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Some new generalizations of Mizoguchi-Takahashi type fixed point theorem

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Abstract

In the light of the paper of Hasanzade Asl *et al.* (Fixed Point Theory Appl. 2012:212, 2012, doi:10.1186/1687-1812-2012-212), we obtain a fixed point theorem for multivalued mappings on a complete metric space. Our result is a generalized version of some results in the literature, including the famous result of Mizoguchi-Takahashi (J. Math. Anal. Appl. 141:177-188, 1989). Also, we give some examples to illustrate our result.

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1 Introduction and preliminaries

Let (X,d) be a metric space, and let $CB(X)$ denote the class of all nonempty, closed and bounded subsets of X. It is well known that $H : CB(X) \times CB(X) \to \mathbb{R}$ defined by

$$H(A,B) = \max\left\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\right\}$$

is a metric on $CB(X)$, which is called a Hausdorff metric, where $d(x,B) = \inf\{d(x,y) : y \in B\}$. Let $T : X \to CB(X)$ be a map, then T is called a multivalued contraction if for all $x,y \in X$, there exists $\lambda \in [0,1)$ such that

$$H(Tx,Ty) \le \lambda d(x,y).$$

In 1969, Nadler [1] proved a fundamental fixed point theorem for multivalued maps: Every multivalued contraction on a complete metric space has a fixed point.

Then, a lot of generalizations of the result of Nadler have been given (see, for example, [2–5]). One of the most important generalizations of it was given by Mizoguchi and Takahashi [6]. We can find both a simple proof of Mizoguchi-Takahashi fixed point theorem and an example showing that it is a real generalization of Nadler's result in [7]. We can also find some important results about this direction in [8–12].

Definition 1 [2] A function $k : [0,\infty) \to [0,1)$ is said to be an $\mathcal{MT}$-function if it satisfies $\limsup_{s\to t^+} k(s) < 1$ for all $t \in [0,\infty)$ (Mizoguchi-Takahashi's condition).

Lemma 1 [9] *Let $k : [0,\infty) \to [0,1)$ be an $\mathcal{MT}$-function, then the function $h : [0,\infty) \to [0,1)$ defined as $h(t) = \frac{1+k(t)}{2}$ is also an $\mathcal{MT}$-function.*

Lemma 2 [9] *$k : [0,\infty) \to [0,1)$ is an $\mathcal{MT}$-function if and only if for each $t \in [0,\infty)$, there exist $r_t \in [0,1)$ and $\varepsilon_t > 0$ such that $k(s) \le r_t$ for all $s \in [t, t+\varepsilon_t)$.*

Theorem 1 [6] *Let (X,d) be a complete metric space, and let $T : X \to CB(X)$ be a multivalued map. Assume*

$$H(Tx,Ty) \le k\bigl(d(x,y)\bigr)d(x,y) \tag{1.1}$$

for all $x,y \in X$, where k is an $\mathcal{MT}$-function. Then T has a fixed point.

Recently, Samet *et al.* [13] introduced the notion of α-ψ-contractive mappings and gave some fixed point results for such mappings. Their results are closely related to some ordered fixed point results. Then, using their idea, some authors presented fixed point results for single and multivalued mappings (see, for example, [13–17]). First, we recall these results. Denote by Ψ the family of nondecreasing functions $\psi : [0,\infty) \to [0,\infty)$ such that $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t > 0$.

Definition 2 [13] Let (X,d) be a metric space, T be a self-map on X, $\psi \in \Psi$ and $\alpha : X \times X \to [0,\infty)$ be a function. Then T is called α-ψ-contractive whenever

$$\alpha(x,y)d(Tx,Ty) \le \psi\bigl(d(x,y)\bigr)$$

for all $x,y \in X$.

Note that every Banach contraction mapping is an α-ψ-contractive mapping with $\alpha(x,y) = 1$ and $\psi(t) = \lambda t$ for some $\lambda \in [0,1)$.

Definition 3 [13] T is called α-admissible whenever $\alpha(x,y) \ge 1$ implies $\alpha(Tx,Ty) \ge 1$.

There exist some examples for α-admissible mappings in [13]. For convenience, we mention in here one of them. Let $X = [0,\infty)$. Define $T : X \to X$ and $\alpha : X \times X \to [0,\infty)$ by $Tx = \sqrt{x}$ for all $x \in X$ and $\alpha(x,y) = e^{x-y}$ for $x \ge y$ and $\alpha(x,y) = 0$ for $x < y$. Then T is α-admissible.

