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New general integral inequalities for quasi-geometrically convex functions via fractional integrals

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Abstract

In this paper, the author introduces the concept of the quasi-geometrically convex functions, gives Hermite-Hadamard's inequalities for GA-convex functions in fractional integral forms and defines a new identity for fractional integrals. By using this identity, the author obtains new estimates on generalization of Hadamard *et al.* type inequalities for quasi-geometrically convex functions via Hadamard fractional integrals.

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1 Introduction

Let a real function f be defined on some nonempty interval I of a real line $\mathbb{R}$. The function f is said to be convex on I if inequality

$$f\big(tx+(1-t)y\big)\leq tf(x)+(1-t)f(y) \tag{1}$$

holds for all $x,y\in I$ and $t\in[0,1]$.

We recall that the notion of quasi-convex function generalizes the notion of convex function. More exactly, a function $f:[a,b]\subset\mathbb{R}\to\mathbb{R}$ is said to be quasi-convex on $[a,b]$ if

$$f\big(tx+(1-t)y\big)\leq\max\big\{f(x),f(y)\big\}$$

for all $x,y\in[a,b]$ and $t\in[0,1]$. Clearly, any convex function is a quasi-convex function. Furthermore, there exist quasi-convex functions which are not convex (see [1]).

The following inequalities are well known in the literature as the Hermite-Hadamard inequality, the Ostrowski inequality and the Simpson inequality, respectively.

Theorem 1.1 *Let $f:I\subseteq\mathbb{R}\to\mathbb{R}$ be a convex function defined on the interval I of real numbers, and let $a,b\in I$ with $a<b$. The following double inequality holds*:

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)\,dx\leq\frac{f(a)+f(b)}{2}. \tag{2}$$

Theorem 1.2 *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a mapping differentiable in I°, the interior of I, and let $a, b \in I^\circ$ with $a < b$. If $|f'(x)| \leq M$, $x \in [a, b]$, then the following inequality holds*:

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \leq \frac{M}{b-a} \left[\frac{(x-a)^2 + (b-x)^2}{2} \right]$$

for all $x \in [a, b]$.

Theorem 1.3 *Let $f : [a, b] \to \mathbb{R}$ be a four times continuously differentiable mapping on (a, b) and $\|f^{(4)}\|_\infty = \sup_{x \in (a,b)} |f^{(4)}(x)| < \infty$. Then the following inequality holds*:

$$\left| \frac{1}{3} \left[\frac{f(a) + f(b)}{2} + 2f\left(\frac{a+b}{2} \right) \right] - \frac{1}{b-a} \int_a^b f(x)\,dx \right| \leq \frac{1}{2{,}880} \|f^{(4)}\|_\infty (b-a)^4.$$

The following definitions are well known in the literature.

Definition 1.1 ([2, 3]) A function $f : I \subseteq (0, \infty) \to \mathbb{R}$ is said to be GA-convex (geometric-arithmetically convex) if

$$f\left(x^t y^{1-t}\right) \leq tf(x) + (1-t)f(y)$$

for all $x, y \in I$ and $t \in [0, 1]$.

Definition 1.2 ([2, 3]) A function $f : I \subseteq (0, \infty) \to (0, \infty)$ is said to be GG-convex (called in [4] a geometrically convex function) if

$$f\left(x^t y^{1-t}\right) \leq f(x)^t f(y)^{(1-t)}$$

for all $x, y \in I$ and $t \in [0, 1]$.

We will now give definitions of the right-hand side and left-hand side Hadamard fractional integrals which are used throughout this paper.

Definition 1.3 Let $f \in L[a, b]$. The right-hand side and left-hand side Hadamard fractional integrals $J^\alpha_{a+}f$ and $J^\alpha_{b-}f$ of order $\alpha > 0$ with $b > a \geq 0$ are defined by

$$J^\alpha_{a+}f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left(\ln \frac{x}{t} \right)^{\alpha-1} f(t) \frac{dt}{t}, \quad a < x < b$$

and

$$J^\alpha_{b-}f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \left(\ln \frac{t}{x} \right)^{\alpha-1} f(t) \frac{dt}{t}, \quad a < x < b,$$

respectively, where $\Gamma(\alpha)$ is the Gamma function defined by $\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1}\,dt$ (see [5]).

In recent years, many authors have studied error estimations for Hermite-Hadamard, Ostrowski and Simpson inequalities; for refinements, counterparts, generalization see [4, 6–20].

In this paper, the concept of the quasi-geometrically convex function is introduced, Hermite-Hadamard's inequalities for GA-convex functions in fractional integral forms are established, and a new identity for Hadamard fractional integrals is defined. By using this identity, author obtains a generalization of Hadamard, Ostrowski and Simpson type inequalities for quasi-geometrically convex functions via Hadamard fractional integrals.

2 Main results

Let $f : I \subseteq (0,\infty) \to \mathbb{R}$ be a differentiable function on I°, the interior of I, throughout this section we will take

$$\begin{aligned} &I_f(x,\lambda,\alpha,a,b) \\ &\quad = (1-\lambda)\left[\ln^\alpha \frac{x}{a} + \ln^\alpha \frac{b}{x}\right] f(x) + \lambda\left[f(a)\ln^\alpha \frac{x}{a} + f(b)\ln^\alpha \frac{b}{x}\right] \\ &\qquad - \Gamma(\alpha+1)\left[J^\alpha_{x-}f(a) + J^\alpha_{x+}f(b)\right], \end{aligned}$$

where $a,b \in I$ with $a<b$, $x \in [a,b]$, $\lambda \in [0,1]$, $\alpha > 0$ and Γ is the Euler Gamma function.

Definition 2.1 A function $f : I \subseteq (0,\infty) \to \mathbb{R}$ is said to be quasi-geometrically convex on I if

$$f\left(x^t y^{1-t}\right) \le \sup\left\{f(x), f(y)\right\},$$

for any $x, y \in I$ and $t \in [0,1]$.

Remark 2.1 Clearly, any GA-convex and geometrically convex functions are quasi-geometrically convex functions. Furthermore, there exist quasi-geometrically convex functions which are neither GA-convex nor geometrically convex. In that context, we point out an elementary example. The function $f : (0,4] \to \mathbb{R}$,

$$f(x) = \begin{cases} 1, & x \in (0,1], \\ (x-2)^2, & x \in [1,4] \end{cases}$$

is neither GA-convex nor geometrically convex on $(0,4]$, but it is a quasi-geometrically convex function on $(0,4]$.

