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An iterative method for approximating the common solutions of a variational inequality, a mixed equilibrium problem and a hierarchical fixed point problem

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Abstract
In this paper, we suggest and analyze an iterative scheme for finding the approximate element of the common set of solutions of a generalized equilibrium problem, a variational inequality problem and a hierarchical fixed point problem in a real Hilbert space. We also consider the strong convergence of the proposed method under some conditions. Results proved in this paper may be viewed as an improvement and refinement of the previously known results.

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1 Introduction

Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$. Let C be a nonempty closed convex subset of H, and A is a mapping from C into H. A classical variational inequality problem, denoted by $\mathrm{VI}(A,C)$, is to find a vector $u\in C$ such that

$$\langle v-u, Au\rangle \geq 0, \quad \forall v\in C. \tag{1.1}$$

The solution of $\mathrm{VI}(A,C)$ is denoted by Ω^*. It is easy to observe that

$$u^*\in\Omega^* \quad\Longleftrightarrow\quad u^* = P_C\left[u^*-\rho Au^*\right], \quad \text{where } \rho>0.$$

We now have a variety of techniques to suggest and analyze various iterative algorithms for solving variational inequalities and the related optimization problems, see [1–22]. The fixed-point theory has played an important role in the development of various algorithms for solving variational inequalities. Using the projection operator technique, one usually establishes an equivalence between the variational inequalities and the fixed-point problem. This alternative equivalent formulation was used by Lions and Stampacchia [8] to study the existence of a solution of the variational inequalities.

We introduce the following definitions, which are useful in the following analysis.

Definition 1.1 The mapping $T : C \to H$ is said to be

(a) monotone if

$$\langle Tx - Ty, x - y\rangle \geq 0, \quad \forall x, y \in C;$$

(b) strongly monotone if there exists an $\alpha > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq \alpha \|x - y\|^2, \quad \forall x, y \in C;$$

(c) α-inverse strongly monotone if there exists an $\alpha > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq \alpha \|Tx - Ty\|^2, \quad \forall x, y \in C;$$

(d) nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C;$$

(e) k-Lipschitz continuous if there exists a constant $k > 0$ such that

$$\|Tx - Ty\| \leq k\|x - y\|, \quad \forall x, y \in C;$$

(f) contraction on C if there exists a constant $0 \leq k < 1$ such that

$$\|Tx - Ty\| \leq k\|x - y\|, \quad \forall x, y \in C.$$

It is easy to observe that every α-inverse strongly monotone T is monotone and Lipschitz continuous. A mapping $T : C \to H$ is called k-strict pseudo-contraction if there exists a constant $0 \leq k < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\big\|(I - T)x - (I - T)y\big\|^2, \quad \forall x, y \in C. \tag{1.2}$$

The fixed point problem for the mapping T is to find $x \in C$ such that

$$Tx = x. \tag{1.3}$$

We denote $F(T)$ the set of solutions of (1.3). It is well known that the class of strict pseudo-contractions strictly includes the class of nonexpansive mappings, then $F(T)$ is closed and convex, and $P_{F(T)}$ is well defined (see [22]).

The mixed equilibrium problem, denoted by *MEP*, is to find $x \in C$ such that

$$F(x, y) + \langle Dx, y - x\rangle \geq 0, \quad \forall y \in C, \tag{1.4}$$

where $F : C \times C \to \mathbb{R}$ is a bifunction, and $D : C \to H$ is a nonlinear mapping. This problem was introduced and studied by Moudafi and Théra [13] and Moudafi [14]. The set of solutions of (1.4) is denoted by

$$MEP(F) := \big\{x \in C : F(x, y) + \langle Dx, y - x\rangle \geq 0, \forall y \in C\big\}. \tag{1.5}$$

If $D = 0$, then it is reduced to the equilibrium problem, which is to find $x \in C$ such that

$$F(x,y) \geq 0, \quad \forall y \in C. \tag{1.6}$$

The solution set of (1.6) is denoted by $EP(F)$. Numerous problems in physics, optimization, and economics reduce to find a solution of (1.6), see [4, 7, 16, 17]. In 1997, Combettes and Hirstoaga [5] introduced an iterative scheme of finding the best approximation to the initial data when $EP(F)$ is nonempty. Recently Plubtieng and Punpaeng [16] introduced an iterative method for finding the common element of the set $F(T) \cap \Omega^* \cap EP(F)$.

Let $S : C \to H$ be a nonexpansive mapping. The following problem is called a hierarchical fixed point problem: Find $x \in F(T)$ such that

$$\langle x - Sx, y - x \rangle \geq 0, \quad \forall y \in F(T). \tag{1.7}$$

It is known that the hierarchical fixed point problem (1.7) links with some monotone variational inequalities and convex programming problems; see [6, 20]. Various methods have been proposed to solve the hierarchical fixed point problem; see Moudafi [15], Mainge and Moudafi in [9], Marino and Xu in [11] and Cianciaruso *et al.* [3]. Very recently, Yao *et al.* [20] introduced the following strong convergence iterative algorithm to solve problem (1.7):

$$\begin{aligned} y_n &= \beta_n S x_n + (1 - \beta_n) x_n, \\ x_{n+1} &= P_C\bigl[\alpha_n f(x_n) + (1 - \alpha_n) T y_n\bigr], \quad \forall n \geq 0, \end{aligned} \tag{1.8}$$

where $f : C \to H$ is a contraction mapping, and $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0,1)$. Under some certain restrictions on parameters, Yao *et al.* proved that the sequence $\{x_n\}$ generated by (1.8) converges strongly to $z \in F(T)$, which is the unique solution of the following variational inequality:

$$\bigl\langle (I - f)z, y - z \bigr\rangle \geq 0, \quad \forall y \in F(T). \tag{1.9}$$

By changing the restrictions on parameters, the authors obtained another result on the iterative scheme (1.8), the sequence $\{x_n\}$ generated by (1.8) converges strongly to a point $z \in F(T)$, which is the unique solution of the following variational inequality:

$$\left\langle \frac{1}{\tau}(I - f)z + (I - S)z, y - z \right\rangle \geq 0, \quad \forall y \in F(T). \tag{1.10}$$

Let $S : C \to H$ be a nonexpansive mapping, and $\{T_i\}_{i=1}^{\infty} : C \to C$ is a countable family of nonexpansive mappings. Very recently, Gu *et al.* [6] introduced the following iterative algorithm:

$$\begin{aligned} y_n &= P_C\bigl[\beta_n S x_n + (1 - \beta_n) x_n\bigr], \\ x_{n+1} &= P_C\left[\alpha_n f(x_n) + \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i) T_i y_n\right], \quad \forall n \geq 1, \end{aligned} \tag{1.11}$$

where $\alpha_0 = 1$, $\{\alpha_n\}$ is a strictly decreasing sequence in $(0,1)$, and $\{\beta_n\}$ is a sequence in $(0,1)$. Under some certain conditions on parameters, Gu *et al.* proved that the sequence $\{x_n\}$ generated by (1.11) converges strongly to $z \in \bigcap_{i=1}^{\infty} F(T_i)$, which is a unique solution of one of the variational inequalities (1.9) and (1.10).

In this paper, motivated by the work of Yao *et al.* [20] and Gu *et al.* [6] and by the recent work going in this direction, we give an iterative method for finding the approximate element of the common set of solutions of (1.1), (1.4) and (1.7) for a strictly pseudo-contraction mapping in a real Hilbert space. We establish a strong convergence theorem based on this method. The presented method improves and generalizes many known results for solving equilibrium problems, variational inequality problems and hierarchical fixed point problems, see, *e.g.*, [3, 6, 9, 20] and relevant references cited therein.

2 Preliminaries

In this section, we list some fundamental lemmas that are useful in the consequent analysis. The first lemma provides some basic properties of projection onto C.

Lemma 2.1 *Let P_C denote the projection of H onto C. Then we have the following inequalities*:

$$\langle z - P_C[z], P_C[z] - v\rangle \geq 0, \quad \forall z \in H, v \in C; \tag{2.1}$$

$$\langle u - v, P_C[u] - P_C[v]\rangle \geq \|P_C[u] - P_C[v]\|^2, \quad \forall u, v \in H; \tag{2.2}$$

$$\|P_C[u] - P_C[v]\| \leq \|u - v\|, \quad \forall u, v \in H; \tag{2.3}$$

$$\|u - P_C[z]\|^2 \leq \|z - u\|^2 - \|z - P_C[z]\|^2, \quad \forall z \in H, u \in C. \tag{2.4}$$

Lemma 2.2 [2] *Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying the following assumptions*:

(i) $F(x,x) = 0$, $\forall x \in C$;
(ii) *F is monotone, i.e.*, $F(x,y) + F(y,x) \leq 0$, $\forall x, y \in C$;
(iii) *For each* $x, y, z \in C$, $\lim_{t\to 0} F(tz + (1-t)x, y) \leq F(x,y)$;
(iv) *For each* $x \in C$, $y \to F(x,y)$ *is convex and lower semicontinuous*.

Let $r > 0$ and $x \in H$. Then there exists $z \in C$ such that

$$F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C.$$

Lemma 2.3 [5] *Assume that $F : C \times C \to \mathbb{R}$ satisfies assumptions* (i)-(iv) *of Lemma* 2.2. *For $r > 0$ and $\forall x \in H$, define a mapping $T_r : H \to C$ as follows*:

$$T_r(x) = \left\{z \in C : F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\right\}.$$

Then the following hold:

(i) *T_r is single-valued*;
(ii) *T_r is firmly nonexpansive, i.e.*,

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle, \quad \forall x, y \in H;$$

(iii) $F(T_r) = EP(F)$;
(iv) *$EP(F)$ is closed and convex.*

Lemma 2.4 [21] *Let C be a nonempty closed convex subset of a real Hilbert space H. If $T : C \to C$ is a k-strict pseudo-contraction, then*

(i) *The mapping $I - T$ is demiclosed at 0, i.e., if $\{x_n\}$ is a sequence in C weakly converging to x, and if $\{(I - T)x_n\}$ converges strongly to 0, then $(I - T)x = 0$;*
(ii) *The set $F(T)$ of T is closed and convex, so that the projection $P_{F(T)}$ is well defined.*

Lemma 2.5 [10] *Let H be a real Hilbert space. Then the following inequality holds*:

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall x, y \in H.$$

Lemma 2.6 [19] *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n,$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$, and δ_n is a sequence such that

(1) $\sum_{n=1}^{\infty} \gamma_n = \infty$;
(2) $\limsup_{n\to\infty} \delta_n/\gamma_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.7 [1] *Let C be a closed convex subset of H. Let $\{x_n\}$ be a bounded sequence in H. Assume that*

(i) *The weak w-limit set $w_w(x_n) \subset C$, where $w_w(x_n) = \{x : x_{n_i} \rightharpoonup x\}$.*
(ii) *For each $z \in C$, $\lim_{n\to\infty} \|x_n - z\|$ exists.*

Then $\{x_n\}$ is weakly convergent to a point in C.

