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# Non-differentiable minimax fractional programming with higher-order type I functions

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**Abstract**
In this article, we are concerned with a class of non-differentiable minimax fractional programming problems and their higher-order dual model. Weak, strong and converse duality theorems are discussed involving generalized higher-order type I functions. The presented results extend some previously known results on non-differentiable minimax fractional programming.

## 1 Introduction

In nonlinear optimization, problems, where minimization and maximization process are performed together, are called minimax (minmax) problems. Frequently, problems of this type arise in many areas like game theory, Chebychev approximation, economics, financial planning and facility location [1].

The optimization problems in which the objective function is a ratio of two functions are commonly known as fractional programming problems. In the past few years, many authors have shown interest in the field of minimax fractional programming problems. Schmittendorf [2] first developed necessary and sufficient optimality conditions for a minimax programming problem. Tanimoto [3] applied the necessary conditions in [2] to formulate a dual problem and discussed the duality results, which were extended to a fractional analogue of the problem considered in [2, 3] by several authors [4–10]. Liu [11] proposed the second-order duality theorems for a minimax programming problem under generalized second-order $B$-invex functions. Husain *et al.* [12] formulated two types of second-order dual models for minimax fractional programming and derived weak, strong and converse duality theorems under $\eta$-convexity assumptions.

Ahmad *et al.* [13] and Husain *et al.* [14] discussed the second-order duality results for the following non-differentiable minimax programming problem:

$$\text{Minimize } \sup_{y\in Y} f(x,y) + \left(x^T Bx\right)^{1/2}, \tag{P}$$

$$\text{subject to } h(x) \leq 0, \quad x \in R^n,$$

where $Y$ is a compact subset of $R^l$, $f(\cdot,\cdot) : R^n \times R^l \to R$ and $h(\cdot) : R^n \to R^m$ are twice differentiable functions. $B$ is an $n \times n$ positive semidefinite symmetric matrices. Ahmad *et al.* [15] formulated a unified higher-order dual of (P) and established appropriate duality theorems under higher-order $(F,\alpha,\rho,d)$-type I assumptions. Recently, Jayswal and Stancu-Minasian [16] obtained higher-order duality results for (P).

In this paper, we formulate a higher-order dual for a non-differentiable minimax fractional programming problem and establish weak, strong and strict converse duality theorems under generalized higher-order $(\mathcal{F},\alpha,\rho,d)$-type I assumptions. This paper generalizes several results that have appeared in the literature [11, 12, 14–22] and references therein.

## 2 Preliminaries

The problem to be considered in the present analysis is the following non-differentiable minimax fractional problem:

$$\begin{aligned} &\text{Minimize } \sup_{y\in Y} \frac{f(x,y)+(x^TBx)^{1/2}}{g(x,y)-(x^TCx)^{1/2}}, \qquad \text{(NP)}\\ &\quad \text{subject to } h(x)\leq 0, \quad x\in R^n, \end{aligned}$$

where $Y$ is a compact subset of $R^l$, $f(\cdot,\cdot),g(\cdot,\cdot):R^n\times R^l\to R$ and $h(\cdot):R^n\to R^m$ are differentiable functions. $B$ and $C$ are $n\times n$ positive semidefinite symmetric matrices. It is assumed that for each $(x,y)$ in $R^n\times R^l$, $f(x,y)+(x^TBx)^{\frac{1}{2}}\geq 0$ and $g(x,y)-(x^TCx)^{\frac{1}{2}}>0$.

Let $\mathcal{X}=\{x\in R^n: h(x)\leq 0\}$ denote the set of all feasible solutions of (NP). Any point $x\in\mathcal{X}$ is called the feasible point of (NP). For each $(x,y)\in\mathcal{X}\times Y$, we define

$$\psi(x,y)=\frac{f(x,y)+(x^TBx)^{1/2}}{g(x,y)-(x^TCx)^{1/2}}$$

such that for each $(x,y)\in\mathcal{X}\times Y$,

$$f(x,y)+\left(x^TBx\right)^{1/2}\geq 0 \quad\text{and}\quad g(x,y)-\left(x^TCx\right)^{1/2}>0.$$