Proposition 2.1 *If $f : I \subseteq (0,\infty) \to \mathbb{R}$ is convex and nondecreasing, then it is quasi-geometrically convex on I.*

Proof This follows from

$$\begin{aligned} f\left(x^t y^{1-t}\right) &\le f\left(tx + (1-t)y\right) \\ &\le tf(x) + (1-t)f(y) \le \sup\left\{f(x), f(y)\right\}, \end{aligned}$$

for all $x, y \in I$ and $t \in [0,1]$. $\square$

Proposition 2.2 *If $f: I \subseteq (0,\infty) \to \mathbb{R}$ is quasi-convex and nondecreasing, then it is quasi-geometrically convex on I. If $f: I \subseteq (0,\infty) \to \mathbb{R}$ is quasi-geometrically convex and nonincreasing, then it is quasi-convex on I.*

Proof These conclusions follows from

$$f(x^t y^{1-t}) \le f(tx + (1-t)y) \le \sup\{f(x), f(y)\}$$

and

$$f(tx + (1-t)y) \le f(x^t y^{1-t}) \le \sup\{f(x), f(y)\}$$

for all $x, y \in I$ and $t \in [0,1]$, respectively. □

Hermite-Hadamard's inequalities can be represented for GA-convex functions in fractional integral forms as follows.

Theorem 2.1 *Let $f: I \subseteq (0,\infty) \to \mathbb{R}$ be a function such that $f \in L[a,b]$, where $a, b \in I$ with $a < b$. If f is a GA-convex function on $[a,b]$, then the following inequalities for fractional integrals hold:*

$$f(\sqrt{ab}) \le \frac{\Gamma(\alpha+1)}{2(\ln\frac{b}{a})^\alpha}\{J^\alpha_{a+}f(b) + J^\alpha_{b-}f(a)\} \le \frac{f(a)+f(b)}{2} \tag{3}$$

with $\alpha > 0$.

Proof Since f is a GA-convex function on $[a,b]$, we have for all $x, y \in [a,b]$ (with $t = 1/2$ in inequality (1)),

$$f(\sqrt{xy}) \le \frac{f(x)+f(y)}{2}.$$

Choosing $x = a^t b^{1-t}$, $y = b^t a^{1-t}$, we get

$$f(\sqrt{ab}) \le \frac{f(a^t b^{1-t}) + f(b^t a^{1-t})}{2}. \tag{4}$$

Multiplying both sides of (4) by $t^{\alpha-1}$, then integrating the resulting inequality with respect to t over $[0,1]$, we obtain

$$\begin{aligned} f(\sqrt{ab}) &\le \frac{\alpha}{2}\left\{\int_0^1 f(a^t b^{1-t})\,dt + \int_0^1 f(b^t a^{1-t})\,dt\right\} \\ &= \frac{\alpha}{2}\left\{\int_a^b \left(\frac{\ln b - \ln u}{\ln b - \ln a}\right)^{\alpha-1} f(u)\frac{du}{u\ln\frac{b}{a}} + \int_a^b \left(\frac{\ln u - \ln a}{\ln b - \ln a}\right)^{\alpha-1} f(u)\frac{du}{u\ln\frac{b}{a}}\right\} \\ &= \frac{\alpha\Gamma(\alpha)}{2(\ln\frac{b}{a})^\alpha}\{J^\alpha_{a+}f(b) + J^\alpha_{b-}f(a)\} \\ &= \frac{\Gamma(\alpha+1)}{2(\ln\frac{b}{a})^\alpha}\{J^\alpha_{a+}f(b) + J^\alpha_{b-}f(a)\}, \end{aligned}$$

and the first inequality is proved.

For the proof of the second inequality in (3), we first note that if f is a convex function, then for $t \in [0,1]$, it yields

$$f\left(a^t b^{1-t}\right) \leq tf(a) + (1-t)f(b)$$

and

$$f\left(b^t a^{1-t}\right) \leq tf(b) + (1-t)f(a).$$

By adding these inequalities, we have

$$f\left(a^t b^{1-t}\right) + f\left(b^t a^{1-t}\right) \leq f(a) + f(b). \tag{5}$$

Then multiplying both sides of (5) by $t^{\alpha-1}$, and integrating the resulting inequality with respect to t over $[0,1]$, we obtain

$$\int_0^1 f\left(a^t b^{1-t}\right) t^{\alpha-1}\,dt + \int_0^1 f\left(b^t a^{1-t}\right) t^{\alpha-1}\,dt \leq \left[f(a) + f(b)\right] \int_0^1 t^{\alpha-1}\,dt,$$

i.e.,

$$\frac{\Gamma(\alpha+1)}{(\ln\frac{b}{a})^\alpha}\left\{J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)\right\} \leq f(a) + f(b).$$

The proof is completed. □

In order to prove our main results, we need the following identity.

Lemma 2.1 *Let $f : I \subseteq (0,\infty) \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. Then for all $x \in [a,b]$, $\lambda \in [0,1]$ and $\alpha > 0$, we have*:

$$\begin{aligned} I_f(x,\lambda,\alpha,a,b) &= a\left(\ln\frac{x}{a}\right)^{\alpha+1}\int_0^1 \left(t^\alpha - \lambda\right)\left(\frac{x}{a}\right)^t f'\left(x^t a^{1-t}\right)dt \\ &\quad - b\left(\ln\frac{b}{x}\right)^{\alpha+1}\int_0^1 \left(t^\alpha - \lambda\right)\left(\frac{x}{b}\right)^t f'\left(x^t b^{1-t}\right)dt. \end{aligned} \tag{6}$$