Definition 4 [14] α is said to have (B) property whenever $\{x_n\}$ is a sequence in X such that $\alpha(x_n,x_{n+1}) \ge 1$ for all $n \in \mathbb{N}$ and $x_n \to x$, then $\alpha(x_n,x) \ge 1$ for all $n \in \mathbb{N}$.

Theorem 2 (Theorem 2.1 of [13]) *Let (X,d) be a complete metric space and $T : X \to X$ be an α-admissible and α-ψ-contractive mapping. If there exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \ge 1$ and T is continuous, then T has a fixed point.*

Remark 1 If we assume that α has (B) property instead of the continuity of T, then again T has a fixed point (Theorem 2.2 of [13]). If for each $x,y \in X$ there exists $z \in X$ such that $\alpha(x,z) \ge 1$ and $\alpha(y,z) \ge 1$, then X is said to have (H) property. Therefore, if X has (H) property in Theorem 2.1 and Theorem 2.2 in [13], then the fixed point of T is unique (Theorem 2.3 of [13]).

Then some generalizations of α-ψ-contractive mappings are given as follows.

Definition 5 [14] T is called a Ćirić type α-ψ-generalized contractive mapping whenever

$$\alpha(x,y)d(Tx,Ty) \leq \psi\bigl(m(x,y)\bigr)$$

for all $x, y \in X$, where

$$m(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}\bigl[d(x,Ty) + d(y,Tx)\bigr]\right\}.$$

Note that every Ćirić type generalized contraction mapping is a Ćirić type α-ψ-generalized contractive mapping with $\alpha(x,y) = 1$ and $\psi(t) = \lambda t$ for some $\lambda \in [0,1)$.

Theorem 3 (Theorem 2.3 of [14]) *Let (X,d) be a complete metric space and $T : X \to X$ be an α-admissible and Ćirić type α-ψ-generalized contractive mapping. If there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and T is continuous or α has* (B) *property, then T has a fixed point. If X has* (H) *property, then the fixed point of T is unique.*

We can find some fixed point results for single-valued mappings in these directions in [15, 17]. Now we recall some multivalued case.

Definition 6 [14, 16] Let (X,d) be a metric space and $T : X \to CB(X)$ be a multivalued mapping. Then T is called multivalued α-ψ-contractive whenever

$$\alpha(x,y)H(Tx,Ty) \leq \psi\bigl(d(x,y)\bigr)$$

for all $x, y \in X$ and T is called multivalued α_*-ψ-contractive whenever

$$\alpha_*(Tx,Ty)H(Tx,Ty) \leq \psi\bigl(d(x,y)\bigr),$$

where $\alpha_*(Tx,Ty) = \inf\{\alpha(a,b) : a \in Tx, b \in Ty\}$. Similarly, if we replace $d(x,y)$ with $m(x,y)$, we can obtain Ćirić type multivalued α-ψ-generalized contractive and Ćirić type multivalued α_*-ψ-generalized contractive mappings on X.

Definition 7 [14, 16] Let (X,d) be a metric space and $T : X \to CB(X)$ be a multivalued mapping.

(a) T is said to be α-admissible whenever for each $x \in X$ and $y \in Tx$ with $\alpha(x,y) \geq 1$ implies $\alpha(y,z) \geq 1$ for all $z \in Ty$.

(b) T is said to be α_*-admissible whenever for each $x \in X$ and $y \in Tx$ with $\alpha(x,y) \geq 1$ implies $\alpha_*(Tx,Ty) \geq 1$.

Remark 2 It is clear that α_*-admissible maps are also α-admissible, but the converse may not be true as shown in the following example.

Example 1 Let $X = [-1,1]$ and $\alpha : X \times X \to [0,\infty)$ be defined by $\alpha(x,x) = 0$ and $\alpha(x,y) = 1$ for $x \neq y$. Define $T : X \to CB(X)$ by

$$Tx = \begin{cases} \{-x\}, & x \notin \{-1,0\}, \\ \{0,1\}, & x = -1, \\ \{1\}, & x = 0. \end{cases}$$

Let $x = -1$ and $y = 0 \in Tx = \{0,1\}$, then $\alpha(x,y) \geq 1$, but $\alpha_*(Tx,Ty) = \alpha_*(\{0,1\},\{1\}) = 0$. Thus T is not α_*-admissible. Now we show that T is α-admissible with the following cases:

Case 1. If $x = 0$, then $y = 1$ and $\alpha(x,y) \geq 1$. Also, $\alpha(y,z) \geq 1$ since $z = -1 \in Ty = \{-1\}$.