For each $(x,y)\in\mathcal{X}\times Y$, we define

$$J(x)=\left\{j\in J: h_j(x)=0\right\},$$

where

$$\begin{aligned} &J=\{1,2,\ldots,m\},\\ &Y(x)=\left\{y\in Y: \frac{f(x,y)+(x^TBx)^{1/2}}{g(x,y)-(x^TCx)^{1/2}}=\sup_{z\in Y}\frac{f(x,z)+(x^TBx)^{1/2}}{g(x,z)-(x^TCx)^{1/2}}\right\},\\ &S(x)=\left\{(s,t,\tilde{y})\in N\times R^s_+\times R^{ls}: 1\leq s\leq n+1, t=(t_1,t_2,\ldots,t_s)\in R^s_+\right.\\ &\qquad\left.\text{with } \sum_{i=1}^s t_i=1, \tilde{y}=(\bar{y}_1,\bar{y}_2,\ldots,\bar{y}_s) \text{ with } \bar{y}_i\in Y(x)\ (i=1,2,\ldots,s)\right\}. \end{aligned}$$

Since $f$ and $g$ are continuously differentiable and $Y$ is compact in $R^l$, it follows that for each $x^* \in \mathcal{X}$, $Y(x^*) \neq \emptyset$, and for any $\bar{y}_i \in Y(x^*)$, we have

$$\lambda_\circ = \phi(x^*, \bar{y}_i) = \frac{f(x^*, \bar{y}_i) + (x^{*T} B x^*)^{1/2}}{g(x^*, \bar{y}_i) - (x^{*T} C x^*)^{1/2}}.$$

**Lemma 2.1** (Generalized Schwarz inequality) *Let $A$ be a positive-semidefinite matrix of order $n$. Then, for all $x, w \in R^n$,*

$$x^T A w \leq (x^T A x)^{\frac{1}{2}} (w^T A w)^{\frac{1}{2}}. \tag{2.1}$$

*The equality $Ax = \xi A w$ holds for some $\xi \geq 0$. Clearly, if $(w^T A w)^{\frac{1}{2}} \leq 1$, we have*

$$x^T A w \leq (x^T A x)^{\frac{1}{2}}.$$

Let $\mathcal{F}$ be a sublinear functional, and let $d(\cdot, \cdot) : R^n \times R^n \to R$. Let $\rho = (\rho^1, \rho^2)$, where $\rho^1 = (\rho_1^1, \rho_2^1, \ldots, \rho_s^1) \in R^s$ and $\rho^2 = (\rho_1^2, \rho_2^2, \ldots, \rho_m^2) \in R^m$, and let $\alpha = (\alpha^1, \alpha^2) : R^n \times R^n \to R_+ \setminus \{0\}$. Let $\psi(\cdot, \cdot) : R^n \times Y \to R$, $h(\cdot) : R^n \to R^m$, $K : R^n \times Y \times R^n \to R$ and $H_j : R^n \times Y \times R^n \to R$, $j = 1, 2, \ldots, m$, be differentiable functions at $\bar{x} \in R^n$.

**Definition 2.1** A functional $\mathcal{F} : R^n \times R^n \times R^n \mapsto R$ is said to be sublinear in its third argument if for all $x, \bar{x} \in R^n$,

(i) $\mathcal{F}(x, \bar{x}; a + b) \leq \mathcal{F}(x, \bar{x}; a) + \mathcal{F}(x, \bar{x}; b)$, $\forall a, b \in R^n$;

(ii) $\mathcal{F}(x, \bar{x}; \beta a) = \beta \mathcal{F}(x, \bar{x}; a)$, $\forall \beta \in R$, $\beta \geq 0$, and $\forall a \in R^n$.

From (ii), it is clear that $\mathcal{F}(x, \bar{x}; 0) = 0$.