Lemma 2.8 [22] *Let H be a Hilbert space, C be a closed and convex subset of H, and $T : C \to C$ be a k-strict pseudo-contraction mapping. Define a mapping $V : C \to H$ by $Vx = \lambda x + (1 - \lambda)Tx$, $\forall x \in C$. Then, as $k \leq \lambda < 1$, V is a nonexpansive mapping such that $F(V) = F(T)$.*

Lemma 2.9 [6] *Let H be a Hilbert space, C be a closed and convex subset of H, and $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Then*

$$\|Tx - x\|^2 \leq 2\langle x - Tx, x - x'\rangle, \quad \forall x' \in F(T), \forall x \in C.$$

3 The proposed method and some properties

In this section, we suggest and analyze our method for finding the common solutions of the variational inequality (1.1), the mixed equilibrium problem (1.4) and the hierarchical fixed point problem (1.7).

Let C be a nonempty closed convex subset of a real Hilbert space H. Let $D, A : C \to H$ be θ, α-inverse strongly monotone mappings, respectively. Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying assumptions (i)-(iv) of Lemma 2.2, $S : C \to H$ be a nonexpansive mapping and $\{T_i\}_{i=1}^{\infty} : C \to C$ is a countable family of k_i-strict pseudo-contraction mappings such that $F(T) \cap \Omega^* \cap MEP(F) \neq \emptyset$, where $F(T) = \bigcap_{i=1}^{\infty} F(T_i)$. Let f be a ρ-contraction mapping.

Algorithm 3.1 For a given $x_0 \in C$ arbitrarily, let the iterative sequences $\{u_n\}$, $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be generated by

$$\begin{aligned} &F(u_n, y) + \langle Dx_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C; \\ &z_n = P_C[u_n - \lambda_n A u_n]; \\ &y_n = P_C\bigl[\beta_n S x_n + (1-\beta_n) z_n\bigr]; \\ &x_{n+1} = P_C\left[\alpha_n f(x_n) + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i) V_i y_n\right], \quad \forall n \geq 0, \end{aligned} \tag{3.1}$$

where $V_i = k_i I + (1-k_i)T_i$, $0 \leq k_i < 1$, $\{\lambda_n\} \subset (0, 2\alpha)$, $\{r_n\} \subset (0, 2\theta]$, $\alpha_0 = 1$, $\{\alpha_n\}$ is a strictly decreasing sequence in $(0,1)$, and $\{\beta_n\}$ is a sequence in $(0,1)$ satisfying the following conditions:

(a) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(b) $\lim_{n\to\infty}(\beta_n/\alpha_n) = 0$,
(c) $\sum_{n=1}^{\infty} |\alpha_{n-1} - \alpha_n| < \infty$ and $\sum_{n=1}^{\infty} |\beta_{n-1} - \beta_n| < \infty$,
(d) $\liminf_{n\to\infty} r_n > 0$ and $\sum_{n=1}^{\infty} |r_{n-1} - r_n| < \infty$,
(e) $\liminf_{n\to\infty} \lambda_n < \limsup_{n\to\infty} \lambda_n < 2\alpha$ and $\sum_{n=1}^{\infty} |\lambda_{n-1} - \lambda_n| < \infty$.

Remark 3.1 It is easy to verify that Algorithm 3.1 includes some existing methods (*e.g.*, [3, 6, 9, 20]) as special cases. Therefore, the new algorithm is expected to be widely applicable.

Lemma 3.1 *Let $x^* \in F(T) \cap \Omega^* \cap MEP(F)$. Then $\{x_n\}$, $\{u_n\}$, $\{z_n\}$ and $\{y_n\}$ are bounded.*

Proof First, we show that the mapping $(I - r_n D)$ is nonexpansive. For any $x, y \in C$,

$$\begin{aligned} &\bigl\|(I - r_n D)x - (I - r_n D)y\bigr\|^2 \\ &\quad = \bigl\|(x - y) - r_n(Dx - Dy)\bigr\|^2 \\ &\quad = \|x - y\|^2 - 2r_n\langle x - y, Dx - Dy\rangle + r_n^2\|Dx - Dy\|^2 \\ &\quad \leq \|x - y\|^2 - r_n(2\theta - r_n)\|Dx - Dy\|^2 \\ &\quad \leq \|x - y\|^2. \end{aligned}$$

Similarly, we can show that the mapping $(I - \lambda_n A)$ is nonexpansive. It follows from Lemma 2.3 that $u_n = T_{r_n}(x_n - r_n D x_n)$. Let $x^* \in F(T) \cap \Omega^* \cap MEP(F)$, we have $x^* = T_{r_n}(x^* - r_n D x^*)$.

$$\begin{aligned} &\bigl\|u_n - x^*\bigr\|^2 \\ &\quad = \bigl\|T_{r_n}(x_n - r_n D x_n) - T_{r_n}\bigl(x^* - r_n D x^*\bigr)\bigr\|^2 \\ &\quad \leq \bigl\|(x_n - r_n D x_n) - \bigl(x^* - r_n D x^*\bigr)\bigr\|^2 \\ &\quad \leq \bigl\|x_n - x^*\bigr\|^2 - r_n(2\theta - r_n)\bigl\|D x_n - D x^*\bigr\|^2 \\ &\quad \leq \bigl\|x_n - x^*\bigr\|^2. \end{aligned} \tag{3.2}$$

Since the mapping A is α-inverse strongly monotone, we have

$$\begin{aligned}\|z_n - x^*\|^2 &= \|P_C[u_n - \lambda_n Au_n) - P_C[x^* - \lambda_n Ax^*]\|^2 \\ &\le \|u_n - x^* - \lambda_n(Au_n - Ax^*)\|^2 \\ &\le \|u_n - x^*\|^2 - \lambda_n(2\alpha - \lambda_n)\|Au_n - Ax^*\|^2 \\ &\le \|u_n - x^*\|^2 \\ &\le \|x_n - x^*\|^2. \end{aligned} \tag{3.3}$$

Next, we prove that the sequence $\{x_n\}$ is bounded, without loss of generality, we can assume that $\beta_n \le \alpha_n$ for all $n \ge 1$. From (3.1), we have

$$\begin{aligned}\|x_{n+1} - x^*\| &\le \left\|\alpha_n f(x_n) + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)V_i y_n - \alpha_n x^* - \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)V_i x^*\right\| \\ &\le \alpha_n\|f(x_n) - f(x^*)\| + \alpha_n\|f(x^*) - x^*\| + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)\|V_i y_n - V_i x^*\| \\ &\le \alpha_n\|f(x_n) - f(x^*)\| + \alpha_n\|f(x^*) - x^*\| + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)\|y_n - x^*\| \\ &= \alpha_n\|f(x_n) - f(x^*)\| + \alpha_n\|f(x^*) - x^*\| + (1-\alpha_n)\|\beta_n Sx_n + (1-\beta_n)z_n - x^*\| \\ &\le \alpha_n\|f(x_n) - f(x^*)\| + \alpha_n\|f(x^*) - x^*\| \\ &\quad + (1-\alpha_n)\big(\beta_n\|Sx_n - Sx^*\| + \beta_n\|Sx^* - x^*\| + (1-\beta_n)\|z_n - x^*\|\big) \\ &\le \alpha_n\rho\|x_n - x^*\| + \alpha_n\|f(x^*) - x^*\| + (1-\alpha_n)\big(\beta_n\|x_n - x^*\| + \beta_n\|Sx^* - x^*\| \\ &\quad + (1-\beta_n)\|x_n - x^*\|\big) \\ &= \big(1 - \alpha_n(1-\rho)\big)\|x_n - x^*\| + \alpha_n\|f(x^*) - x^*\| + (1-\alpha_n)\beta_n\|Sx^* - x^*\| \\ &\le \big(1 - \alpha_n(1-\rho)\big)\|x_n - x^*\| + \alpha_n\|f(x^*) - x^*\| + \beta_n\|Sx^* - x^*\| \\ &\le \big(1 - \alpha_n(1-\rho)\big)\|x_n - x^*\| + \alpha_n\big(\|f(x^*) - x^*\| + \|Sx^* - x^*\|\big) \\ &= \big(1 - \alpha_n(1-\rho)\big)\|x_n - x^*\| + \frac{\alpha_n(1-\rho)}{1-\rho}\big(\|f(x^*) - x^*\| + \|Sx^* - x^*\|\big) \\ &\le \max\left\{\|x_n - x^*\|, \frac{1}{1-\rho}\big(\|f(x^*) - x^*\| + \|Sx^* - x^*\|\big)\right\}. \end{aligned} \tag{3.4}$$

By induction on n, we obtain $\|x_n - x^*\| \le \max\{\|x_0 - x^*\|, \frac{1}{1-\rho}(\|f(x^*) - x^*\| + \|Sx^* - x^*\|)\}$ for $n \ge 0$ and $x_0 \in C$. Hence, $\{x_n\}$ is bounded and consequently, we deduce that $\{u_n\}$, $\{z_n\}$ and $\{y_n\}$ are bounded. □

Lemma 3.2 *Let* $x^* \in F(T) \cap \Omega^* \cap MEP(F)$ *and* $\{x_n\}$ *be the sequence generated by Algorithm* 3.1. *Then we have*

(a) $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$.

(b) *The weak w-limit set* $w_w(x_n) \subset F(T)$, $(w_w(x_n) = \{x : x_{n_i} \rightharpoonup x\})$.