Proof By integration by parts and twice changing the variable, for $x \neq a$, we can state that

$$\begin{aligned} & a\ln\frac{x}{a}\int_0^1 \left(t^\alpha - \lambda\right)\left(\frac{x}{a}\right)^t f'\left(x^t a^{1-t}\right)dt \\ &\quad = \int_0^1 \left(t^\alpha - \lambda\right) df\left(x^t a^{1-t}\right) \\ &\quad = \left(t^\alpha - \lambda\right)f\left(x^t a^{1-t}\right)\Big|_0^1 - \frac{\alpha}{(\ln\frac{x}{a})^\alpha}\int_a^x \left(\ln\frac{u}{a}\right)^{\alpha-1}\frac{f(u)}{u}\,du \\ &\quad = (1-\lambda)f(x) + \lambda f(a) - \frac{\Gamma(\alpha+1)}{(\ln\frac{x}{a})^\alpha} J_{x-}^\alpha f(a), \end{aligned} \tag{7}$$

and for $x \neq b$, similarly, we get

$$\begin{aligned}
&-b\ln\frac{b}{x}\int_0^1 \left(t^\alpha-\lambda\right)\left(\frac{x}{b}\right)^t f'\left(x^t b^{1-t}\right)dt \\
&\quad= \int_0^1 \left(t^\alpha-\lambda\right)df\left(x^t b^{1-t}\right) \\
&\quad= \left(t^\alpha-\lambda\right)f\left(x^t b^{1-t}\right)\Big|_0^1 - \frac{\alpha}{(\ln\frac{b}{x})^\alpha}\int_x^b\left(\ln\frac{b}{u}\right)^{\alpha-1}\frac{f(u)}{u}\,du \\
&\quad= (1-\lambda)f(x)+\lambda f(b) - \frac{\Gamma(\alpha+1)}{(\ln\frac{b}{x})^\alpha}J_{x+}^\alpha f(b). \qquad (8)
\end{aligned}$$

Multiplying both sides of (7) and (8) by $(\ln\frac{x}{a})^\alpha$ and $(\ln\frac{b}{x})^\alpha$, respectively, and adding the resulting identities, we obtain the desired result. For $x=a$ and $x=b$, the identities

$$I_f(a,\lambda,\alpha;a,b) = b\left(\ln\frac{b}{a}\right)^{\alpha+1}\int_0^1\left(t^\alpha-\lambda\right)\left(\frac{a}{b}\right)^t f'\left(a^t b^{1-t}\right)dt,$$

and

$$I_f(b,\lambda,\alpha;a,b) = a\left(\ln\frac{b}{a}\right)^{\alpha+1}\int_0^1\left(t^\alpha-\lambda\right)\left(\frac{b}{a}\right)^t f'\left(b^t a^{1-t}\right),$$

can be proved respectively easily by performing an integration by parts in the integrals from the right-hand side and changing the variable. □

Theorem 2.2 *Let $f: I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a,b \in I^\circ$ with $a<b$. If $|f'|^q$ is quasi-geometrically convex on $[a,b]$ for some fixed $q \geq 1$, $x \in [a,b]$, $\lambda \in [0,1]$ and $\alpha > 0$, then the following inequality for fractional integrals holds*:

$$\begin{aligned}
&\left|I_f(x,\lambda,\alpha,a,b)\right| \\
&\quad\leq A_1^{1-\frac{1}{q}}(\alpha,\lambda)\left\{a\left(\ln\frac{x}{a}\right)^{\alpha+1}\left(\sup\left\{\left|f'(x)\right|^q,\left|f'(a)\right|^q\right\}\right)^{\frac{1}{q}}B_1^{\frac{1}{q}}(x,\alpha,\lambda,q)\right. \\
&\qquad\left.+\,b\left(\ln\frac{b}{x}\right)^{\alpha+1}\left(\sup\left\{\left|f'(x)\right|^q,\left|f'(b)\right|^q\right\}\right)^{\frac{1}{q}}B_2^{\frac{1}{q}}(x,\alpha,\lambda,q)\right\}, \qquad (9)
\end{aligned}$$

where

$$\begin{aligned}
A_1(\alpha,\lambda) &= \frac{2\alpha\lambda^{1+\frac{1}{\alpha}}+1}{\alpha+1}-\lambda, \\
B_1(x,\alpha,\lambda,q) &= \int_0^1\left|t^\alpha-\lambda\right|\left(\frac{x}{a}\right)^{qt}dt, \\
B_2(x,\alpha,\lambda,q) &= \int_0^1\left|t^\alpha-\lambda\right|\left(\frac{x}{b}\right)^{qt}dt.
\end{aligned}$$

Proof Since $|f'|^q$ is quasi-geometrically convex on $[a,b]$, for all $t \in [0,1]$,

$$\left|f'\left(x^t a^{1-t}\right)\right|^q \leq \sup\left\{\left|f'(x)\right|^q,\left|f'(a)\right|^q\right\}$$

and

$$\left|f'\left(x^t b^{1-t}\right)\right|^q \le \sup\left\{\left|f'(x)\right|^q, \left|f'(b)\right|^q\right\}.$$

Hence, using Lemma 2.1 and power mean inequality, we get

$$\begin{aligned}
&\left|I_f(x,\lambda,\alpha,a,b)\right| \\
&\quad\le a\left(\ln\frac{x}{a}\right)^{\alpha+1}\left(\int_0^1\left|t^\alpha-\lambda\right|dt\right)^{1-\frac{1}{q}}\left(\int_0^1\left|t^\alpha-\lambda\right|\left(\frac{x}{a}\right)^{qt}\sup\left\{\left|f'(x)\right|^q,\left|f'(a)\right|^q\right\}dt\right)^{\frac{1}{q}} \\
&\qquad + b\left(\ln\frac{b}{x}\right)^{\alpha+1}\left(\int_0^1\left|t^\alpha-\lambda\right|dt\right)^{1-\frac{1}{q}}\left(\int_0^1\left|t^\alpha-\lambda\right|\left(\frac{x}{b}\right)^{qt}\sup\left\{\left|f'(x)\right|^q,\left|f'(b)\right|^q\right\}dt\right)^{\frac{1}{q}}, \\
&\left|I_f(x,\lambda,\alpha,a,b)\right| \le \left(\int_0^1\left|t^\alpha-\lambda\right|dt\right)^{1-\frac{1}{q}} \\
&\qquad\times\left\{a\left(\ln\frac{x}{a}\right)^{\alpha+1}\left(\sup\left\{\left|f'(x)\right|^q,\left|f'(a)\right|^q\right\}\right)^{\frac{1}{q}}\left(\int_0^1\left|t^\alpha-\lambda\right|\left(\frac{x}{a}\right)^{qt}dt\right)^{\frac{1}{q}}\right. \\
&\qquad\left. + b\left(\ln\frac{b}{x}\right)^{\alpha+1}\left(\sup\left\{\left|f'(x)\right|^q,\left|f'(b)\right|^q\right\}\right)^{\frac{1}{q}}\left(\int_0^1\left|t^\alpha-\lambda\right|\left(\frac{x}{b}\right)^{qt}dt\right)^{\frac{1}{q}}\right\} \\
&\qquad\le A_1^{1-\frac{1}{q}}(\alpha,\lambda)\left\{a\left(\ln\frac{x}{a}\right)^{\alpha+1}\left(\sup\left\{\left|f'(x)\right|^q,\left|f'(a)\right|^q\right\}\right)^{\frac{1}{q}}B_1^{\frac{1}{q}}(x,\alpha,\lambda,q)\right. \\
&\qquad\left. + b\left(\ln\frac{b}{x}\right)^{\alpha+1}\left(\sup\left\{\left|f'(x)\right|^q,\left|f'(b)\right|^q\right\}\right)^{\frac{1}{q}}B_2^{\frac{1}{q}}(x,\alpha,\lambda,q)\right\},
\end{aligned}$$