**Definition 2.2** [15] For each $j \in J$, $(\psi, h_j)$ is said to be higher-order $(\mathcal{F}, \alpha, \rho, d)$-pseudoquasi-type I at $\bar{x} \in R^n$ if for all $x \in \mathcal{X}$, $p \in R^n$ and $\bar{y}_i \in Y(x)$,

$$\begin{aligned}
&\psi(x, \bar{y}_i) < \psi(\bar{x}, \bar{y}_i) + K(\bar{x}, \bar{y}_i, p) - p^T \nabla_p K(\bar{x}, \bar{y}_i, p) \\
&\quad \Rightarrow \quad \mathcal{F}\big(x, \bar{x}; \alpha^1(x, \bar{x})\big(\nabla_p K(\bar{x}, \bar{y}_i, p)\big)\big) < -\rho_i^1 d^2(x, \bar{x}), \quad i = 1, 2, \ldots, s, \\
&-\big[h_j(\bar{x}) + H_j(\bar{x}, p) - p^T \nabla_p H_j(\bar{x}, p)\big] \leq 0 \\
&\quad \Rightarrow \quad \mathcal{F}\big(x, \bar{x}; \alpha^2(x, \bar{x})\big(\nabla_p H_j(\bar{x}, p)\big)\big) \leq -\rho_j^2 d^2(x, \bar{x}), \quad j = 1, 2, \ldots, m.
\end{aligned}$$

In the above definition, if

$$\begin{aligned}
&\mathcal{F}\big(x, \bar{x}; \alpha^1(x, \bar{x})\big(\nabla_p K(\bar{x}, \bar{y}_i, p)\big)\big) \geq -\rho_i^1 d^2(x, \bar{x}) \\
&\quad \Rightarrow \quad \psi(x, \bar{y}_i) > \psi(\bar{x}, \bar{y}_i) + K(\bar{x}, \bar{y}_i, p) - p^T \nabla_p K(\bar{x}, \bar{y}_i, p), \quad i = 1, 2, \ldots, s,
\end{aligned}$$

then we say that $(\psi, g_j)$ is higher-order $(\mathcal{F}, \alpha, \rho, d)$-strictly pseudoquasi-type I at $\bar{x}$.

If the functions $f$, $g$ and $h$ in problem (NP) are continuously differentiable with respect to $x \in R^n$, then Liu [11] derived the following necessary conditions for optimality of (NP).

**Theorem 2.1** (Necessary conditions) *If $x^*$ is a solution of* (NP) *satisfying $x^{*T}Bx^* > 0$, $x^{*T}Cx^* > 0$, and $\nabla h_j(x^*)$, $j \in J(x^*)$ are linearly independent, then there exist $(s,t^*,\tilde{y}) \in S(x^*)$, $\lambda_0 \in R_+$, $w, v \in R^n$, and $\mu^* \in R_+^m$ such that*

$$\sum_{i=1}^{s} t_i^* \{\nabla f(x^*,\bar{y}_i) + Bw - \lambda_0(\nabla g(x^*,\bar{y}_i) - Cv)\} + \nabla \sum_{j=1}^{m} \mu_j^* h_j(x^*) = 0,$$

$$f(x^*,\bar{y}_i) + (x^{*T}Bx^*)^{\frac{1}{2}} - \lambda_0\big(g(x^*,\bar{y}_i) - (x^{*T}Cx^*)^{\frac{1}{2}}\big) = 0, \quad i = 1,2,\ldots,s,$$

$$\sum_{j=1}^{m} \mu_j^* h_j(x^*) = 0,$$

$$t_i^* \geq 0 \quad (i = 1,2,\ldots,s), \qquad \sum_{i=1}^{s} t_i^* = 1,$$

$$w^T Bw \leq 1, \qquad v^T Cv \leq 1,$$

$$(x^{*T}Bx^*)^{1/2} = x^{*T}Bw, \qquad (x^{*T}Cx^*)^{1/2} = x^{*T}Cv.$$

In the above theorem, both matrices $B$ and $C$ are positive semidefinite. If either $x^{*T}Bx^*$ or $x^{*T}Cx^*$ is zero, then the functions involved in the objective function of problem (NP) are not differentiable. To derive these necessary conditions under this situation, for $(s,t^*,\tilde{y}) \in S(x^*)$, we define

$$U_{\tilde{y}}(x^*) = \{u \in R^n : u^t \nabla h_j(x^*) \leq 0, j \in J(x^*) \text{ satisfying one of the following conditions:}$$