Proof From the nonexpansivity of the mapping $(I - \lambda_n A)$ and P_C, we have

$$\begin{aligned}\|z_n - z_{n-1}\| &\le \big\|(u_n - \lambda_n Au_n) - (u_{n-1} - \lambda_{n-1}Au_{n-1})\big\| \\ &= \big\|(u_n - u_{n-1}) - \lambda_n(Au_n - Au_{n-1}) - (\lambda_n - \lambda_{n-1})Au_{n-1}\big\| \\ &\le \big\|(u_n - u_{n-1}) - \lambda_n(Au_n - Au_{n-1})\big\| + |\lambda_n - \lambda_{n-1}|\|Au_{n-1}\| \\ &\le \|u_n - u_{n-1}\| + |\lambda_n - \lambda_{n-1}|\|Au_{n-1}\|. \end{aligned} \tag{3.5}$$

Next, we estimate that

$$\begin{aligned}\|y_n - y_{n-1}\| &\le \big\|\beta_n Sx_n + (1-\beta_n)z_n - \big(\beta_{n-1}Sx_{n-1} + (1-\beta_{n-1})z_{n-1}\big)\big\| \\ &= \big\|\beta_n(Sx_n - Sx_{n-1}) + (\beta_n - \beta_{n-1})Sx_{n-1} \\ &\quad + (1-\beta_n)(z_n - z_{n-1}) + (\beta_{n-1} - \beta_n)z_{n-1}\big\| \\ &\le \beta_n\|x_n - x_{n-1}\| + (1-\beta_n)\|z_n - z_{n-1}\| \\ &\quad + |\beta_n - \beta_{n-1}|\big(\|Sx_{n-1}\| + \|z_{n-1}\|\big). \end{aligned} \tag{3.6}$$

It follows from (3.5) and (3.6) that

$$\begin{aligned}&\|y_n - y_{n-1}\| \\ &\quad \le \beta_n\|x_n - x_{n-1}\| + (1-\beta_n)\big\{\|u_n - u_{n-1}\| + |\lambda_n - \lambda_{n-1}|\|Au_{n-1}\|\big\} \\ &\qquad + |\beta_n - \beta_{n-1}|\big(\|Sx_{n-1}\| + \|z_{n-1}\|\big). \end{aligned} \tag{3.7}$$

On the other hand, $u_n = T_{r_n}(x_n - r_n Dx_n)$ and $u_{n-1} = T_{r_{n-1}}(x_{n-1} - r_{n-1}Dx_{n-1})$, we have

$$F(u_n, y) + \langle Dx_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \quad \forall y \in C \tag{3.8}$$

and

$$F(u_{n-1}, y) + \langle Dx_{n-1}, y - u_{n-1}\rangle + \frac{1}{r_{n-1}}\langle y - u_{n-1}, u_{n-1} - x_{n-1}\rangle \ge 0, \quad \forall y \in C. \tag{3.9}$$

Take $y = u_{n-1}$ in (3.8) and $y = u_n$ in (3.9), we get

$$F(u_n, u_{n-1}) + \langle Dx_n, u_{n-1} - u_n\rangle + \frac{1}{r_n}\langle u_{n-1} - u_n, u_n - x_n\rangle \ge 0 \tag{3.10}$$

and

$$F(u_{n-1}, u_n) + \langle Dx_{n-1}, u_n - u_{n-1}\rangle + \frac{1}{r_{n-1}}\langle u_n - u_{n-1}, u_{n-1} - x_{n-1}\rangle \ge 0. \tag{3.11}$$

Adding (3.10) and (3.11) and using the monotonicity of F, we have

$$\langle Dx_{n-1} - Dx_n, u_n - u_{n-1}\rangle + \left\langle u_n - u_{n-1}, \frac{u_{n-1} - x_{n-1}}{r_{n-1}} - \frac{u_n - x_n}{r_n}\right\rangle \ge 0,$$

which implies that

$$\begin{aligned}
0 &\le \left\langle u_n - u_{n-1}, r_n(Dx_{n-1} - Dx_n) + \frac{r_n}{r_{n-1}}(u_{n-1} - x_{n-1}) - (u_n - x_n)\right\rangle \\
&= \left\langle u_{n-1} - u_n, u_n - u_{n-1} + \left(1 - \frac{r_n}{r_{n-1}}\right)u_{n-1} \right.\\
&\quad \left. + (x_{n-1} - r_nDx_{n-1}) - (x_n - r_nDx_n) - x_{n-1} + \frac{r_n}{r_{n-1}}x_{n-1}\right\rangle \\
&= \left\langle u_{n-1} - u_n, \left(1 - \frac{r_n}{r_{n-1}}\right)u_{n-1} + (x_{n-1} - r_nDx_{n-1}) - (x_n - r_nDx_n) - x_{n-1} + \frac{r_n}{r_{n-1}}x_{n-1}\right\rangle \\
&\quad - \|u_n - u_{n-1}\|^2 \\
&= \left\langle u_{n-1} - u_n, \left(1 - \frac{r_n}{r_{n-1}}\right)(u_{n-1} - x_{n-1}) + (x_{n-1} - r_nDx_{n-1}) - (x_n - r_nDx_n)\right\rangle \\
&\quad - \|u_n - u_{n-1}\|^2 \\
&\le \|u_{n-1} - u_n\|\left\{\left|1 - \frac{r_n}{r_{n-1}}\right|\|u_{n-1} - x_{n-1}\| + \big\|(x_{n-1} - r_nDx_{n-1}) - (x_n - r_nDx_n)\big\|\right\} \\
&\quad - \|u_n - u_{n-1}\|^2 \\
&\le \|u_{n-1} - u_n\|\left\{\left|1 - \frac{r_n}{r_{n-1}}\right|\|u_{n-1} - x_{n-1}\| + \|x_{n-1} - x_n\|\right\} - \|u_n - u_{n-1}\|^2,
\end{aligned}$$

and then

$$\|u_{n-1} - u_n\| \le \left|1 - \frac{r_n}{r_{n-1}}\right|\|u_{n-1} - x_{n-1}\| + \|x_{n-1} - x_n\|.$$

Without loss of generality, let us assume that there exists a real number μ such that $r_n > \mu > 0$ for all positive integers n. Then we get

$$\|u_{n-1} - u_n\| \le \|x_{n-1} - x_n\| + \frac{1}{\mu}|r_{n-1} - r_n|\|u_{n-1} - x_{n-1}\|. \tag{3.12}$$

It follows from (3.7) and (3.12) that

$$\begin{aligned}
\|y_n - y_{n-1}\| &\le \beta_n\|x_n - x_{n-1}\| + (1 - \beta_n)\left\{\|x_n - x_{n-1}\| + \frac{1}{\mu}|r_n - r_{n-1}|\|u_{n-1} - x_{n-1}\|\right.\\
&\quad \left. + |\lambda_n - \lambda_{n-1}|\|Au_{n-1}\|\right\} + |\beta_n - \beta_{n-1}|\big(\|Sx_{n-1}\| + \|z_{n-1}\|\big) \\
&= \|x_n - x_{n-1}\| + (1 - \beta_n)\left\{\frac{1}{\mu}|r_n - r_{n-1}|\|u_{n-1} - x_{n-1}\| + |\lambda_n - \lambda_{n-1}|\|Au_{n-1}\|\right\} \\
&\quad + |\beta_n - \beta_{n-1}|\big(\|Sx_{n-1}\| + \|z_{n-1}\|\big).
\end{aligned} \tag{3.13}$$

Next, we estimate that

$$\begin{aligned}
&\|x_{n+1} - x_n\| \\
&\quad \le \left\|\alpha_n f(x_n) + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)V_iy_n - \left(\alpha_{n-1}f(x_{n-1}) + \sum_{i=1}^{n-1}(\alpha_{i-1} - \alpha_i)V_iy_{n-1}\right)\right\|
\end{aligned}$$

$$
\begin{aligned}
&= \Bigg\| \alpha_n\big(f(x_n) - f(x_{n-1})\big) + (\alpha_n - \alpha_{n-1}) f(x_{n-1}) + \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i)(V_i y_n - V_i y_{n-1}) \\
&\quad + (\alpha_{n-1} - \alpha_n) V_n y_{n-1} \Bigg\| \\
&\leq \alpha_n \big\| f(x_n) - f(x_{n-1}) \big\| + \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i) \| V_i y_n - V_i y_{n-1} \| \\
&\quad + |\alpha_n - \alpha_{n-1}| \big(\big\| f(x_{n-1}) \big\| + \| V_n y_{n-1} \| \big) \\
&\leq \alpha_n \rho \| x_n - x_{n-1} \| + \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i) \| y_n - y_{n-1} \| \\
&\quad + |\alpha_n - \alpha_{n-1}| \big(\big\| f(x_{n-1}) \big\| + \| V_n y_{n-1} \| \big) \\
&= \alpha_n \rho \| x_n - x_{n-1} \| + (1 - \alpha_n) \| y_n - y_{n-1} \| \\
&\quad + |\alpha_n - \alpha_{n-1}| \big(\big\| f(x_{n-1}) \big\| + \| V_n y_{n-1} \| \big). \qquad (3.14)
\end{aligned}
$$

From (3.13) and (3.14), we have

$$
\begin{aligned}
\| x_{n+1} - x_n \| &\leq \alpha_n \rho \| x_n - x_{n-1} \| \\
&\quad + (1 - \alpha_n) \Bigg\{ \| x_n - x_{n-1} \| \\
&\quad + (1 - \beta_n) \left(\frac{1}{\mu} |r_n - r_{n-1}| \| u_{n-1} - x_{n-1} \| + |\lambda_n - \lambda_{n-1}| \| A u_{n-1} \| \right) \\
&\quad + |\beta_n - \beta_{n-1}| \big(\| S x_{n-1} \| + \| z_{n-1} \| \big) \Bigg\} + |\alpha_n - \alpha_{n-1}| \big(\big\| f(x_{n-1}) \big\| + \| V_n y_{n-1} \| \big) \\
&\leq \big(1 - (1 - \rho)\alpha_n\big) \| x_n - x_{n-1} \| + \frac{1}{\mu} |r_n - r_{n-1}| \| u_{n-1} - x_{n-1} \| \\
&\quad + |\lambda_n - \lambda_{n-1}| \| A u_{n-1} \| \\
&\quad + |\beta_n - \beta_{n-1}| \big(\| S x_{n-1} \| + \| z_{n-1} \| \big) + |\alpha_n - \alpha_{n-1}| \big(\big\| f(x_{n-1}) \big\| + \| V_n y_{n-1} \| \big) \\
&\leq \big(1 - (1 - \rho)\alpha_n\big) \| x_n - x_{n-1} \| \\
&\quad + M \left(\frac{1}{\mu} |r_n - r_{n-1}| + |\lambda_n - \lambda_{n-1}| + |\beta_n - \beta_{n-1}| + |\alpha_n - \alpha_{n-1}| \right). \qquad (3.15)
\end{aligned}
$$

Where

$$
\begin{aligned}
M = \max \Big\{ &\sup_{n \geq 1} \| u_{n-1} - x_{n-1} \|, \sup_{n \geq 1} \| A u_{n-1} \|, \sup_{n \geq 1} \big(\| S x_{n-1} \| + \| z_{n-1} \| \big), \\
&\sup_{n \geq 1} \big(\big\| f(x_{n-1}) \big\| + \| V_n y_{n-1} \| \big) \Big\}.
\end{aligned}
$$

It follows by conditions (a)-(e) of Algorithm 3.1 and Lemma 2.6 that

$$
\lim_{n \to \infty} \| x_{n+1} - x_n \| = 0.
$$

Next, we show that $\lim_{n\to\infty}\|u_n - x_n\| = 0$. Since $x^* \in F(T)\cap\Omega^*\cap MEP(F)$ and $\alpha_n + \sum_{i=1}^n(\alpha_{i-1}-\alpha_i) = 1$, by using (3.2) and (3.3), we obtain