which completes the proof. □

Corollary 2.1 *Under the assumptions of Theorem* 2.2 *with* $q = 1$, *inequality* (9) *reduces to the following inequality*:

$$\begin{aligned}
\left|I_f(x,\lambda,\alpha,a,b)\right| \le{}& \left\{a\left(\ln\frac{x}{a}\right)^{\alpha+1}B_1(x,\alpha,\lambda,1)\sup\left\{\left|f'(x)\right|,\left|f'(a)\right|\right\}\right. \\
&\left. + b\left(\ln\frac{b}{x}\right)^{\alpha+1}B_2(x,\alpha,\lambda,1)\sup\left\{\left|f'(x)\right|,\left|f'(b)\right|\right\}\right\}.
\end{aligned}$$

Corollary 2.2 *Under the assumptions of Theorem* 2.2 *with* $\alpha = 1$, *inequality* (9) *reduces to the following inequality*:

$$\begin{aligned}
&\left(\ln\frac{b}{a}\right)^{-1}\left|I_f(x,\lambda,\alpha,a,b)\right| \\
&\quad\le\left|(1-\lambda)f(x)+\lambda\left[\frac{f(a)\ln\frac{x}{a}+f(b)\ln\frac{b}{x}}{\ln\frac{b}{a}}\right]-\frac{1}{\ln\frac{b}{a}}\int_a^b\frac{f(u)}{u}du\right| \\
&\quad\le\left(\ln\frac{b}{a}\right)^{-1}\left(\frac{2\lambda^2-2\lambda+1}{2}\right)^{1-\frac{1}{q}}\left\{a\left(\ln\frac{x}{a}\right)^2B_1^{\frac{1}{q}}(x,1,\lambda,q)\left(\sup\left\{\left|f'(x)\right|^q,\left|f'(a)\right|^q\right\}\right)^{\frac{1}{q}}\right. \\
&\qquad\left. + b\left(\ln\frac{b}{x}\right)^2B_2^{\frac{1}{q}}(x,1,\lambda,q)\left(\sup\left\{\left|f'(x)\right|^q,\left|f'(b)\right|^q\right\}\right)^{\frac{1}{q}}\right\},
\end{aligned}$$

where

$$B_1(x,1,\lambda,q) = h_\lambda\left(\left(\frac{x}{a}\right)^q\right), \qquad B_2(x,1,\lambda,q) = h_\lambda\left(\left(\frac{x}{b}\right)^q\right),$$
$$h(u,\lambda) = \frac{2u^\lambda - u - 1}{(\ln u)^2} + \frac{(1-\lambda)u - \lambda}{\ln u}, \quad u \in (0,\infty)\backslash\{1\}, \tag{10}$$

specially for $x = \sqrt{ab}$, we get

$$\left|(1-\lambda)f(\sqrt{ab}) + \lambda\left(\frac{f(a)+f(b)}{2}\right) - \frac{1}{\ln\frac{b}{a}}\int_a^b \frac{f(u)}{u}\,du\right|$$
$$\leq \frac{\ln\frac{b}{a}}{4}\left(\frac{2\lambda^2 - 2\lambda + 1}{2}\right)^{1-\frac{1}{q}}\left\{ah^{\frac{1}{q}}\left(\left(\frac{b}{a}\right)^{\frac{q}{2}},\lambda\right)\left(\sup\left\{\left|f'(\sqrt{ab})\right|^q, \left|f'(a)\right|^q\right\}\right)^{\frac{1}{q}}\right.$$
$$\left. + bh^{\frac{1}{q}}\left(\left(\frac{a}{b}\right)^{\frac{q}{2}},\lambda\right)\left(\sup\left\{\left|f'(\sqrt{ab})\right|^q, \left|f'(b)\right|^q\right\}\right)^{\frac{1}{q}}\right\}. \tag{11}$$

Corollary 2.3 *In Theorem* 2.2,

1. *If we take $x = \sqrt{ab}$, $\lambda = \frac{1}{3}$, then we get the following Simpson-type inequality for fractional integrals*:

$$\left|\frac{1}{6}\left[f(a) + 4f(\sqrt{ab}) + f(b)\right] - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\ln\frac{b}{a})^\alpha}\left[J^\alpha_{\sqrt{ab}-}f(a) + J^\alpha_{\sqrt{ab}+}f(b)\right]\right|$$
$$\leq \frac{\ln\frac{b}{a}}{4}A_1^{1-\frac{1}{q}}\left(\alpha,\frac{1}{3}\right)\left\{a\left[\sup\left\{\left|f'(\sqrt{ab})\right|^q, \left|f'(a)\right|^q\right\}\right]^{\frac{1}{q}}B_1^{\frac{1}{q}}\left(\sqrt{ab},\alpha,\frac{1}{3},q\right)\right.$$
$$\left. + b\left[\sup\left\{\left|f'(\sqrt{ab})\right|^q, \left|f'(b)\right|^q\right\}\right]^{\frac{1}{q}}B_2^{\frac{1}{q}}\left(\sqrt{ab},\alpha,\frac{1}{3},q\right)\right\},$$

specially for $\alpha = 1$, we get

$$\left|\frac{1}{6}\left[f(a) + 4f(\sqrt{ab}) + f(b)\right] - \frac{1}{\ln\frac{b}{a}}\int_a^b \frac{f(u)}{u}\,du\right|$$
$$\leq \frac{\ln\frac{b}{a}}{4}\left(\frac{5}{18}\right)^{1-\frac{1}{q}}\left\{a\left[\sup\left\{\left|f'(\sqrt{ab})\right|, \left|f'(a)\right|\right\}\right]^{\frac{1}{q}}h^{\frac{1}{q}}\left(\left(\frac{b}{a}\right)^{\frac{q}{2}},\frac{1}{3}\right)\right.$$
$$\left. + b\left[\sup\left\{\left|f'(\sqrt{ab})\right|^q, \left|f'(b)\right|^q\right\}\right]^{\frac{1}{q}}h^{\frac{1}{q}}\left(\left(\frac{a}{b}\right)^{\frac{q}{2}},\frac{1}{3}\right)\right\},$$

where h is defined as in (10).