(i) $x^{*T}Bx^* > 0$, $x^{*T}Cx^* = 0$

$$\Rightarrow \quad u^T\left(\sum_{i=1}^{s} t_i\left\{\nabla f(x^*,\bar{y}_i) + \frac{Bx^*}{(x^{*t}Bx^*)^{\frac{1}{2}}} - \lambda_\circ \nabla g(x^*,\bar{y}_i)\right\}\right) + \big(u^T(\lambda_\circ^2 C)u\big)^{\frac{1}{2}} < 0,$$

(ii) $x^{*T}Bx^* = 0$, $x^{*T}Cx^* > 0$

$$\Rightarrow \quad u^T\left(\sum_{i=1}^{s} t_i\left\{\nabla f(x^*,\bar{y}_i) - \lambda_\circ\left(\nabla g(x^*,\bar{y}_i) - \frac{Cx^*}{(x^{*T}Cx^*)^{\frac{1}{2}}}\right)\right\}\right) + \big(u^T Bu\big)^{\frac{1}{2}} < 0,$$

(iii) $x^{*T}Bx^* = 0$, $x^{*T}Cx^* = 0$

$$\Rightarrow \quad u^T\left(\sum_{i=1}^{s} t_i\{\nabla f(x^*,\bar{y}_i) - \lambda_\circ \nabla g(x^*,\bar{y}_i)\}\right) + \big(u^T(\lambda_\circ^2 C)u\big)^{\frac{1}{2}} + \big(u^T Bu\big)^{\frac{1}{2}} < 0,$$

(iv) $x^{*T}Bx^* > 0$, $x^{*T}Cx^* > 0$

$$\Rightarrow \quad u^T\left(\sum_{i=1}^{s} t_i\{\nabla f(x^*,\bar{y}_i) - \lambda_\circ \nabla g(x^*,\bar{y}_i)\}\right) + \big(u^T(\lambda_\circ^2 C)u\big)^{\frac{1}{2}} + \big(u^T Bu\big)^{\frac{1}{2}} < 0\}.$$

If in addition, we insert $U_{\tilde{y}}(x^*) = \emptyset$, then the results of Theorem 2.1 still hold.

## 3 Higher-order non-differentiable fractional duality

In this section, we consider the following dual problem to (NP):

$$\max_{(s,t,\tilde{y})\in S(z)} \sup_{(z,\mu,\lambda,\nu,w,p)\in L(s,t,\tilde{y})} \lambda, \tag{ND}$$

where $L(s,t,\tilde{y})$ denotes the set of all $(z,\mu,\lambda,\nu,w,p)\in R^n\times R_+^m\times R_+\times R^n\times R^n\times R^n$ satisfying

$$\sum_{i=1}^{s} t_i\bigl[\nabla_p\bigl(F(z,\bar{y}_i,p)-\lambda G(z,\bar{y}_i,p)\bigr)\bigr]+Bw+\lambda Cv+\sum_{j=1}^{m}\mu_j\nabla_p H_j(z,p)=0, \tag{3.1}$$

$$\begin{aligned}&\sum_{i=1}^{s} t_i\Bigl[f(z,\bar{y}_i)+z^TBw-\lambda\bigl(g(z,\bar{y}_i)-z^TCv\bigr)+F(z,\bar{y}_i,p)-\lambda G(z,\bar{y}_i,p)+\sum_{j\in J_0}\mu_j h_j(z)\\&\quad-p^T\nabla_p\bigl\{F(z,\bar{y}_i,p)-\lambda G(z,\bar{y}_i,p)\bigr\}\Bigr]\\&\quad+\sum_{j\in J_0}\mu_j H_j(z,p)-p^T\sum_{j\in J_0}\mu_j\nabla_p H_j(z,p)\geq 0,\end{aligned} \tag{3.2}$$

$$\sum_{j\in J_\beta}\mu_j\bigl[h_j(z)+H_j(z,p)-p^T\nabla_p H_j(z,p)\bigr]\geq 0,\quad \beta=1,2,\dots,r, \tag{3.3}$$

$$w^TBw\leq 1,\qquad v^TCv\leq 1, \tag{3.4}$$

where $F:R^n\times Y\times R^n\to R$, $G:R^n\times Y\times R^n\to R$, $J_\beta\subseteq M=\{1,2,\dots,m\}$, $\beta=0,1,2,\dots,r$ with $\bigcup_{\beta=0}^{r}J_\beta=M$ and $J_\beta\cap J\alpha=\emptyset$ if $\beta\neq\alpha$. If for a triplet $(s,t,\tilde{y})\in S(z)$, the set $L(s,t,\tilde{y})=\varnothing$, then we define the supremum over it to be $\infty$.