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &\le \left\|\alpha_n f(x_n) + \sum_{i=1}^n(\alpha_{i-1}-\alpha_i)V_i y_n - \alpha_n x^* - \sum_{i=1}^n(\alpha_{i-1}-\alpha_i)V_i x^*\right\|^2 \\
&\le \alpha_n\|f(x_n)-x^*\|^2 + \sum_{i=1}^n(\alpha_{i-1}-\alpha_i)\|V_i y_n - V_i x^*\|^2 \\
&\le \alpha_n\|f(x_n)-x^*\|^2 + \sum_{i=1}^n(\alpha_{i-1}-\alpha_i)\|y_n - x^*\|^2 \\
&\le \alpha_n\|f(x_n)-x^*\|^2 + (1-\alpha_n)\big(\beta_n\|Sx_n-x^*\|^2 \\
&\quad + (1-\beta_n)\|z_n-x^*\|^2\big) \\
&\le \alpha_n\|f(x_n)-x^*\|^2 + (1-\alpha_n)\beta_n\|Sx_n-x^*\|^2 \\
&\quad + (1-\alpha_n)(1-\beta_n)\big\{\|x_n-x^*\|^2 - r_n(2\theta-r_n)\|Dx_n-Dx^*\|^2 \\
&\quad - \lambda_n(2\alpha-\lambda_n)\|Au_n-Ax^*\|^2\big\} \\
&\le \alpha_n\|f(x_n)-x^*\|^2 + \beta_n\|Sx_n-x^*\|^2 + \|x_n-x^*\|^2 \\
&\quad - (1-\alpha_n)(1-\beta_n)\big\{r_n(2\theta-r_n)\|Dx_n-Dx^*\|^2 \\
&\quad + \lambda_n(2\alpha-\lambda_n)\|Au_n-Ax^*\|^2\big\}.
\end{aligned}\tag{3.16}$$

Then from the inequality above, we get

$$\begin{aligned}
&(1-\alpha_n)(1-\beta_n)\big\{r_n(2\theta-r_n)\|Dx_n-Dx^*\|^2 + \lambda_n(2\alpha-\lambda_n)\|Au_n-Ax^*\|^2\big\} \\
&\quad\le \alpha_n\|f(x_n)-x^*\|^2 + \beta_n\|Sx_n-x^*\|^2 + \|x_n-x^*\|^2 - \|x_{n+1}-x^*\|^2 \\
&\quad\le \alpha_n\|f(x_n)-x^*\|^2 + \beta_n\|Sx_n-x^*\|^2 + \big(\|x_n-x^*\| + \|x_{n+1}-x^*\|\big)\|x_{n+1}-x_n\|.
\end{aligned}$$

Since $\liminf_{n\to\infty}\lambda_n \le \limsup_{n\to\infty}\lambda_n < 2\alpha$, $\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0$, $\alpha_n \to 0$ and $\beta_n \to 0$, we obtain $\lim_{n\to\infty}\|Dx_n - Dx^*\| = 0$ and $\lim_{n\to\infty}\|Au_n - Ax^*\| = 0$.

Since T_{r_n} is firmly nonexpansive, we have

$$\begin{aligned}
\|u_n-x^*\|^2 &= \|T_{r_n}(x_n-r_nDx_n) - T_{r_n}(x^*-r_nDx^*)\|^2 \\
&\le \langle u_n-x^*, (x_n-r_nDx_n) - (x^*-r_nDx^*)\rangle \\
&= \frac{1}{2}\big\{\|u_n-x^*\|^2 + \|(x_n-r_nDx_n)-(x^*-r_nDx^*)\|^2 \\
&\quad - \|u_n-x^*-[(x_n-r_nDx_n)-(x^*-r_nDx^*)]\|^2\big\}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\|u_n-x^*\|^2 &\le \|(x_n-r_nDx_n)-(x^*-r_nDx^*)\|^2 - \|u_n-x_n+r_n(Dx_n-Dx^*)\|^2 \\
&\le \|x_n-x^*\|^2 - \|u_n-x_n+r_n(Dx_n-Dx^*)\|^2 \\
&\le \|x_n-x^*\|^2 - \|u_n-x_n\|^2 + 2r_n\|u_n-x_n\|\|Dx_n-Dx^*\|.
\end{aligned}$$

From (3.16), (3.3) and the inequality above, we have

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &\le \alpha_n\|f(x_n)-x^*\|^2 + (1-\alpha_n)\big(\beta_n\|Sx_n-x^*\|^2 + (1-\beta_n)\|z_n-x^*\|^2\big) \\
&\le \alpha_n\|f(x_n)-x^*\|^2 + (1-\alpha_n)\big(\beta_n\|Sx_n-x^*\|^2 + (1-\beta_n)\|u_n-x^*\|^2\big) \\
&\le \alpha_n\|f(x_n)-x^*\|^2 + (1-\alpha_n)\big\{\beta_n\|Sx_n-x^*\|^2 \\
&\quad + (1-\beta_n)\big(\|x_n-x^*\|^2 - \|u_n-x_n\|^2 + 2r_n\|u_n-x_n\|\|Dx_n-Dx^*\|\big)\big\} \\
&\le \alpha_n\|f(x_n)-x^*\|^2 + \beta_n\|Sx_n-x^*\|^2 + \|x_n-x^*\|^2 \\
&\quad - (1-\alpha_n)(1-\beta_n)\|u_n-x_n\|^2 + 2r_n\|u_n-x_n\|\|Dx_n-Dx^*\|.
\end{aligned}$$

Hence,

$$\begin{aligned}
&(1-\alpha_n)(1-\beta_n)\|u_n-x_n\|^2 \\
&\quad\le \alpha_n\|f(x_n)-x^*\|^2 + \beta_n\|Sx_n-x^*\|^2 + \|x_n-x^*\|^2 - \|x_{n+1}-x^*\|^2 \\
&\qquad + 2r_n\|u_n-x_n\|\|Dx_n-Dx^*\| \\
&\quad\le \alpha_n\|f(x_n)-x^*\|^2 + \beta_n\|Sx_n-x^*\|^2 + \big(\|x_n-x^*\| + \|x_{n+1}-x^*\|\big)\|x_{n+1}-x_n\| \\
&\qquad + 2r_n\|u_n-x_n\|\|Dx_n-Dx^*\|.
\end{aligned}$$

Since $\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0$, $\alpha_n \to 0$, $\beta_n \to 0$ and $\lim_{n\to\infty}\|Dx_n-Dx^*\| = 0$, we obtain

$$\lim_{n\to\infty}\|u_n-x_n\| = 0. \tag{3.17}$$

From (2.2), we get

$$\begin{aligned}
\|z_n-x^*\|^2 &= \|P_C[u_n-\lambda_nAu_n] - P_C[x^*-\lambda_nAx^*]\|^2 \\
&\le \big\langle z_n-x^*, (u_n-\lambda_nAu_n) - (x^*-\lambda_nAx^*)\big\rangle \\
&= \frac{1}{2}\big\{\|z_n-x^*\|^2 + \|u_n-x^*-\lambda_n(Au_n-Ax^*)\|^2 \\
&\quad - \|u_n-x^*-\lambda_n(Au_n-Ax^*) - (z_n-x^*)\|^2\big\} \\
&\le \frac{1}{2}\big\{\|z_n-x^*\|^2 + \|u_n-x^*\|^2 - \|u_n-z_n-\lambda_n(Au_n-Ax^*)\|^2\big\} \\
&\le \frac{1}{2}\big\{\|z_n-x^*\|^2 + \|u_n-x^*\|^2 - \|u_n-z_n\|^2 + 2\lambda_n\langle u_n-z_n, Au_n-Ax^*\rangle\big\} \\
&\le \frac{1}{2}\big\{\|z_n-x^*\|^2 + \|u_n-x^*\|^2 - \|u_n-z_n\|^2 \\
&\quad + 2\lambda_n\|u_n-z_n\|\|Au_n-Ax^*\|\big\}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\|z_n-x^*\|^2 &\le \|u_n-x^*\|^2 - \|u_n-z_n\|^2 + 2\lambda_n\|u_n-z_n\|\|Au_n-Ax^*\| \\
&\le \|x_n-x^*\|^2 - \|u_n-z_n\|^2 + 2\lambda_n\|u_n-z_n\|\|Au_n-Ax^*\|.
\end{aligned}$$

From (3.16) and the inequality above, we have

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &\le \alpha_n\|f(x_n)-x^*\|^2+(1-\alpha_n)\big(\beta_n\|Sx_n-x^*\|^2+(1-\beta_n)\|z_n-x^*\|^2\big)\\
&\le \alpha_n\|f(x_n)-x^*\|^2+(1-\alpha_n)\big\{\beta_n\|Sx_n-x^*\|^2+(1-\beta_n)\big(\|x_n-x^*\|^2\\
&\quad -\|u_n-z_n\|^2+2\lambda_n\|u_n-z_n\|\|Au_n-Ax^*\|\big)\big\}\\
&\le \alpha_n\|f(x_n)-x^*\|^2+\beta_n\|Sx_n-x^*\|^2+\|x_n-x^*\|^2\\
&\quad -(1-\alpha_n)(1-\beta_n)\|u_n-z_n\|^2+2\lambda_n\|u_n-z_n\|\|Au_n-Ax^*\|.
\end{aligned}$$

Hence,

$$\begin{aligned}
&(1-\alpha_n)(1-\beta_n)\|u_n-z_n\|^2\\
&\quad\le \alpha_n\|f(x_n)-x^*\|^2+\beta_n\|Sx_n-x^*\|^2+\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2\\
&\qquad +2\lambda_n\|u_n-z_n\|\|Au_n-Ax^*\|\\
&\quad\le \alpha_n\|f(x_n)-x^*\|^2+\beta_n\|Sx_n-x^*\|^2+\big(\|x_n-x^*\|+\|x_{n+1}-x^*\|\big)\|x_{n+1}-x_n\|\\
&\qquad +2\lambda_n\|u_n-z_n\|\|Au_n-Ax^*\|.
\end{aligned}$$

Since $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$, $\alpha_n\to 0$, $\beta_n\to 0$ and $\lim_{n\to\infty}\|Au_n-Ax^*\|=0$, we obtain

$$\lim_{n\to\infty}\|u_n-z_n\|=0. \tag{3.18}$$

It follows from (3.17) and (3.18) that

$$\lim_{n\to\infty}\|x_n-z_n\|=0. \tag{3.19}$$

Now, let $z\in F(T)\cap\Omega^*\cap MEP(F)$, since for each $i\ge 1$, $V_ix_n\in C$ and $\alpha_n+\sum_{i=1}^n(\alpha_{i-1}-\alpha_i)=1$, we have $\sum_{i=1}^n(\alpha_{i-1}-\alpha_i)V_ix_n+\alpha_nz\in C$. And

$$\begin{aligned}
\sum_{i=1}^n(\alpha_{i-1}-\alpha_i)(x_n-V_ix_n) &= P_C\left[\alpha_nf(x_n)+\sum_{i=1}^n(\alpha_{i-1}-\alpha_i)V_iy_n\right]+(1-\alpha_n)x_n\\
&\quad -\left(\sum_{i=1}^n(\alpha_{i-1}-\alpha_i)V_ix_n+\alpha_nz\right)+\alpha_nz-x_{n+1}\\
&= P_C\left[\alpha_nf(x_n)+\sum_{i=1}^n(\alpha_{i-1}-\alpha_i)V_iy_n\right]+\alpha_n(z-x_{n+1})\\
&\quad -P_C\left[\sum_{i=1}^n(\alpha_{i-1}-\alpha_i)V_ix_n+\alpha_nz\right]+(1-\alpha_n)(x_n-x_{n+1}).
\end{aligned}$$