Case 2. If $x = -1$, then $y \in \{0,1\}$ and $\alpha(x,y) \geq 1$. Also, $\alpha(y,z) \geq 1$ for all $z \in Ty$.

Case 3. If $x \notin \{-1,0\}$, then $y = -x$ and $\alpha(x,y) \geq 1$. Also, $\alpha(y,z) \geq 1$ since $z = x \in Ty = \{x\}$.

The purpose of this work is to present some generalizations of Mizoguchi-Takahashi's fixed point theorem using this new idea.

2 Main results

Theorem 4 *Let (X,d) be a complete metric space, and let $T : X \to CB(X)$ be an α-admissible multivalued mapping such that*

$$\alpha(x,y)H(Tx,Ty) \leq k\bigl(d(x,y)\bigr)d(x,y) \tag{2.1}$$

for all $x,y \in X$, where k is an $\mathcal{MT}$-function. Suppose that there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0,x_1) \geq 1$. If T is continuous or α has (B) *property, then T has a fixed point.*

Proof Define $h(t) = \frac{1+k(t)}{2}$, then from Lemma 1, $h : [0,\infty) \to [0,1)$ is an $\mathcal{MT}$-function. Let x_0 and x_1 be as mentioned in the hypothesis. If $x_0 = x_1$, then x_0 is a fixed point of T. Assume $x_0 \neq x_1$, then $\frac{1-k(d(x_0,x_1))}{2}d(x_0,x_1) > 0$. Therefore there exists $x_2 \in Tx_1$ such that

$$\begin{aligned} d(x_1,x_2) &\leq H(Tx_0,Tx_1) + \frac{1-k(d(x_0,x_1))}{2}d(x_0,x_1) \\ &\leq \alpha(x_0,x_1)H(Tx_0,Tx_1) + \frac{1-k(d(x_0,x_1))}{2}d(x_0,x_1) \\ &\leq k\bigl(d(x_0,x_1)\bigr)d(x_0,x_1) + \frac{1-k(d(x_0,x_1))}{2}d(x_0,x_1) \\ &= \frac{1+k(d(x_0,x_1))}{2}d(x_0,x_1) \\ &= h\bigl(d(x_0,x_1)\bigr)d(x_0,x_1). \end{aligned}$$

Since T is α-admissible, $x_1 \in Tx_0$ and $\alpha(x_0,x_1) \geq 1$, then $\alpha(x_1,u) \geq 1$ for all $u \in Tx_1$. Thus $\alpha(x_1,x_2) \geq 1$ since $x_2 \in Tx_1$. If $x_1 = x_2$, then x_1 is a fixed point of T. Assume $x_1 \neq x_2$, then $\frac{1-k(d(x_1,x_2))}{2}d(x_1,x_2) > 0$. Therefore there exists $x_3 \in Tx_2$ such that

$$\begin{aligned} d(x_2,x_3) &\leq H(Tx_1,Tx_2) + \frac{1-k(d(x_1,x_2))}{2}d(x_1,x_2) \\ &\leq \alpha(x_1,x_2)H(Tx_1,Tx_2) + \frac{1-k(d(x_1,x_2))}{2}d(x_1,x_2) \end{aligned}$$

$$\begin{aligned}&\leq k\big(d(x_1,x_2)\big)d(x_1,x_2)+\frac{1-k(d(x_1,x_2))}{2}d(x_1,x_2)\\&=\frac{1+k(d(x_1,x_2))}{2}d(x_1,x_2)\\&=h\big(d(x_1,x_2)\big)d(x_1,x_2).\end{aligned}$$

Again, since T is α-admissible, then $\alpha(x_2,x_3)\geq 1$. In this way, we can construct a sequence $\{x_n\}$ in X such that $x_{n+1}\in Tx_n$, $\alpha(x_n,x_{n+1})\geq 1$ and

$$d(x_n,x_{n+1})\leq h\big(d(x_{n-1},x_n)\big)d(x_{n-1},x_n)$$

for all $n\in\mathbb{N}$. Since $h(t)<1$ for all $t\in[0,\infty)$, then $\{d(x_n,x_{n+1})\}$ is a nonincreasing sequence in $[0,\infty)$ and so there exists $\lambda\geq 0$ such that $\lim_{n\to\infty}d(x_n,x_{n+1})=\lambda$. Now since h is an $\mathcal{MT}$-function, then $\limsup_{s\to\lambda^+}h(s)<1$ and $h(\lambda)<1$. Therefore from Lemma 2 there exist $r\in[0,1)$ and $\varepsilon>0$ such that $h(s)\leq r$ for all $s\in[\lambda,\lambda+\varepsilon)$. Since $\lim_{n\to\infty}d(x_n,x_{n+1})=\lambda$, then there exists $n_0\in\mathbb{N}$ such that $\lambda\leq d(x_n,x_{n+1})<\lambda+\varepsilon$ for all $n\geq n_0$ and so