Remark 2.2

1. If we take $x = \sqrt{ab}$, $\lambda = 0$, then we get the following midpoint- type inequality for fractional integrals:

$$\left|f(\sqrt{ab}) - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\ln\frac{b}{a})^\alpha}\left[J^\alpha_{\sqrt{ab}-}f(a) + J^\alpha_{\sqrt{ab}+}f(b)\right]\right|$$
$$\leq \frac{\ln\frac{b}{a}}{4}\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\left\{a\left[\sup\left\{\left|f'(\sqrt{ab})\right|^q, \left|f'(a)\right|^q\right\}\right]^{\frac{1}{q}}B_1^{\frac{1}{q}}(\sqrt{ab},1,0,q)\right.$$
$$\left. + b\left[\sup\left\{\left|f'(\sqrt{ab})\right|^q, \left|f'(b)\right|^q\right\}\right]^{\frac{1}{q}}B_2^{\frac{1}{q}}(\sqrt{ab},1,0,q)\right\},$$

specially for $\alpha = 1$, we get

$$\begin{aligned}
&\left| f(\sqrt{ab}) - \frac{1}{\ln\frac{b}{a}} \int_a^b \frac{f(u)}{u}\,du \right| \\
&\quad \leq \frac{2^{\frac{1}{q}} \ln\frac{b}{a}}{8} \left\{ a\left[\sup\left\{\left|f'(\sqrt{ab})\right|^q, \left|f'(a)\right|^q\right\}\right]^{\frac{1}{q}} h^{\frac{1}{q}}\left(\left(\frac{b}{a}\right)^{\frac{q}{2}}, 0\right) \right. \\
&\qquad \left. + b\left[\sup\left\{\left|f'(\sqrt{ab})\right|^q, \left|f'(b)\right|^q\right\}\right]^{\frac{1}{q}} h^{\frac{1}{q}}\left(\left(\frac{a}{b}\right)^{\frac{q}{2}}, 0\right) \right\},
\end{aligned}$$

where h is defined as in (10).

2. If we take $x = \sqrt{ab}$, $\lambda = 1$, then we get the following trapezoid-type inequality for fractional integrals:

$$\begin{aligned}
&\left| \frac{f(a)+f(b)}{2} - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\ln\frac{b}{a})^\alpha}\left[J^\alpha_{\sqrt{ab}-} f(a) + J^\alpha_{\sqrt{ab}+} f(b)\right] \right| \\
&\quad \leq \frac{\ln\frac{b}{a}}{4}\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}} \left\{ a\left[\sup\left\{\left|f'(\sqrt{ab})\right|^q, \left|f'(a)\right|^q\right\}\right]^{\frac{1}{q}} B_1^{\frac{1}{q}}(\sqrt{ab}, \alpha, 1, q) \right. \\
&\qquad \left. + b\left[\sup\left\{\left|f'(\sqrt{ab})\right|^q, \left|f'(b)\right|^q\right\}\right]^{\frac{1}{q}} B_2^{\frac{1}{q}}(\sqrt{ab}, \alpha, 1, q) \right\},
\end{aligned}$$

specially for $\alpha = 1$, we get

$$\begin{aligned}
&\left| \frac{f(a)+f(b)}{2} - \frac{1}{\ln\frac{b}{a}} \int_a^b \frac{f(u)}{u}\,du \right| \\
&\quad \leq \frac{2^{\frac{1}{q}} \ln\frac{b}{a}}{8} \left\{ a\left[\sup\left\{\left|f'(\sqrt{ab})\right|^q, \left|f'(a)\right|^q\right\}\right]^{\frac{1}{q}} h^{\frac{1}{q}}\left(\left(\frac{b}{a}\right)^{\frac{q}{2}}, 1\right) \right. \\
&\qquad \left. + b\left[\sup\left\{\left|f'(\sqrt{ab})\right|^q, \left|f'(b)\right|^q\right\}\right]^{\frac{1}{q}} h^{\frac{1}{q}}\left(\left(\frac{a}{b}\right)^{\frac{q}{2}}, 1\right) \right\},
\end{aligned}$$

where h is defined as in (10).

Corollary 2.4 *Let the assumptions of Theorem* 2.2 *hold. If* $|f'(x)| \leq M$ *for all* $x \in [a,b]$ *and* $\lambda = 0$, *then we get the following Ostrowski-type inequality for fractional integrals from inequality* (9):

$$\begin{aligned}
&\left| \left[\left(\ln\frac{x}{a}\right)^\alpha + \left(\ln\frac{b}{x}\right)^\alpha\right] f(x) - \Gamma(\alpha+1)\left[J^\alpha_{\sqrt{ab}-} f(a) + J^\alpha_{\sqrt{ab}+} f(b)\right] \right| \\
&\quad \leq \frac{M}{(\alpha+1)^{1-\frac{1}{q}}} \left[a\left(\ln\frac{x}{a}\right)^{\alpha+1} B_1^{\frac{1}{q}}(x, \alpha, 0, q) + b\left(\ln\frac{b}{x}\right)^{\alpha+1} B_2^{\frac{1}{q}}(x, \alpha, 0, q) \right].
\end{aligned}$$