It follows that

$$\begin{aligned}
&\sum_{i=1}^n(\alpha_{i-1}-\alpha_i)\langle x_n-V_ix_n,x_n-x^*\rangle\\
&\quad=\left\langle P_C\left[\alpha_nf(x_n)+\sum_{i=1}^n(\alpha_{i-1}-\alpha_i)V_iy_n\right]\right.
\end{aligned}$$

$$
\begin{aligned}
&- P_C\left[\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)V_i x_n + \alpha_n z\right], x_n - x^*\Big\rangle \\
&+ \alpha_n\langle z - x_{n+1}, x_n - x^*\rangle + (1-\alpha_n)\langle x_n - x_{n+1}, x_n - x^*\rangle \\
\le{}& \left\|\alpha_n\big(f(x_n)-z\big) + \sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)(V_i y_n - V_i x_n)\right\|\|x_n - x^*\| \\
&+ \alpha_n\|z - x_{n+1}\|\|x_n - x^*\| + (1-\alpha_n)\|x_n - x_{n+1}\|\|x_n - x^*\| \\
\le{}& \alpha_n\|f(x_n) - z\|\|x_n - x^*\| + \sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\|y_n - x_n\|\|x_n - x^*\| \\
&+ \alpha_n\|z - x_{n+1}\|\|x_n - x^*\| + (1-\alpha_n)\|x_n - x_{n+1}\|\|x_n - x^*\| \\
={}& \alpha_n\|f(x_n) - z\|\|x_n - x^*\| + (1-\alpha_n)\|y_n - x_n\|\|x_n - x^*\| \\
&+ \alpha_n\|z - x_{n+1}\|\|x_n - x^*\| + (1-\alpha_n)\|x_n - x_{n+1}\|\|x_n - x^*\| \\
\le{}& \alpha_n\|f(x_n) - z\|\|x_n - x^*\| + (1-\alpha_n)\|\beta_n S x_n + (1-\beta_n)z_n - x_n\|\|x_n - x^*\| \\
&+ \alpha_n\|z - x_{n+1}\|\|x_n - x^*\| + (1-\alpha_n)\|x_n - x_{n+1}\|\|x_n - x^*\| \\
\le{}& \alpha_n\|f(x_n) - z\|\|x_n - x^*\| + (1-\alpha_n)\beta_n\|S x_n - x_n\|\|x_n - x^*\| \\
&+ (1-\alpha_n)(1-\beta_n)\|z_n - x_n\|\|x_n - x^*\| \\
&+ \alpha_n\|z - x_{n+1}\|\|x_n - x^*\| + (1-\alpha_n)\|x_n - x_{n+1}\|\|x_n - x^*\|.
\end{aligned}
$$

From Lemma 2.9 and the inequality above, we get

$$
\begin{aligned}
&\frac{1}{2}\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\|x_n - V_i x_n\|^2 \\
&\quad\le \sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\langle x_n - V_i x_n, x_n - x^*\rangle \\
&\quad\le \alpha_n\|f(x_n) - z\|\|x_n - x^*\| + (1-\alpha_n)\beta_n\|S x_n - x_n\|\|x_n - x^*\| \\
&\quad\quad+ (1-\alpha_n)(1-\beta_n)\|z_n - x_n\|\|x_n - x^*\| + \alpha_n\|z - x_{n+1}\|\|x_n - x^*\| \\
&\quad\quad+ (1-\alpha_n)\|x_n - x_{n+1}\|\|x_n - x^*\|.
\end{aligned}
$$

Since $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$, $\alpha_n \to 0$, $\beta_n \to 0$ and $\lim_{n\to\infty}\|x_n - z_n\| = 0$, we obtain

$$
\lim_{n\to\infty}\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\|x_n - V_i x_n\|^2 = 0.
$$

Since $(\alpha_{i-1}-\alpha_i)\|x_n - V_i x_n\|^2 \le \sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\|x_n - V_i x_n\|^2$ and $\{\alpha_n\}$ is strictly decreasing, we have

$$
\lim_{n\to\infty}\|x_n - V_i x_n\| = 0.
$$

Hence, we obtain

$$\lim_{n\to\infty} \|x_n - T_i x_n\| = \lim_{n\to\infty} \frac{\|x_n - V_i x_n\|}{(1-k_i)} = 0, \quad \forall i \geq 1.$$

Since $\{x_n\}$ is bounded, without loss of generality, we can assume that $x_n \rightharpoonup w \in C$. It follows from Lemma 2.4 that $w \in F(T)$. Therefore, $w_w(x_n) \subset F(T)$. □

Theorem 3.1 *The sequence* $\{x_n\}$ *generated by Algorithm* 3.1 *converges strongly to* $z = P_{\Omega^*\cap MEP(F)\cap F(T)} f(z)$, *which is the unique solution of the variational inequality*

$$\langle (I-f)z, x-z\rangle \geq 0, \quad \forall x \in \Omega^* \cap MEP(F) \cap F(T). \tag{3.20}$$

Proof Since $\{x_n\}$ is bounded $x_n \rightharpoonup w$ and from Lemma 3.2, we have $w \in F(T)$. Next, we show that $w \in MEP(F)$. Since $u_n = T_{r_n}(x_n - r_n D x_n)$, we have

$$F(u_n, y) + \langle Dx_n, y-u_n\rangle + \frac{1}{r_n}\langle y-u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C.$$

It follows from monotonicity of F that

$$\langle Dx_n, y-u_n\rangle + \frac{1}{r_n}\langle y-u_n, u_n - x_n\rangle \geq F(y, u_n), \quad \forall y \in C$$

and

$$\langle Dx_{n_k}, y-u_{n_k}\rangle + \left\langle y-u_{n_k}, \frac{u_{n_k} - x_{n_k}}{r_{n_k}}\right\rangle \geq F(y, u_{n_k}), \quad \forall y \in C. \tag{3.21}$$

Since $\lim_{n\to\infty} \|u_n - x_n\| = 0$ and $x_n \rightharpoonup w$, it easy to observe that $u_{n_k} \to w$. For any $0 < t \leq 1$ and $y \in C$, let $y_t = ty + (1-t)w$, we have $y_t \in C$. Then from (3.21), we obtain

$$\begin{aligned}
\langle Dy_t, y_t - u_{n_k}\rangle &\geq \langle Dy_t, y_t - u_{n_k}\rangle - \langle Dx_{n_k}, y_t - u_{n_k}\rangle - \left\langle y_t - u_{n_k}, \frac{u_{n_k} - x_{n_k}}{r_{n_k}}\right\rangle + F(y_t, u_{n_k}) \\
&= \langle Dy_t - Du_{n_k}, y_t - u_{n_k}\rangle + \langle Du_{n_k} - Dx_{n_k}, y_t - u_{n_k}\rangle \\
&\quad - \left\langle y_t - u_{n_k}, \frac{u_{n_k} - x_{n_k}}{r_{n_k}}\right\rangle + F(y_t, u_{n_k}).
\end{aligned} \tag{3.22}$$

Since D is Lipschitz continuous and $\lim_{n\to\infty} \|u_n - x_n\| = 0$, we obtain $\lim_{k\to\infty} \|Du_{n_k} - Dx_{n_k}\| = 0$. From the monotonicity of D and $u_{n_k} \to w$, it follows from (3.22) that

$$\langle Dy_t, y_t - w\rangle \geq F(y_t, w). \tag{3.23}$$

Hence, from assumptions (i)-(iv) of Lemma 2.2 and (3.23), we have

$$\begin{aligned}
0 = F(y_t, y_t) &\leq tF(y_t, y) + (1-t)F(y_t, w) \\
&\leq tF(y_t, y) + (1-t)\langle Dy_t, y_t - w\rangle \\
&\leq tF(y_t, y) + (1-t)t\langle Dy_t, y - w\rangle,
\end{aligned} \tag{3.24}$$

which implies that $F(y_t, y) + (1-t)\langle Dy_t, y - w\rangle \geq 0$. Letting $t \to 0_+$, we have

$$F(w, y) + \langle Dw, y - w\rangle \geq 0, \quad \forall y \in C,$$

which implies that $w \in MEP(F)$.

Furthermore, we show that $w \in \Omega^*$. Let

$$Tv = \begin{cases} Av + N_C v, & \forall v \in C, \\ \emptyset, & \text{otherwise}, \end{cases}$$

where $N_C v := \{w \in H : \langle w, v - u\rangle \geq 0, \forall u \in C\}$ is the normal cone to C at $v \in C$. Then T is maximal monotone and $0 \in Tv$ if and only if $v \in \Omega^*$ (see [18]). Let $G(T)$ denote the graph of T, and let $(v, u) \in G(T)$, since $u - Av \in N_C v$ and $z_n \in C$, we have

$$\langle v - z_n, u - Av\rangle \geq 0. \tag{3.25}$$

On the other hand, it follows from $z_n = P_C[u_n - \lambda_n A u_n]$ and $v \in C$ that

$$\bigl\langle v - z_n, z_n - (u_n - \lambda_n A u_n)\bigr\rangle \geq 0$$

and

$$\left\langle v - z_n, \frac{z_n - u_n}{\lambda_n} + Au_n\right\rangle \geq 0.$$

Therefore, from (3.25) and inverse strongly monotonicity of A, we have

$$\begin{aligned} \langle v - z_{n_k}, u\rangle &\geq \langle v - z_{n_k}, Av\rangle \\ &\geq \langle v - z_{n_k}, Av\rangle - \left\langle v - z_{n_k}, \frac{z_{n_k} - u_{n_k}}{\lambda_{n_k}} + Au_{n_k}\right\rangle \\ &\geq \langle v - z_{n_k}, Av - Az_{n_k}\rangle + \langle v - z_{n_k}, Az_{n_k} - Au_{n_k}\rangle - \left\langle v - z_{n_k}, \frac{z_{n_k} - u_{n_k}}{\lambda_{n_k}}\right\rangle \\ &\geq \langle v - z_{n_k}, Az_{n_k} - Au_{n_k}\rangle - \left\langle v - z_{n_k}, \frac{z_{n_k} - u_{n_k}}{\lambda_{n_k}}\right\rangle. \end{aligned}$$

Since $\lim_{n\to\infty} \|u_n - z_n\| = 0$ and $u_{n_k} \to w$, it easy to observe that $z_{n_k} \to w$. Hence, we obtain $\langle v - w, u\rangle \geq 0$. Since T is maximal monotone, we have $w \in T^{-1}0$, and hence, $w \in \Omega^*$. Thus, we have

$$w \in \Omega^* \cap MEP(F) \cap F(T).$$

Next, we claim that $\limsup_{n\to\infty}\langle f(z) - z, x_n - z\rangle \leq 0$, where $z = P_{\Omega^*\cap MEP(F)\cap F(T)} f(z)$.