$$\begin{aligned}d(x_{n+1},x_{n+2})&\leq h\big(d(x_n,x_{n+1})\big)d(x_n,x_{n+1})\\&\leq rd(x_n,x_{n+1})\end{aligned}$$

for all $n\geq n_0$. Thus, we have

$$\begin{aligned}\sum_{n=1}^{\infty}d(x_n,x_{n+1})&=\sum_{n=1}^{n_0}d(x_n,x_{n+1})+\sum_{n=n_0+1}^{\infty}d(x_n,x_{n+1})\\&=\sum_{n=1}^{n_0}d(x_n,x_{n+1})+\sum_{n=n_0}^{\infty}d(x_{n+1},x_{n+2})\\&\leq\sum_{n=1}^{n_0}d(x_n,x_{n+1})+\sum_{n=n_0}^{\infty}rd(x_n,x_{n+1})\\&\leq\sum_{n=1}^{n_0}d(x_n,x_{n+1})+\sum_{n=1}^{\infty}r^nd(x_{n_0},x_{n_0+1})\\&<\infty\end{aligned}$$

and so $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists $z\in X$ such that $\lim_{n\to\infty}x_n=z$.

If T is continuous, then from the inequality $d(x_{n+1},Tz)\leq H(Tx_n,Tz)$, we have $d(z,Tz)=0$ and so $z\in Tz$.

Now assume that α has (B) property. Then $\alpha(x_n,z)\geq 1$ for all $n\in\mathbb{N}$. Therefore

$$\begin{aligned}d(x_{n+1},Tz)&\leq H(Tx_n,Tz)\\&\leq\alpha(x_n,z)H(Tx_n,Tz)\\&\leq k\big(d(x_n,z)\big)d(x_n,z)\\&\leq d(x_n,z)\end{aligned}$$

and, taking limit $n\to\infty$, we have $d(z,Tz)=0$ and so $z\in Tz$. □

Although α_*-admissibility implies α-admissibility of T, we will give the following theorem. However, the contractive condition is slightly different from (2.1).

Theorem 5 *Let (X,d) be a complete metric space, and let $T : X \to CB(X)$ be an α_*-admissible multivalued mapping such that*

$$\alpha_*(Tx,Ty)H(Tx,Ty) \le k\big(d(x,y)\big)d(x,y)$$

for all $x,y \in X$, where k is an $\mathcal{MT}$-function. Suppose that there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0,x_1) \ge 1$. If T is continuous or α has (B) *property, then T has a fixed point.*

Proof Define $h(t) = \frac{1+k(t)}{2}$, then from Lemma 1, $h : [0,\infty) \to [0,1)$ is an $\mathcal{MT}$-function. Let x_0 and x_1 be as mentioned in the hypothesis. If $x_0 \in Tx_0$, then x_0 is a fixed point of T. Let $x_0 \notin Tx_0$. Since $x_0 \neq x_1$, then $\frac{1-k(d(x_0,x_1))}{2}d(x_0,x_1) > 0$. If $x_1 \in Tx_1$, x_1 is a fixed point of T. Let $x_1 \notin Tx_1$. Also, since T is α_*-admissible, $\alpha_*(Tx_0,Tx_1) \ge 1$. Therefore, there exists $x_2 \in Tx_1$ such that

$$\begin{aligned} d(x_1,x_2) &\le H(Tx_0,Tx_1) + \frac{1-k(d(x_0,x_1))}{2}d(x_0,x_1) \\ &\le \alpha_*(Tx_0,Tx_1)H(Tx_0,Tx_1) + \frac{1-k(d(x_0,x_1))}{2}d(x_0,x_1) \\ &\le k\big(d(x_0,x_1)\big)d(x_0,x_1) + \frac{1-k(d(x_0,x_1))}{2}d(x_0,x_1) \\ &= \frac{1+k(d(x_0,x_1))}{2}d(x_0,x_1) \\ &= h\big(d(x_0,x_1)\big)d(x_0,x_1). \end{aligned}$$