Theorem 2.3 *Let* $f : I \subset (0,\infty) \to \mathbb{R}$ *be a differentiable function on* I° *such that* $f' \in L[a,b]$, *where* $a, b \in I^\circ$ *with* $a < b$. *If* $|f'|^q$ *is quasi-geometrically convex on* $[a,b]$ *for some fixed* $q > 1$, $x \in [a,b]$, $\lambda \in [0,1]$ *and* $\alpha > 0$, *then the following inequality for fractional inte-*

grals holds:

$$\begin{aligned}
&\left|I_f(x,\lambda,\alpha,a,b)\right| \\
&\quad\le A_2^{\frac{1}{p}}(\alpha,\lambda,p)\left\{a\left(\ln\frac{x}{a}\right)^{\alpha+\frac{1}{p}}\left(\sup\left\{\left|f'(x)\right|^q,\left|f'(a)\right|^q\right\}\right)^{\frac{1}{q}}\left(\frac{x^q-a^q}{q}\right)^{\frac{1}{q}}\right. \\
&\qquad\left.+\,b\left(\ln\frac{b}{x}\right)^{\alpha+\frac{1}{p}}\left(\sup\left\{\left|f'(x)\right|^q,\left|f'(b)\right|^q\right\}\right)^{\frac{1}{q}}\left(\frac{b^q-x^q}{q}\right)^{\frac{1}{q}}\right\}, \qquad (12)
\end{aligned}$$

where

$$\begin{aligned}
&A_2(\alpha,\lambda,p) \\
&\quad=\begin{cases}
\frac{1}{\alpha p+1}, & \lambda=0,\\
\frac{\lambda^{\frac{\alpha p+1}{\alpha}}}{\alpha}\{\beta(\frac{1}{\alpha},p+1)+\frac{(1-\lambda)^{p+1}}{p+1} & \\
\quad\times{}_2F_1(\frac{1}{\alpha}+p+1,p+1,p+2;1-\lambda)\}, & 0<\lambda<1,\\
\frac{1}{\alpha}\beta(p+1,\frac{1}{\alpha}), & \lambda=1,
\end{cases}
\end{aligned}$$

${}_2F_1$ *is hypergeometric function defined by*

$$ {}_2F_1(a,b;c;z)=\frac{1}{\beta(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\,dt,\quad c>b>0,|z|<1\ (\textit{see}\ [21]),$$

and $\frac{1}{p}+\frac{1}{q}=1$.

Proof Using Lemma 2.1, the Hölder inequality and quasi-geometrical convexity of $|f'|^q$, we get

$$\begin{aligned}
&\left|I_f(x,\lambda,\alpha,a,b)\right| \\
&\quad\le a\left(\ln\frac{x}{a}\right)^{\alpha+1}\left(\int_0^1\left|t^\alpha-\lambda\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left(\frac{x}{a}\right)^{qt}\sup\left\{\left|f'(x)\right|^q,\left|f'(a)\right|^q\right\}dt\right)^{\frac{1}{q}} \\
&\qquad+b\left(\ln\frac{b}{x}\right)^{\alpha+1}\left(\int_0^1\left|t^\alpha-\lambda\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left(\frac{x}{b}\right)^{qt}\sup\left\{\left|f'(x)\right|^q,\left|f'(b)\right|^q\right\}dt\right)^{\frac{1}{q}}, \\
&\left|I_f(x,\lambda,\alpha,a,b)\right|\le\left(\int_0^1\left|t^\alpha-\lambda\right|^p dt\right)^{\frac{1}{p}} \\
&\qquad\times\left\{a\left(\ln\frac{x}{a}\right)^{\alpha+1}\left(\sup\left\{\left|f'(x)\right|^q,\left|f'(a)\right|^q\right\}\right)^{\frac{1}{q}}\left(\int_0^1\left(\frac{x}{a}\right)^{qt}dt\right)^{\frac{1}{q}}\right. \\
&\qquad\left.+\,b\left(\ln\frac{b}{x}\right)^{\alpha+1}\left(\sup\left\{\left|f'(x)\right|^q,\left|f'(b)\right|^q\right\}\right)^{\frac{1}{q}}\left(\int_0^1\left(\frac{x}{b}\right)^{qt}dt\right)^{\frac{1}{q}}\right\} \\
&\quad\le A_2^{\frac{1}{p}}(\alpha,\lambda,p)\left\{a\left(\ln\frac{x}{a}\right)^{\alpha+1-\frac{1}{q}}\left(\sup\left\{\left|f'(x)\right|^q,\left|f'(a)\right|^q\right\}\right)^{\frac{1}{q}}\left(\frac{x^q-a^q}{q}\right)^{\frac{1}{q}}\right. \\
&\qquad\left.+\,b\left(\ln\frac{b}{x}\right)^{\alpha+1-\frac{1}{q}}\left(\sup\left\{\left|f'(x)\right|^q,\left|f'(b)\right|^q\right\}\right)^{\frac{1}{q}}\left(\frac{b^q-x^q}{q}\right)^{\frac{1}{q}}\right\},
\end{aligned}$$

here, it is seen by a simple computation that

$$
\begin{aligned}
A_2(\alpha,\lambda,p) &= \int_0^1 \left|t^\alpha - \lambda\right|^p \, dt \\
&= \begin{cases}
\frac{1}{\alpha p+1}, & \lambda = 0, \\
\frac{\lambda^{\frac{\alpha p+1}{\alpha}}}{\alpha}\{\beta(\frac{1}{\alpha}, p+1) + \frac{(1-\lambda)^{p+1}}{p+1} & \\
\quad \times {}_2F_1(\frac{1}{\alpha}+p+1, p+1, 2+p; 1-\lambda)\}, & 0 < \lambda < 1, \\
\frac{1}{\alpha}\beta(p+1, \frac{1}{\alpha}), & \lambda = 1.
\end{cases}
\end{aligned}
$$