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\bigl\langle f(z) - z, x_n - z\bigr\rangle = \limsup_{k\to\infty}\bigl\langle f(z) - z, x_{n_k} - z\bigr\rangle = \bigl\langle f(z) - z, w - z\bigr\rangle \leq 0.$$

Next, we show that $x_n \to z$.

$$\begin{aligned}
\|x_{n+1}-z\|^2 &= \left\langle x_{n+1}-\alpha_n f(x_n)-\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)V_i y_n, x_{n+1}-z\right\rangle \\
&\quad+\left\langle \alpha_n f(x_n)+\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)V_i y_n - z, x_{n+1}-z\right\rangle \\
&\le \left\langle \alpha_n f(x_n)+\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)V_i y_n - z, x_{n+1}-z\right\rangle \\
&\le \alpha_n\langle f(x_n)-f(z), x_{n+1}-z\rangle+\alpha_n\langle f(z)-z, x_{n+1}-z\rangle \\
&\quad+\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\langle V_i y_n - z, x_{n+1}-z\rangle \\
&\le \alpha_n\|f(x_n)-f(z)\|\|x_{n+1}-z\|+\alpha_n\langle f(z)-z, x_{n+1}-z\rangle \\
&\quad+\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\|V_i y_n - z\|\|x_{n+1}-z\| \\
&\le \alpha_n\rho\|x_n-z\|\|x_{n+1}-z\|+\alpha_n\langle f(z)-z, x_{n+1}-z\rangle \\
&\quad+\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\|y_n - z\|\|x_{n+1}-z\| \\
&\le \alpha_n\rho\|x_n-z\|\|x_{n+1}-z\|+\alpha_n\langle f(z)-z, x_{n+1}-z\rangle \\
&\quad+(1-\alpha_n)\bigl\{\beta_n\|Sx_n-Sz\|+\beta_n\|Sz-z\|+(1-\beta_n)\|z_n-z\|\bigr\}\|x_{n+1}-z\| \\
&\le \alpha_n\rho\|x_n-z\|\|x_{n+1}-z\|+\alpha_n\langle f(z)-z, x_{n+1}-z\rangle \\
&\quad+(1-\alpha_n)\bigl\{\beta_n\|x_n-z\|+\beta_n\|Sz-z\|+(1-\beta_n)\|x_n-z\|\bigr\}\|x_{n+1}-z\| \\
&\le \bigl(1-\alpha_n(1-\rho)\bigr)\|x_n-z\|\|x_{n+1}-z\|+\alpha_n\langle f(z)-z, x_{n+1}-z\rangle \\
&\quad+(1-\alpha_n)\beta_n\|Sz-z\|\|x_{n+1}-z\| \\
&\le \frac{1-\alpha_n(1-\rho)}{2}\bigl(\|x_n-z\|^2+\|x_{n+1}-z\|^2\bigr)+\alpha_n\langle f(z)-z, x_{n+1}-z\rangle \\
&\quad+(1-\alpha_n)\beta_n\|Sz-z\|\|x_{n+1}-z\|,
\end{aligned}$$

which implies that

$$\begin{aligned}
\|x_{n+1}-z\|^2 &\le \left(1-\frac{2\alpha_n(1-\rho)}{1+\alpha_n(1-\rho)}\right)\|x_n-z\|^2+\frac{2\alpha_n}{1+\alpha_n(1-\rho)}\langle f(z)-z, x_{n+1}-z\rangle \\
&\quad+\frac{2(1-\alpha_n)\beta_n}{1+\alpha_n(1-\rho)}\|Sz-z\|\|x_{n+1}-z\| \\
&\le \left(1-\frac{2\alpha_n(1-\rho)}{1+\alpha_n(1-\rho)}\right)\|x_n-z\|^2+\frac{2\alpha_n(1-\rho)}{1+\alpha_n(1-\rho)}\left\{\frac{1}{1-\rho}\langle f(z)-z, x_{n+1}-z\rangle\right. \\
&\quad\left.+\frac{(1-\alpha_n)\beta_n}{\alpha_n(1-\rho)}\|Sz-z\|\|x_{n+1}-z\|\right\}.
\end{aligned}$$

Let $\gamma_n = \frac{2\alpha_n(1-\rho)}{1+\alpha_n(1-\rho)}$ and $\delta_n = \frac{2\alpha_n(1-\rho)}{1+\alpha_n(1-\rho)}\{\frac{1}{1-\rho}\langle f(z)-z, x_{n+1}-z\rangle + \frac{(1-\alpha_n)\beta_n}{\alpha_n(1-\rho)}\|Sz-z\|\|x_{n+1}-z\|\}$.

Since

$$\sum_{n=1}^{\infty}\alpha_n=\infty,\qquad 1+\alpha_n(1-\rho)\leq 2\quad\text{and}$$

$$\limsup_{n\to\infty}\left\{\frac{1}{1-\rho}\langle f(z)-z,x_{n+1}-z\rangle+\frac{(1-\alpha_n)\beta_n}{\alpha_n(1-\rho)}\|Sz-z\|\|x_{n+1}-z\|\right\}\leq 0.$$

It follows that

$$\sum_{n=1}^{\infty}\gamma_n=\infty\quad\text{and}\quad\limsup_{n\to\infty}\frac{\delta_n}{\gamma_n}\leq 0.$$

Thus, all the conditions of Lemma 2.6 are satisfied. Hence, we deduce that $x_n\to z$.

Since $P_{\Omega^*\cap MEP(F)\cap F(T)}f$ is a contraction, there exists a unique $z\in C$ such that $z=P_{\Omega^*\cap MEP(F)\cap F(T)}f(z)$. From (2.1), it follows that z is the unique solution of problem (3.20). This completes the proof. □

Theorem 3.2 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $D,A:C\to H$ be θ,α-inverse strongly monotone mappings, respectively. Let $F:C\times C\to\mathbb{R}$ be a bifunction satisfying the assumptions* (i)-(iv) *of Lemma* 2.2, *$S:C\to H$ be a nonexpansive mapping, and $\{T_i\}_{i=1}^{\infty}:C\to C$ is a countable family of k_i-strict pseudo-contraction mappings such that $F(T)\cap\Omega^*\cap MEP(F)\neq\emptyset$, where $F(T)=\bigcap_{i=1}^{\infty}F(T_i)$. Let f be a ρ-contraction mapping. For a given $x_0\in C$ arbitrarily, let the iterative sequences $\{u_n\}$, $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be generated by*

$$\begin{aligned}&F(u_n,y)+\langle Dx_n,y-u_n\rangle+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\geq 0,\quad\forall y\in C;\\&z_n=P_C[u_n-\lambda_nAu_n];\\&y_n=\beta_nSx_n+(1-\beta_n)z_n;\\&x_{n+1}=P_C\left[\alpha_nf(x_n)+\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)V_iy_n\right],\quad\forall n\geq 0,\end{aligned}\tag{3.26}$$

where $V_i=k_iI+(1-k_i)T_i$, $0\leq k_i<1$, $\alpha_0=1$, $\{\alpha_n\}$ is a strictly decreasing sequence in $(0,1)$, *and $\{\beta_n\}$ is a sequence in* $(0,1)$ *satisfying the following conditions*:

(a) $\lim_{n\to\infty}\alpha_n=0$ *and* $\sum_{n=1}^{\infty}\alpha_n=\infty$,
(b) $\lim_{n\to\infty}\frac{\beta_n}{\alpha_n}=\tau\in(0,\infty)$,
(c) $\sum_{n=1}^{\infty}(\alpha_{n-1}-\alpha_n)<\infty$ *and* $\sum_{n=1}^{\infty}|\beta_{n-1}-\beta_n|<\infty$,
(d) $\lim_{n\to\infty}\frac{\frac{1}{\mu}|r_n-r_{n-1}|+|\lambda_n-\lambda_{n-1}|+|\alpha_{n-1}-\alpha_n|+|\beta_{n-1}-\beta_n|}{\alpha_n\beta_n}=0$,
(e) *there exists a constant $K>0$ such that* $\frac{1}{\alpha_n}|\frac{1}{\beta_n}-\frac{1}{\beta_{n-1}}|\leq K$,
(f) $\liminf_{n\to\infty}r_n>0$ *and* $\sum_{n=1}^{\infty}|r_{n-1}-r_n|<\infty$,
(g) $\liminf_{n\to\infty}\lambda_n<\limsup_{n\to\infty}\lambda_n<2\alpha$ *and* $\sum_{n=1}^{\infty}|\lambda_{n-1}-\lambda_n|<\infty$.

Then sequence $\{x_n\}$ generated by (3.26) *converges strongly to $x^*\in\Omega^*\cap MEP(F)\cap F(T)$, which is the unique solution of the variational inequality*

$$\left\langle\frac{1}{\tau}(I-f)x^*+(I-S)x^*,x-x^*\right\rangle\geq 0,\quad\forall x\in\Omega^*\cap MEP(F)\cap F(T).\tag{3.27}$$

Proof From $\lim_{n\to\infty}(\beta_n/\alpha_n) = \tau \in (0,\infty)$, without loss of generality, we can assume that $\beta_n \le (1+\tau)\alpha_n$ for all $n \ge 1$. Hence, $\beta_n \to 0$. By similar argument as that in Lemmas 3.1 and 3.2, we can deduce that $\{x_n\}$ is bounded, $\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0$, $\lim_{n\to\infty}\|x_n - z_n\| = 0$ (see (3.19)) and $(I - V_i)x_n \to 0$. Then we have

$$\|y_n - x_n\| \le \beta_n\|x_n - Sx_n\| + (1-\beta_n)\|x_n - z_n\| \to 0 \quad \text{as } n \to \infty. \tag{3.28}$$

It follows that for all $i \ge 1$,

$$\|y_n - V_i x_n\| \le \|y_n - x_n\| + \|x_n - V_i x_n\| \to 0 \quad \text{as } n \to \infty. \tag{3.29}$$

From (3.28) and (3.29), we have

$$\begin{aligned}\|y_n - V_i y_n\| &\le \|y_n - V_i x_n\| + \|V_i x_n - V_i y_n\| \\ &\le \|y_n - V_i x_n\| + \|y_n - x_n\| \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