Since $\alpha(x_1,x_2) \ge \alpha_*(Tx_0,Tx_1) \ge 1$, then $\alpha_*(Tx_1,Tx_2) \ge 1$. Therefore there exists $x_3 \in Tx_2$ such that

$$\begin{aligned} d(x_2,x_3) &\le H(Tx_1,Tx_2) + \frac{1-k(d(x_1,x_2))}{2}d(x_1,x_2) \\ &\le \alpha_*(Tx_1,Tx_2)H(Tx_1,Tx_2) + \frac{1-k(d(x_1,x_2))}{2}d(x_1,x_2) \\ &\le k\big(d(x_1,x_2)\big)d(x_1,x_2) + \frac{1-k(d(x_1,x_2))}{2}d(x_1,x_2) \\ &= \frac{1+k(d(x_1,x_2))}{2}d(x_1,x_2) \\ &= h\big(d(x_1,x_2)\big)d(x_1,x_2). \end{aligned}$$

Again, if $x_2 \in Tx_2$, x_2 is a fixed point of T. Let $x_2 \notin Tx_2$. Since $\alpha(x_2,x_3) \ge \alpha_*(Tx_1,Tx_2) \ge 1$, then $\alpha_*(Tx_2,Tx_3) \ge 1$. In this way, we can construct a sequence $\{x_n\}$ in X such that $x_{n+1} \in Tx_n$, $\alpha(x_n,x_{n+1}) \ge 1$ and

$$d(x_n,x_{n+1}) \le h\big(d(x_{n-1},x_n)\big)d(x_{n-1},x_n)$$

for all $n \in \mathbb{N}$. As in the proof of Theorem 4, we can show that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $z \in X$ such that $\lim_{n\to\infty} x_n = z$.

If T is continuous, then from the inequality $d(x_{n+1}, Tz) \leq H(Tx_n, Tz)$, we have $d(z, Tz) = 0$ and so $z \in Tz$.

Now assume that α has (B) property. Then $\alpha(x_n, z) \geq 1$ for all $n \in \mathbb{N}$. Since T is α_*-admissible, $\alpha_*(Tx_n, Tz) \geq 1$. Therefore

$$\begin{aligned} d(x_{n+1}, Tz) &\leq H(Tx_n, Tz) \\ &\leq \alpha_*(Tx_n, Tz)H(Tx_n, Tz) \\ &\leq k\big(d(x_n, z)\big)d(x_n, z) \\ &\leq d(x_n, z) \end{aligned}$$

and, taking limit $n \to \infty$, we have $d(z, Tz) = 0$ and so $z \in Tz$. □

Now we give an example to illustrate our main theorems. Note that Theorem 1 cannot be applied to this example.

Example 2 Let $X = [-1, 1]$ and $d(x, y) = |x - y|$. Define $T : X \to CB(X)$ by

$$Tx = \begin{cases} \{2x + 1\}, & x \in [-1, -\frac{3}{4}), \\ \{2x - 1\}, & x \in (\frac{3}{4}, 1], \\ [-\frac{1}{2}, \frac{1}{2}], & x \in [-\frac{3}{4}, \frac{3}{4}] \end{cases}$$

and $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1, & x, y \in [-\frac{1}{2}, \frac{1}{2}], \\ 0, & \text{otherwise}. \end{cases}$$

Then T is α_*-admissible and

$$\alpha_*(Tx, Ty)H(Tx, Ty) \leq k\big(d(x, y)\big)d(x, y) \tag{2.2}$$

for all $x, y \in X$, where k is any $\mathcal{MT}$-function. Indeed, first we show that T is α_*-admissible. If $\alpha(x, y) \geq 1$, then $x, y \in [-\frac{1}{2}, \frac{1}{2}]$ and hence

$$\begin{aligned} \alpha_*(Tx, Ty) &= \alpha_*\left(\left[-\frac{1}{2}, \frac{1}{2}\right], \left[-\frac{1}{2}, \frac{1}{2}\right]\right) \\ &= \inf\left\{\alpha(a, b) : a, b \in \left[-\frac{1}{2}, \frac{1}{2}\right]\right\} \\ &= 1. \end{aligned}$$

Therefore T is α_*-admissible.

Now we consider the following cases:

Case 1. Let $x, y \in X$ with $\{x, y\} \cap \{[-1, -\frac{3}{4}) \cup (\frac{3}{4}, 1]\} \neq \emptyset$, then $\alpha_*(Tx, Ty) = 0$. Thus (2.2) is satisfied.