**Theorem 3.1** (Weak duality) *Let $x$ and $(z,\mu,\lambda,s,t,v,w,\tilde{y},p)$ be feasible solutions of* (NP) *and* (ND), *respectively. Suppose that*

$$\Biggl[\sum_{i=1}^{s} t_i\bigl\{f(\cdot,\bar{y}_i)+(\cdot)^TBw-\lambda\bigl(g(\cdot,\bar{y}_i)-z^TCv\bigr)\bigr\}+\sum_{j\in J_0}\mu_j h_j(\cdot),\sum_{j\in j_\beta}\mu_j h_j(\cdot),\beta=1,2,\dots,r\Biggr]$$

*is higher-order $(F,\alpha,\rho,d)$-pseudoquasi-type I at $z$ and*

$$\frac{\rho_1^1}{\alpha^1(x,z)}+\sum_{\beta=1}^{r}\frac{\rho_\beta^2}{\alpha^2(x,z)}\geq 0.$$

*Then*

$$\sup_{y\in Y}\frac{f(x,y)+(x^tBx)^{1/2}}{g(x,y)-(x^tCx)^{1/2}}\geq\lambda.$$

*Proof* Suppose to the contrary that

$$\sup_{y\in Y}\frac{f(x,y)+(x^TBx)^{1/2}}{g(x,y)-(x^TCx)^{1/2}}<\lambda.$$

Then we have

$$f(x,\bar{y}_i)+\left(x^TBx\right)^{1/2}-\lambda\left(g(x,\bar{y}_i)-\left(x^TCx\right)^{1/2}\right)<0,\quad \text{for all } \bar{y}_i\in Y, i=1,2,\ldots,s.$$

It follows from $t_i \geq 0$, $i=1,2,\ldots,s$, that

$$t_i\left[f(x,\bar{y}_i)+\left(x^TBx\right)^{1/2}-\lambda\left(g(x,\bar{y}_i)-\left(x^TCx\right)^{1/2}\right)\right]\leq 0,\quad i=1,2,\ldots,s,$$

with at least one strict inequality, since $t=(t_1,t_2,\ldots,t_s)\neq 0$. Taking summation over $i$ and using $\sum_{i=1}^{s} t_i = 1$, we have

$$\sum_{i=1}^{s} t_i\left[f(x,\bar{y}_i)+\left(x^TBx\right)^{1/2}-\lambda\left(g(x,\bar{y}_i)-\left(x^TCx\right)^{1/2}\right)\right]<0.$$

It follows from the generalized Schwarz inequality and (3.4) that

$$\sum_{i=1}^{s} t_i\left[f(x,\bar{y}_i)+x^TBw-\lambda\left(g(x,\bar{y}_i)-x^TCv\right)\right]<0. \tag{3.5}$$

By the feasibility of $x$ for (NP) and $\mu \geq 0$, we obtain

$$\sum_{j\in J_0}\mu_j h_j(x)\leq 0. \tag{3.6}$$

The above inequality with (3.5) gives

$$\sum_{i=1}^{s} t_i\left[f(x,\bar{y}_i)+x^TBw-\lambda\left(g(x,\bar{y}_i)-x^TCv\right)\right]+\sum_{j\in J_0}\mu_j h_j(x)<0. \tag{3.7}$$