Set $w_n = \alpha_n f(x_n) + \sum_{i=1}^n (\alpha_{i-1} - \alpha_i)V_i y_n$. From (3.14) and (3.15), we obtain

$$\begin{aligned}\frac{\|x_{n+1}-x_n\|}{\beta_n} &\le \frac{\|w_n - w_{n-1}\|}{\beta_n} \\ &\le \bigl(1-(1-\rho)\alpha_n\bigr)\frac{\|x_n - x_{n-1}\|}{\beta_n} \\ &\quad + M\biggl(\frac{1}{\mu}\frac{|r_n - r_{n-1}|}{\beta_n} + \frac{|\lambda_n - \lambda_{n-1}|}{\beta_n} + \frac{|\beta_n - \beta_{n-1}|}{\beta_n} + \frac{|\alpha_n - \alpha_{n-1}|}{\beta_n}\biggr) \\ &= \bigl(1-(1-\rho)\alpha_n\bigr)\frac{\|x_n - x_{n-1}\|}{\beta_{n-1}} + \bigl(1-(1-\rho)\alpha_n\bigr)\|x_n - x_{n-1}\|\biggl(\frac{1}{\beta_n} - \frac{1}{\beta_{n-1}}\biggr) \\ &\quad + M\biggl(\frac{1}{\mu}\frac{|r_n - r_{n-1}|}{\beta_n} + \frac{|\lambda_n - \lambda_{n-1}|}{\beta_n} + \frac{|\beta_n - \beta_{n-1}|}{\beta_n} + \frac{|\alpha_n - \alpha_{n-1}|}{\beta_n}\biggr) \\ &\le \bigl(1-(1-\rho)\alpha_n\bigr)\frac{\|x_n - x_{n-1}\|}{\beta_{n-1}} + \|x_n - x_{n-1}\|\biggl|\frac{1}{\beta_n} - \frac{1}{\beta_{n-1}}\biggr| \\ &\quad + M\biggl(\frac{1}{\mu}\frac{|r_n - r_{n-1}|}{\beta_n} + \frac{|\lambda_n - \lambda_{n-1}|}{\beta_n} + \frac{|\beta_n - \beta_{n-1}|}{\beta_n} + \frac{|\alpha_n - \alpha_{n-1}|}{\beta_n}\biggr) \\ &\le \bigl(1-(1-\rho)\alpha_n\bigr)\frac{\|x_n - x_{n-1}\|}{\beta_{n-1}} + \alpha_n K\|x_n - x_{n-1}\| \\ &\quad + M\biggl(\frac{1}{\mu}\frac{|r_n - r_{n-1}|}{\beta_n} + \frac{|\lambda_n - \lambda_{n-1}|}{\beta_n} + \frac{|\beta_n - \beta_{n-1}|}{\beta_n} + \frac{|\alpha_n - \alpha_{n-1}|}{\beta_n}\biggr) \\ &\le \bigl(1-(1-\rho)\alpha_n\bigr)\frac{\|w_{n-1} - w_{n-2}\|}{\beta_{n-1}} + \alpha_n K\|x_n - x_{n-1}\| \\ &\quad + M\biggl(\frac{1}{\mu}\frac{|r_n - r_{n-1}|}{\beta_n} + \frac{|\lambda_n - \lambda_{n-1}|}{\beta_n} + \frac{|\beta_n - \beta_{n-1}|}{\beta_n} + \frac{|\alpha_n - \alpha_{n-1}|}{\beta_n}\biggr).\end{aligned}$$

Let $\gamma_n = (1-\rho)\alpha_n$ and $\delta_n = \alpha_n K\|x_n - x_{n-1}\| + M(\frac{1}{\mu}\frac{|r_n - r_{n-1}|}{\beta_n} + \frac{|\lambda_n - \lambda_{n-1}|}{\beta_n} + \frac{|\beta_n - \beta_{n-1}|}{\beta_n} + \frac{|\alpha_n - \alpha_{n-1}|}{\beta_n})$. From conditions (a) and (d) of Theorem 3.2, we have

$$\sum_{n=1}^{\infty}\gamma_n = \infty \quad \text{and} \quad \lim_{n\to\infty}\frac{\delta_n}{\gamma_n} = 0.$$

By Lemma 2.6, we obtain

$$\lim_{n\to\infty}\frac{\|x_{n+1}-x_n\|}{\beta_n}=0,\qquad \lim_{n\to\infty}\frac{\|w_{n+1}-w_n\|}{\beta_n}=\lim_{n\to\infty}\frac{\|w_{n+1}-w_n\|}{\alpha_n}=0.$$

From (3.26), we have

$$x_{n+1}=P_C[w_n]-w_n+\alpha_n f(x_n)+\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)(V_iy_n-y_n)+(1-\alpha_n)y_n.$$

Hence, it follows that

$$\begin{aligned}x_n-x_{n+1}&=(1-\alpha_n)x_n+\alpha_nx_n\\&\quad-\left(P_C[w_n]-w_n+\alpha_nf(x_n)+\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)(V_iy_n-y_n)+(1-\alpha_n)y_n\right)\\&=(1-\alpha_n)\big[\beta_n(x_n-Sx_n)+(1-\beta_n)(x_n-z_n)\big]+\big(w_n-P_C[w_n]\big)\\&\quad+\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)(y_n-V_iy_n)+\alpha_n\big(x_n-f(x_n)\big),\end{aligned}$$

and hence,

$$\begin{aligned}\frac{x_n-x_{n+1}}{(1-\alpha_n)\beta_n}&=x_n-Sx_n+\frac{(1-\beta_n)}{\beta_n}(x_n-z_n)+\frac{1}{(1-\alpha_n)\beta_n}\big(w_n-P_C[w_n]\big)\\&\quad+\frac{1}{(1-\alpha_n)\beta_n}\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)(y_n-V_iy_n)+\frac{\alpha_n}{(1-\alpha_n)\beta_n}\big(x_n-f(x_n)\big).\end{aligned}$$

Let $v_n=\frac{x_n-x_{n+1}}{(1-\alpha_n)\beta_n}$. For any $z\in\Omega^*\cap MEP(F)\cap F(T)$, we have

$$\begin{aligned}\langle v_n,x_n-z\rangle&=\frac{1}{(1-\alpha_n)\beta_n}\big\langle w_n-P_C[w_n],P_C[w_{n-1}]-z\big\rangle+\frac{\alpha_n}{(1-\alpha_n)\beta_n}\big\langle (I-f)x_n,x_n-z\big\rangle\\&\quad+\langle x_n-Sx_n,x_n-z\rangle+\frac{(1-\beta_n)}{\beta_n}\langle x_n-z_n,x_n-z\rangle\\&\quad+\frac{1}{(1-\alpha_n)\beta_n}\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\langle y_n-V_iy_n,x_n-z\rangle.\end{aligned}\tag{3.30}$$

Since S is a nonexpansive mapping, f is a ρ-contraction mapping, and V_i is a k_i-strict pseudo-contraction mapping. Then $(I-S)$ and $(I-V_i)$ are monotones, and f is strongly monotone with coefficient $(1-\rho)$. We can deduce

$$\begin{aligned}\langle x_n-Sx_n,x_n-z\rangle&=\big\langle (I-S)x_n-(I-S)z,x_n-z\big\rangle+\big\langle (I-S)z,x_n-z\big\rangle\\&\ge\big\langle (I-S)z,x_n-z\big\rangle,\end{aligned}\tag{3.31}$$

$$\begin{aligned}\big\langle (I-f)x_n,x_n-z\big\rangle&=\big\langle (I-f)x_n-(I-f)z,x_n-z\big\rangle+\big\langle (I-f)z,x_n-z\big\rangle\\&\ge(1-\rho)\|x_n-z\|^2+\big\langle (I-f)z,x_n-z\big\rangle,\end{aligned}\tag{3.32}$$

$$\begin{aligned}\langle (I-V_i)y_n, x_n - z\rangle &= \langle (I-V_i)y_n - (I-V_i)z, x_n - y_n\rangle \\ &\quad + \langle (I-V_i)y_n - (I-V_i)z, y_n - z\rangle \\ &\geq \langle (I-V_i)y_n - (I-V_i)z, x_n - y_n\rangle \\ &= \langle (I-V_i)y_n, x_n - y_n\rangle \\ &= \langle (I-V_i)y_n, \beta_n(x_n - Sx_n) + (1-\beta_n)(x_n - z_n)\rangle. \end{aligned} \tag{3.33}$$

From (2.1), we get

$$\begin{aligned}&\langle w_n - P_C[w_n], P_C[w_{n-1}] - z\rangle \\ &\quad = \langle w_n - P_C[w_n], P_C[w_{n-1}] - P_C[w_n]\rangle + \langle w_n - P_C[w_n], P_C[w_n] - z\rangle \\ &\quad \geq \langle w_n - P_C[w_n], P_C[w_{n-1}] - P_C[w_n]\rangle.\end{aligned}$$

Then from (3.30)-(3.32), we have

$$\begin{aligned}\langle v_n, x_n - z\rangle &\geq \frac{1}{(1-\alpha_n)\beta_n}\langle w_n - P_C[w_n], P_C[w_{n-1}] - P_C[w_n]\rangle + \frac{\alpha_n}{(1-\alpha_n)\beta_n}\langle (I-f)z, x_n - z\rangle \\ &\quad + \langle (I-S)z, x_n - z\rangle + \frac{(1-\beta_n)}{\beta_n}\langle x_n - z_n, x_n - z\rangle \\ &\quad + \frac{(1-\beta_n)}{(1-\alpha_n)\beta_n}\sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)\langle (I-V_i)y_n, x_n - z_n\rangle \\ &\quad + \frac{1}{(1-\alpha_n)}\sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)\langle (I-V_i)y_n, x_n - Sx_n\rangle + \frac{(1-\rho)\alpha_n}{(1-\alpha_n)\beta_n}\|x_n - z\|^2.\end{aligned}$$

Then we obtain

$$\begin{aligned}\|x_n - z\|^2 &\leq \frac{1}{(1-\rho)\alpha_n}\|w_n - P_C[w_n]\|\|w_{n-1} - w_n\| - \frac{1}{(1-\rho)}\langle (I-f)z, x_n - z\rangle \\ &\quad + \frac{(1-\alpha_n)\beta_n}{(1-\rho)\alpha_n}\big(\langle v_n, x_n - z\rangle - \langle (I-S)z, x_n - z\rangle\big) \\ &\quad - \frac{(1-\beta_n)(1-\alpha_n)}{(1-\rho)\alpha_n}\langle x_n - z_n, x_n - z\rangle \\ &\quad - \frac{(1-\beta_n)}{(1-\rho)\alpha_n}\sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)\langle (I-V_i)y_n, x_n - z_n\rangle \\ &\quad - \frac{\beta_n}{(1-\rho)\alpha_n}\sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)\langle (I-V_i)y_n, x_n - Sx_n\rangle. \\ &\leq \frac{\|w_{n-1} - w_n\|}{(1-\rho)\alpha_n}\|w_n - P_C[w_n]\| - \frac{1}{(1-\rho)}\langle (I-f)z, x_n - z\rangle \\ &\quad + \frac{(1-\alpha_n)\beta_n}{(1-\rho)\alpha_n}\big(\langle v_n, x_n - z\rangle - \langle (I-S)z, x_n - z\rangle\big) \\ &\quad + \frac{1}{(1-\rho)}\frac{(1-\beta_n)}{\beta_n}\frac{\beta_n}{\alpha_n}\|x_n - z_n\|\|x_n - z\|\end{aligned}$$

$$+\frac{1}{(1-\rho)}\frac{(1-\beta_n)}{\beta_n}\frac{\beta_n}{\alpha_n}\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\big\|(I-V_i)y_n\big\|\|x_n-z_n\|$$
$$-\frac{\beta_n}{(1-\rho)\alpha_n}\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\big\langle(I-V_i)y_n,x_n-Sx_n\big\rangle.$$

By condition (e) of Theorem 3.2, there exists a constant $N>0$ such that $\frac{1-\beta_n}{\beta_n}\le N$. Since $\lim_{n\to\infty}\|x_n-z_n\|=0$, $v_n\to 0$, $(I-V_i)y_n\to 0$ and $\frac{\|w_{n-1}-w_n\|}{\alpha_n}\to 0$ as $n\to\infty$, then every weak cluster point of $\{x_n\}$ is also a strong cluster point. Since $\{x_n\}$ is bounded, by Lemma 3.2, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converging to a point $x^*\in F(T)$, by a similar argument as that in Theorem 3.1, we can show that $x^*\in\Omega^*\cap MEP(F)\cap F(T)$.