Hence, the proof is completed. □

Corollary 2.5 *Under the assumptions of Theorem* 2.3 *with* $\alpha = 1$, *inequality* (12) *reduces to the following inequality*:

$$
\begin{aligned}
&\left|(1-\lambda)f(x) + \lambda\left[\frac{f(a)\ln\frac{x}{a} + f(b)\ln\frac{b}{x}}{\ln\frac{b}{a}}\right] - \frac{1}{\ln\frac{b}{a}}\int_a^b \frac{f(u)}{u}\, du\right| \\
&\quad \le \left(\ln\frac{b}{a}\right)^{-1}\left(\frac{\lambda^{p+1} + (1-\lambda)^{p+1}}{p+1}\right)^{\frac{1}{p}} \\
&\qquad \times \left\{a\left(\ln\frac{x}{a}\right)^{1+\frac{1}{p}}\left(\sup\left\{\left|f'(x)\right|^q, \left|f'(a)\right|^q\right\}\right)^{\frac{1}{q}}\left(\frac{x^q - a^q}{q}\right)^{\frac{1}{q}}\right. \\
&\qquad \left. + b\left(\ln\frac{b}{x}\right)^{1+\frac{1}{p}}\left(\sup\left\{\left|f'(x)\right|^q, \left|f'(b)\right|^q\right\}\right)^{\frac{1}{q}}\left(\frac{b^q - x^q}{q}\right)^{\frac{1}{q}}\right\},
\end{aligned}
$$

specially for $x = \sqrt{ab}$, *we get*

$$
\begin{aligned}
&\left|(1-\lambda)f(\sqrt{ab}) + \lambda\left(\frac{f(a)+f(b)}{2}\right) - \frac{1}{\ln\frac{b}{a}}\int_a^b \frac{f(u)}{u}\, du\right| \\
&\quad \le \frac{1}{2}\left(\frac{\ln\frac{b}{a}(\lambda^{p+1} + (1-\lambda)^{p+1})}{2(p+1)}\right)^{\frac{1}{p}}\left\{a\left(\sup\left\{\left|f'(\sqrt{ab})\right|^q, \left|f'(a)\right|^q\right\}\right)^{\frac{1}{q}}\left(\frac{\sqrt{ab}^q - a^q}{q}\right)^{\frac{1}{q}}\right. \\
&\qquad \left. + b\left(\sup\left\{\left|f'(\sqrt{ab})\right|^q, \left|f'(b)\right|^q\right\}\right)^{\frac{1}{q}}\left(\frac{b^q - \sqrt{ab}^q}{q}\right)^{\frac{1}{q}}\right\}. \qquad (13)
\end{aligned}
$$

Corollary 2.6 *In Theorem* 2.3,

1. *If we take* $x = \sqrt{ab}$, $\lambda = \frac{1}{3}$, *then we get the following Simpson-type inequality for fractional integrals*:

$$
\begin{aligned}
&\left|\frac{1}{6}\left[f(a) + 4f(\sqrt{ab}) + f(b)\right] - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\ln\frac{b}{a})^\alpha}\left[J^\alpha_{\sqrt{ab}-}f(a) + J^\alpha_{\sqrt{ab}+}f(b)\right]\right| \\
&\quad \le \frac{1}{2}\left(\frac{\ln\frac{b}{a}(1+2^{p+1})}{3^{p+1}(p+1)2}\right)^{\frac{1}{p}}\left\{a\left[\sup\left\{\left|f'(\sqrt{ab})\right|^q, \left|f'(a)\right|^q\right\}\right]^{\frac{1}{q}}\left(\frac{\sqrt{ab}^q - a^q}{q}\right)^{\frac{1}{q}}\right. \\
&\qquad \left. + b\left[\sup\left\{\left|f'(\sqrt{ab})\right|^q, \left|f'(b)\right|^q\right\}\right]^{\frac{1}{q}}\left(\frac{b^q - \sqrt{ab}^q}{q}\right)^{\frac{1}{q}}\right\},
\end{aligned}
$$

specially for $\alpha = 1$, *we get*

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f(\sqrt{ab})+f(b)\right]-\frac{1}{\ln\frac{b}{a}}\int_a^b\frac{f(u)}{u}\,du\right| \\
&\quad\le\frac{1}{2}\left(\frac{\ln\frac{b}{a}(1+2^{p+1})}{3^{p+1}(p+1)2}\right)^{\frac{1}{p}} \\
&\qquad\times\left\{a\left[\sup\left\{\left|f'(\sqrt{ab})\right|^q,\left|f'(a)\right|^q\right\}\right]^{\frac{1}{q}}\left(\frac{\sqrt{ab}^q-a^q}{q}\right)^{\frac{1}{q}}\right. \\
&\qquad\left.+b\left[\sup\left\{\left|f'(\sqrt{ab})\right|^q,\left|f'(b)\right|^q\right\}\right]^{\frac{1}{q}}\left(\frac{b^q-\sqrt{ab}^q}{q}\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

Remark 2.3

1. If we take $x = \sqrt{ab}$, $\lambda = 0$, then we get the following midpoint- type inequality for fractional integrals:

$$\begin{aligned}
&\left|f(\sqrt{ab})-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\ln\frac{b}{a})^\alpha}\left[J^\alpha_{\sqrt{ab}-}f(a)+J^\alpha_{\sqrt{ab}+}f(b)\right]\right| \\
&\quad\le\frac{1}{2}\left(\frac{\ln\frac{b}{a}}{2(\alpha p+1)}\right)^{\frac{1}{p}}\left\{a\left[\sup\left\{\left|f'(\sqrt{ab})\right|^q,\left|f'(a)\right|^q\right\}\right]^{\frac{1}{q}}\left(\frac{\sqrt{ab}^q-a^q}{q}\right)^{\frac{1}{q}}\right. \\
&\qquad\left.+b\left[\sup\left\{\left|f'(\sqrt{ab})\right|^q,\left|f'(b)\right|^q\right\}\right]^{\frac{1}{q}}\left(\frac{b^q-\sqrt{ab}^q}{q}\right)^{\frac{1}{q}}\right\},
\end{aligned}$$

specially for $\alpha = 1$, we get

$$\begin{aligned}
&\left|f(\sqrt{ab})-\frac{1}{\ln\frac{b}{a}}\int_a^b\frac{f(u)}{u}\,du\right| \\
&\quad\le\frac{1}{2}\left(\frac{\ln\frac{b}{a}}{2(p+1)}\right)^{\frac{1}{p}}\left\{a\left[\sup\left\{\left|f'(\sqrt{ab})\right|^q,\left|f'(a)\right|^q\right\}\right]^{\frac{1}{q}}\left(\frac{\sqrt{ab}^q-a^q}{q}\right)^{\frac{1}{q}}\right. \\
&\qquad\left.+b\left[\sup\left\{\left|f'(\sqrt{ab})\right|^q,\left|f'(b)\right|^q\right\}\right]^{\frac{1}{q}}\left(\frac{b^q-\sqrt{ab}^q}{q}\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

