpha+1}+(1-t)^{\alpha+1}\big)-r(\alpha+1)\big]^p\,dt \\
&\quad\le\frac{[r+1-(r+1)2^{-\alpha}]^{p+1}-[1+0.5r(1-\alpha)+(1+r)2^{-\alpha}]^{p+1}}{r(\alpha+1)(p+1)},
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{\frac{1}{2}}^{1}\big[-r-1-\mathrm{tr}(\alpha+1)+(r+1)\big(t^{\alpha+1}+(1-t)^{\alpha+1}\big)+r(\alpha+1)\big]^p\,dt \\
&\quad\le\frac{[0.5r(\alpha+1)]^{p+1}}{r(\alpha+1)(p+1)}.
\end{aligned}
$$

The proof is done. □

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References

1. Baleanu, D, Machado, JAT, Luo, ACJ: Fractional Dynamics and Control. Springer, New York (2012)
2. Diethelm, K: The Analysis of Fractional Differential Equations. Lecture Notes in Mathematics. (2010)
3. Kilbas, AA, Srivastava, HM, Trujillo, JJ: Theory and Applications of Fractional Differential Equations. Elsevier, Amsterdam (2006)
4. Lakshmikantham, V, Leela, S, Devi, JV: Theory of Fractional Dynamic Systems. Cambridge Scientific Publishers, Cambridge (2009)
5. Miller, KS, Ross, B: An Introduction to the Fractional Calculus and Differential Equations. Wiley, New York (1993)
6. Michalski, MW: Derivatives of Noninteger Order and Their Applications. Diss. Math. CCCXXVIII. Inst. Math., Polish Acad. Sci. (1993)
7. Podlubny, I: Fractional Differential Equations. Academic Press, San Diego (1999)
8. Tarasov, VE: Fractional Dynamics: Application of Fractional Calculus to Dynamics of Particles, Fields and Media. Springer, Berlin (2011)
9. Sarikaya, MZ, Set, E, Yaldiz, H, Başak, N: Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities. Math. Comput. Model. **57**, 2403-2407 (2013)
10. Zhu, C, Fečkan, M, Wang, J: Fractional integral inequalities for differentiable convex mappings and applications to special means and a midpoint formula. J. Appl. Math. Statist. Inform. **8**, 21-28 (2012)
11. Latif, MA, Dragomir, SS, Matouk, AE: New inequalities of Ostrowski type for co-ordinated convex functions via fractional integrals. J. Frac. Calc. Appl. **2**, 1-15 (2012)
12. Wang, J, Li, X, Zhu, C: Refinements of Hermite-Hadamard type inequalities involving fractional integrals. Bull. Belg. Math. Soc. Simon Stevin (2013, in press)
13. Zhang, Y, Wang, J: On some new Hermite-Hadamard inequalities involving Riemann-Liouville fractional integrals. J. Inequal. Appl. **2013**, 220 (2013)
14. Set, E: New inequalities of Ostrowski type for mappings whose derivatives are s-convex in the second sense via fractional integrals. Comput. Math. Appl. **63**, 1147-1154 (2012)
15. Wang, J, Deng, J, Fečkan, M: Hermite-Hadamard type inequalities for r-convex functions via Riemann-Liouville fractional integrals. Ukr. Math. J. **65**, 193-211 (2013)
16. Wang, J, Li, X, Fečkan, M, Zhou, Y: Hermite-Hadamard-type inequalities for Riemann-Liouville fractional integrals via two kinds of convexity. Appl. Anal. (2012). doi:10.1080/00036811.2012.727986
17. Wang, J, Deng, J, Fečkan, M: Exploring s-e-condition and applications to some Ostrowski type inequalities via Hadamard fractional integrals. Math. Slovaca (2013, in press)
18. Deng, J, Wang, J: Fractional Hermite-Hadamard inequalities for (α, m)-logarithmically convex functions. J. Inequal. Appl. **2013**, 364 (2013). doi:10.1186/1029-242X-2013-364
19. Shuang, Y, Yin, HP, Qi, F: Hermite-Hadamard type integral inequalities for geometric-arithmetically s-convex functions. Analysis **33**, 197-208 (2013)
20. Liao, Y, Deng, J, Wang, J: Riemann-Liouville fractional Hermite-Hadamard inequalities. Part I: for once differentiable Geometric-Arithmetically s-convex functions. J. Inequal. Appl. **2013**, 443 (2013)
21. DiDonato, AR, Jarnagin, MP: The efficient calculation of the incomplete beta-function ratio for half-integer values of the parameters. Math. Comput. **21**, 652-662 (1967)

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