From (3.30)-(3.32), it follows that for any $z\in\Omega^*\cap MEP(F)\cap F(T)$,

$$\begin{aligned}
&\big\langle(I-f)x_{n_k},x_{n_k}-z\big\rangle\\
&\quad=\frac{(1-\alpha_{n_k})\beta_{n_k}}{\alpha_{n_k}}\langle v_{n_k},x_{n_k}-z\rangle-\frac{1}{\alpha_{n_k}}\big\langle w_{n_k}-P_C[w_{n_k}],P_C[w_{n_k-1}]-z\big\rangle\\
&\qquad-\frac{(1-\alpha_{n_k})\beta_{n_k}}{\alpha_{n_k}}\langle x_{n_k}-Sx_{n_k},x_{n_k}-z\rangle-\frac{(1-\alpha_{n_k})(1-\beta_{n_k})}{\alpha_{n_k}}\langle x_{n_k}-z_{n_k},x_{n_k}-z\rangle\\
&\qquad-\frac{1}{\alpha_{n_k}}\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\langle y_{n_k}-V_iy_{n_k},x_{n_k}-z\rangle\\
&\quad\le\frac{(1-\alpha_{n_k})\beta_{n_k}}{\alpha_{n_k}}\langle v_{n_k},x_{n_k}-z\rangle+\frac{1}{\alpha_{n_k}}\big\|w_{n_k}-P_C[w_{n_k}]\big\|\|w_{n_k-1}-w_{n_k}\|\\
&\qquad-\frac{(1-\alpha_{n_k})\beta_{n_k}}{\alpha_{n_k}}\langle x_{n_k}-Sx_{n_k},x_{n_k}-z\rangle+\frac{(1-\beta_{n_k})}{\beta_{n_k}}\frac{\beta_{n_k}}{\alpha_{n_k}}\|x_{n_k}-z_{n_k}\|\|x_{n_k}-z\|\\
&\qquad+\frac{(1-\beta_{n_k})}{\beta_{n_k}}\frac{\beta_{n_k}}{\alpha_{n_k}}\sum_{i=1}^{n_k}(\alpha_{i-1}-\alpha_i)\big\|(I-V_i)y_{n_k}\big\|\|x_{n_k}-z_{n_k}\|\\
&\qquad-\frac{\beta_{n_k}}{\alpha_{n_k}}\sum_{i=1}^{n_k}(\alpha_{i-1}-\alpha_i)\big\langle(I-V_i)y_{n_k},x_{n_k}-Sx_{n_k}\big\rangle.
\end{aligned}\tag{3.34}$$

Since $\lim_{n\to\infty}\|x_n-z_n\|=0$, $v_n\to 0$, $(I-V_i)y_n\to 0$ and $\frac{\|w_{n-1}-w_n\|}{\alpha_n}\to 0$, letting $k\to\infty$ in (3.34), we obtain

$$\big\langle(I-f)x^*,x^*-z\big\rangle\le-\tau\big\langle x^*-Sx^*,x^*-z\big\rangle$$

i.e.,

$$\left\langle\frac{1}{\tau}(I-f)x^*+(I-S)x^*,z-x^*\right\rangle\ge 0.$$

In the following, we show that (3.27) has a unique solution. Assume that x' is another solution. Then we have

$$\big\langle(I-f)x',x'-x^*\big\rangle\le-\tau\big\langle x'-Sx',x'-x^*\big\rangle,\tag{3.35}$$
$$\big\langle(I-f)x^*,x^*-x'\big\rangle\le-\tau\big\langle x^*-Sx^*,x^*-x'\big\rangle.\tag{3.36}$$

Adding (3.35) and (3.36), we get

$$\begin{aligned}(1-\rho)\|x'-x^*\|^2 &\le \langle (I-f)x'-(I-f)x^*, x'-x^*\rangle \\ &\le -\tau\langle (I-S)x'-(I-S)x^*, x'-x^*\rangle \\ &\le 0.\end{aligned}$$

Then $x' = x^*$. Since (3.27) has a unique solution, it follows that $w_w(x_n) = \{x^*\}$. Since every weak cluster point of $\{x_n\}$ is also a strong cluster point, we conclude that $\{x_n\} \to x^*$. This completes the proof. □

4 Applications

In this section, we obtain the following results by using a special case of the proposed method. The first result can be viewed as extension and improvement of the method of Gu *et al.* [6] for finding the approximate element of the common set of solutions of a generalized equilibrium problem and a hierarchical fixed point problem in a real Hilbert space.

Corollary 4.1 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $D : C \to H$ be θ-inverse strongly monotone mappings, respectively. Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying the assumptions* (i)-(iv) *of Lemma* 2.2, *$S : C \to H$ be a nonexpansive mapping, and $\{T_i\}_{i=1}^{\infty} : C \to C$ is a countable family of k_i-strict pseudo-contraction mappings such that $F(T) \cap MEP(F) \neq \emptyset$, where $F(T) = \bigcap_{i=1}^{\infty} F(T_i)$. Let f be a ρ-contraction mapping. For a given $x_0 \in C$ arbitrarily, let the iterative sequences $\{u_n\}$, $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be generated by*

$$\begin{aligned}&F(u_n, y) + \langle Dx_n, y-u_n\rangle + \frac{1}{r_n}\langle y-u_n, u_n-x_n\rangle \ge 0, \quad \forall y \in C; \\ &y_n = \beta_n Sx_n + (1-\beta_n)u_n; \\ &x_{n+1} = P_C\left[\alpha_n f(x_n) + \sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)T_i y_n\right], \quad \forall n \ge 0,\end{aligned} \tag{4.1}$$

where $\alpha_0 = 1$, $\{\alpha_n\}$ is a strictly decreasing sequence in $(0,1)$, *and $\{\beta_n\}$ is a sequence in* $(0,1)$ *satisfying the following conditions*:

(a) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$,

(b) $\lim_{n\to\infty}\frac{\beta_n}{\alpha_n} = \tau \in (0,\infty)$,

(c) $\sum_{n=1}^{\infty}(\alpha_{n-1}-\alpha_n) < \infty$ *and* $\sum_{n=1}^{\infty}|\beta_{n-1}-\beta_n| < \infty$,

(d) $\lim_{n\to\infty}\frac{\frac{1}{\mu}|r_n-r_{n-1}|+|\alpha_{n-1}-\alpha_n|+|\beta_{n-1}-\beta_n|}{\alpha_n\beta_n} = 0$,

(e) *there exists a constant $K > 0$ such that* $\frac{1}{\alpha_n}|\frac{1}{\beta_n}-\frac{1}{\beta_{n-1}}| \le K$,

(f) $\liminf_{n\to\infty} r_n > 0$ *and* $\sum_{n=1}^{\infty}|r_{n-1}-r_n| < \infty$.

Then sequence $\{x_n\}$ generated by algorithm (4.1) *converges strongly to $x^* \in MEP(F) \cap F(T)$, which is the unique solution of the variational inequality*

$$\left\langle \frac{1}{\tau}(I-f)x^* + (I-S)x^*, x-x^*\right\rangle \ge 0, \quad \forall x \in MEP(F) \cap F(T).$$

Proof Putting $A = 0$ and $k_i = 0$, $\forall i \geq 1$ in Theorem 3.2. Then conclusion of Corollary 4.1 is obtained. □

The following result can be viewed as extension and improvement of the method of Yao *et al.* [20] for finding the approximate element of the common set of solutions of a generalized equilibrium problem and a hierarchical fixed point problem in a real Hilbert space.

Corollary 4.2 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $D: C \to H$ be θ-inverse strongly monotone mappings, respectively. Let $F: C \times C \to \mathbb{R}$ be a bifunction satisfying assumptions* (i)-(iv) *of Lemma* 2.2, *$S: C \to H$ be a nonexpansive mapping, and $T: C \to C$ is a countable family of k-strict pseudo-contraction mappings such that $F(T) \cap MEP(F) \neq \emptyset$. Let f be a ρ-contraction mapping. For a given $x_0 \in C$ arbitrarily, let the iterative sequences $\{u_n\}$, $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be generated by*

$$
\begin{aligned}
&F(u_n, y) + \langle Dx_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C; \\
&y_n = \beta_n S x_n + (1 - \beta_n) u_n; \\
&x_{n+1} = P_C\left[\alpha_n f(x_n) + (1 - \alpha_n) T y_n\right], \quad \forall n \geq 0,
\end{aligned}
\tag{4.2}
$$

where $\alpha_0 = 1$, $\{\alpha_n\}$ is a strictly decreasing sequence in $(0,1)$, *and $\{\beta_n\}$ is a sequence in* $(0,1)$ *satisfying the following conditions*:

(a) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(b) $\lim_{n\to\infty} \frac{\beta_n}{\alpha_n} = \tau \in (0,\infty)$,
(c) $\sum_{n=1}^{\infty} (\alpha_{n-1} - \alpha_n) < \infty$ *and* $\sum_{n=1}^{\infty} |\beta_{n-1} - \beta_n| < \infty$,
(d) $\lim_{n\to\infty} \frac{\frac{1}{\mu}|r_n - r_{n-1}| + |\alpha_{n-1} - \alpha_n| + |\beta_{n-1} - \beta_n|}{\alpha_n \beta_n} = 0$,
(e) *there exists a constant $K > 0$ such that* $\frac{1}{\alpha_n}|\frac{1}{\beta_n} - \frac{1}{\beta_{n-1}}| \leq K$,
(f) $\liminf_{n\to\infty} r_n > 0$ *and* $\sum_{n=1}^{\infty} |r_{n-1} - r_n| < \infty$.

Then sequence $\{x_n\}$ generated by algorithm (4.2) *converges strongly to $x^* \in MEP(F) \cap F(T)$, which is the unique solution of the variational inequality*

$$
\left\langle \frac{1}{\tau}(I - f)x^* + (I - S)x^*, x - x^* \right\rangle \geq 0, \quad \forall x \in MEP(F) \cap F(T).
$$

Proof Putting $A = 0$, $k_i = 0$ and $T_i = T$ $\forall i \geq 1$ in Theorem 3.2. Then conclusion of Corollary 4.2 is obtained. □

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

AB carried out the main part of this article. All authors read and approved the final manuscript.

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