2. If we take $x = \sqrt{ab}$, $\lambda = 1$, then we get the following trapezoid-type inequality for fractional integrals $\frac{1}{\alpha}\beta(p+1,\frac{1}{\alpha})$:

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\ln\frac{b}{a})^\alpha}\left[J^\alpha_{\sqrt{ab}-}f(a)+J^\alpha_{\sqrt{ab}+}f(b)\right]\right| \\
&\quad\le\frac{1}{2}\left(\frac{\ln\frac{b}{a}\beta(p+1,\frac{1}{\alpha})}{2\alpha}\right)^{\frac{1}{p}} \\
&\qquad\times\left\{a\left[\sup\left\{\left|f'(\sqrt{ab})\right|^q,\left|f'(a)\right|^q\right\}\right]^{\frac{1}{q}}\left(\frac{\sqrt{ab}^q-a^q}{q}\right)^{\frac{1}{q}}\right. \\
&\qquad\left.+b\left[\sup\left\{\left|f'(\sqrt{ab})\right|^q,\left|f'(b)\right|^q\right\}\right]^{\frac{1}{q}}\left(\frac{b^q-\sqrt{ab}^q}{q}\right)^{\frac{1}{q}}\right\},
\end{aligned}$$

specially for $\alpha = 1$, we get

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2} - \frac{1}{\ln\frac{b}{a}}\int_a^b \frac{f(u)}{u}\,du\right| \\ &\quad\leq \frac{1}{2}\left(\frac{\ln\frac{b}{a}}{2(p+1)}\right)^{\frac{1}{p}}\left\{a\left[\sup\left\{\left|f'(\sqrt{ab})\right|^q, \left|f'(a)\right|^q\right\}\right]^{\frac{1}{q}}\left(\frac{\sqrt{ab}^q - a^q}{q}\right)^{\frac{1}{q}}\right. \\ &\qquad\left. + b\left[\sup\left\{\left|f'(\sqrt{ab})\right|^q, \left|f'(b)\right|^q\right\}\right]^{\frac{1}{q}}\left(\frac{b^q - \sqrt{ab}^q}{q}\right)^{\frac{1}{q}}\right\}.\end{aligned}$$

Corollary 2.7 *Let the assumptions of Theorem* 2.3 *hold. If* $|f'(x)| \leq M$ *for all* $x \in [a,b]$ *and* $\lambda = 0$, *then we get the following Ostrowski-type inequality for fractional integrals from inequality* (12):

$$\begin{aligned}&\left|\left[\left(\ln\frac{x}{a}\right)^\alpha + \left(\ln\frac{b}{x}\right)^\alpha\right]f(x) - \Gamma(\alpha+1)\left[J^\alpha_{\sqrt{ab}-}f(a) + J^\alpha_{\sqrt{ab}+}f(b)\right]\right| \\ &\quad\leq \frac{M}{(\alpha p+1)^{\frac{1}{p}}}\left[a\left(\ln\frac{x}{a}\right)^{\alpha+\frac{1}{p}}\left(\frac{x^q - a^q}{q}\right)^{\frac{1}{q}} + b\left(\ln\frac{b}{x}\right)^{\alpha+\frac{1}{p}}\left(\frac{b^q - x^q}{q}\right)^{\frac{1}{q}}\right].\end{aligned}$$

3 Application to special means

Let us recall the following special means of two nonnegative numbers a, b with $b > a$:

1. The arithmetic mean

$$A = A(a,b) := \frac{a+b}{2}.$$

2. The geometric mean

$$G = G(a,b) := \sqrt{ab}.$$

3. The logarithmic mean

$$L = L(a,b) := \frac{b-a}{\ln b - \ln a}.$$

4. The p-logarithmic mean

$$L_p = L_p(a,b) := \left(\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}}, \quad p \in \mathbb{R}\backslash\{-1,0\}.$$

Proposition 3.1 *For* $b > a > 0$, $n > 0$ *and* $q \geq 1$, *we have*

$$\begin{aligned}&\left|(1-\lambda)G^{n+1}(a,b) + \lambda A\left(a^{n+1}, b^{n+1}\right) - (n+1)L(a,b)L_n^n(a,b)\right| \\ &\quad\leq \frac{(n+1)\ln\frac{b}{a}}{4}\left(\frac{2\lambda^2 - 2\lambda + 1}{2}\right)^{1-\frac{1}{q}}\left\{aG^n(a,b)h^{\frac{1}{q}}\left(\left(\frac{b}{a}\right)^{\frac{q}{2}}, \lambda\right)\right. \\ &\quad\left. + b^{n+1}h^{\frac{1}{q}}\left(\left(\frac{a}{b}\right)^{\frac{q}{2}}, \lambda\right)\right\},\end{aligned}$$

where h *is defined as in* (10).

Proof Let $f(x) = \frac{x^{n+1}}{n+1}$, $x > 0$, $n > 0$ and $q \geq 1$. Then the function $|f'(x)|^q = x^{nq}$ is quasi-geometrically convex on $(0, \infty)$. Thus, by inequality (11), Proposition 3.1 is proved. □

Proposition 3.2 *For $b > a > 0$, $n > 0$ and $q > 1$, we have*

$$\begin{aligned}&\left|(1-\lambda)G^{n+1}(a,b) + \lambda A\left(a^{n+1}, b^{n+1}\right) - (n+1)L(a,b)L_n^n(a,b)\right| \\ &\quad \leq \frac{n+1}{2}\left(\frac{\ln\frac{b}{a}(\lambda^{p+1} + (1-\lambda)^{p+1})}{2(p+1)}\right)^{\frac{1}{p}}\left\{aG^n(a,b)\left(\frac{G^q(a,b) - a^q}{q}\right)^{\frac{1}{q}}\right. \\ &\qquad \left. + b^{n+1}\left(\frac{b^q - G^q(a,b)}{q}\right)^{\frac{1}{q}}\right\}.\end{aligned}$$

Proof Let $f(x) = \frac{x^{n+1}}{n+1}$, $x > 0$, $n > 0$ and $q > 1$. Then the function $|f'(x)|^q = x^{nq}$ is quasi-geometrically convex on $(0, \infty)$. Thus, by inequality (13), Proposition 3.2 is proved. □

Competing interests

The author declares that he has no competing interests.

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