Case 2. Let $x, y \in X$ with $x, y \in [-\frac{3}{4}, \frac{3}{4}]$, then

$$\begin{aligned} H(Tx, Ty) &= H\left(\left[-\frac{1}{2}, \frac{1}{2}\right], \left[-\frac{1}{2}, \frac{1}{2}\right]\right) \\ &= 0 \end{aligned}$$

and so again (2.2) is satisfied.

Now, if $x, y \in (\frac{3}{4}, 1]$ with $x \neq y$, we have

$$\begin{aligned} H(Tx, Ty) &= H\big(\{2x - 1\}, \{2y - 1\}\big) \\ &= 2d(x, y). \end{aligned}$$

Therefore there is no $\mathcal{MT}$-function satisfying (1.1).

Remark 3 If we take $\alpha : X \times X \to [0, \infty)$ by $\alpha(x, y) = 1$, then any multivalued mappings $T : X \to CB(X)$ are α-admissible as well as α_*-admissible. Therefore, Mizoguchi-Takahashi's fixed point theorem is a special case of Theorem 4 and Theorem 5.

We can obtain some ordered fixed point results from our theorems as follows. First we recall some ordered notions. Let X be a nonempty set and $\preceq$ be a partial order on X.

Definition 8 [18] Let A, B be two nonempty subsets of X, the relations between A and B are defined as follows:

(r_1) If for every $a \in A$ there exists $b \in B$ such that $a \preceq b$, then $A \prec_1 B$.
(r_2) If for every $b \in B$ there exists $a \in A$ such that $a \preceq b$, then $A \prec_2 B$.
(r_3) If $A \prec_1 B$ and $A \prec_2 B$, then $A \prec B$.

Remark 4 [18] $\prec_1$ and $\prec_2$ are different relations between A and B. For example, let $X = \mathbb{R}$, $A = [\frac{1}{2}, 1]$, $B = [0, 1]$, $\preceq$ be the usual order on X, then $A \prec_1 B$ but $A \not\prec_2 B$; if $A = [0, 1]$, $B = [0, \frac{1}{2}]$, then $A \prec_2 B$ while $A \not\prec_1 B$.

Remark 5 [18] $\prec_1$, $\prec_2$ and $\prec$ are reflexive and transitive, but are not antisymmetric. For instance, let $X = \mathbb{R}$, $A = [0, 3]$, $B = [0, 1] \cup [2, 3]$, $\preceq$ be the usual order on X, then $A \prec B$ and $B \prec A$, but $A \neq B$. Hence, they are not partial orders.

Corollary 1 *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $T : X \to CB(X)$ be a multivalued mapping such that*

$$H(Tx, Ty) \leq k\big(d(x, y)\big)d(x, y)$$

for all $x, y \in X$ with $x \preceq y$, where k is an $\mathcal{MT}$-function. Suppose that there exists $x_0 \in X$ such that $\{x_0\} \prec_1 Tx_0$. Assume that for each $x \in X$ and $y \in Tx$ with $x \preceq y$, we have $y \preceq z$ for all $z \in Ty$. If T is continuous or X satisfies the following condition:

$$\begin{cases} \{x_n\} \subset X \text{ is a nondecreasing sequence with } x_n \to x \text{ in } X, \\ \text{then } x_n \preceq x \text{ for all } n, \end{cases} \tag{2.3}$$

then T has a fixed point.

Proof Define the mapping $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x,y) = \begin{cases} 1, & x \preceq y, \\ 0, & \text{otherwise.} \end{cases}$$

Then we have

$$\alpha(x,y)H(Tx,Ty) \leq k\big(d(x,y)\big)d(x,y)$$

for all $x, y \in X$. Also, since $\{x_0\} \prec_1 Tx_0$, then there exists $x_1 \in Tx_0$ such that $x_0 \preceq x_1$ and so $\alpha(x_0, x_1) \geq 1$. Now let $x \in X$ and $y \in Tx$ with $\alpha(x,y) \geq 1$, then $x \preceq y$ and so, by the hypotheses, we have $y \preceq z$ for all $z \in Ty$. Therefore, $\alpha(y,z) \geq 1$ for all $z \in Ty$. This shows that T is α-admissible. Finally, if T is continuous or X satisfies (2.3), then T is continuous or α has (B) property. Therefore, from Theorem 4, T has a fixed point. □

Remark 6 We can give a similar corollary using $\prec_2$ instead of $\prec_1$.

Competing interests
The authors declare that they have no competing interest.

Authors' contributions
The authors read and approved the final manuscript.

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