From (3.2) and (3.7), we have

$$\begin{aligned}
&\sum_{i=1}^{s} t_i\left[f(x,\bar{y}_i)+x^TBw-\lambda\left(g(x,\bar{y}_i)-x^TCv\right)\right]+\sum_{j\in J_0}\mu_j h_j(x)\\
&\quad<\sum_{i=1}^{s} t_i\Bigg[f(z,\bar{y}_i)+z^TBw-\lambda\left(g(z,\bar{y}_i)-z^TCv\right)+F(z,\bar{y}_i,p)-\lambda G(z,\bar{y}_i,p)+\sum_{j\in J_0}\mu_j h_j(z)\\
&\qquad-p^T\nabla_p\left\{F(z,\bar{y}_i,p)-\lambda G(z,\bar{y}_i,p)\right\}\Bigg]+\sum_{j\in J_0}\mu_j H_j(z,p)-p^T\sum_{j\in J_0}\mu_j\nabla_p H_j(z,p).
\end{aligned} \tag{3.8}$$

Also, from (3.3), we have

$$\sum_{j\in j_\beta}\mu_j\left[h_j(z)+H_j(z,p)-p^T\nabla_p H_j(z,p)\right]\geq 0,\quad \beta=1,2,\ldots,r. \tag{3.9}$$

The higher second-order $(\mathcal{F},\alpha,\rho,d)$-pseudoquasi-type I assumption on

$$\left[\sum_{i=1}^{s} t_i\left\{f(\cdot,\bar{y}_i)+(\cdot)^TBw-\lambda\left(g(\cdot,\bar{y}_i)-(\cdot)^TCv\right)\right\}+\sum_{j\in J_0}\mu_j h_j(\cdot),\sum_{j\in j_\beta}\mu_j h_j(\cdot),\beta=1,2,\ldots,r\right]$$

at $z$, with (3.8) and (3.9), implies

$$\mathcal{F}\left(x, z; \alpha^1(x,z) \sum_{i=1}^{s} t_i \left\{\nabla_p \left(F(z, \bar{y}_i, p) - \lambda G(z, \bar{y}_i, p)\right)\right\} + Bw + \lambda Cv\right) < -\rho_1^1 d^2(x,z),$$

$$\mathcal{F}\left(x, z; \alpha^2(x,z) \sum_{j \in j_\beta} \mu_j \nabla_p H_j(z,p)\right) \leq -\rho_\beta^2 d^2(x,z), \quad \beta = 1, 2, \ldots, r.$$

By using $\alpha^1(x,z) > 0$, $\alpha^2(x,z) > 0$, and the sublinearity of $\mathcal{F}$ in the above inequalities, we summarize to get

$$\mathcal{F}\left(x, z; \sum_{i=1}^{s} t_i \left\{\nabla_p \left(F(z, \bar{y}_i, p) - \lambda G(z, \bar{y}_i, p)\right)\right\} + Bw + \lambda Cv + \sum_{\beta=1}^{r} \sum_{j \in j_\beta} \mu_j \nabla_p H_j(z,p)\right)$$
$$< -\left(\frac{\rho_1^1}{\alpha^1(x,z)} + \sum_{\beta=1}^{r} \frac{\rho_\beta^2}{\alpha^2(x,z)}\right) d^2(x,z).$$

Since $(\frac{\rho_1^1}{\alpha^1(x,z)} + \sum_{\beta=1}^{r} \frac{\rho_\beta^2}{\alpha^2(x,z)}) \geq 0$, therefore

$$\mathcal{F}\left(x, z; \sum_{i=1}^{s} t_i \left\{\nabla_p \left(F(z, \bar{y}_i, p) - \lambda G(z, \bar{y}_i, p)\right)\right\} + Bw + \lambda Cv + \sum_{j=1}^{m} \mu_j \nabla_p H_j(z,p)\right) < 0,$$

which contradicts (3.1), as $\mathcal{F}(x, z; 0) = 0$. □

**Theorem 3.2** (Strong duality) *Let $x^*$ be an optimal solution of* (NP) *and let $\nabla h_j(x^*)$, $j \in J(x^*)$ be linearly independent. Assume that*

$$F\left(x^*, \bar{y}_i^*, 0\right) = 0; \qquad \nabla_p F\left(x^*, \bar{y}_i^*, 0\right) = \nabla f\left(x^*, \bar{y}_i^*\right), \quad i = 1, 2, \ldots, s,$$
$$G\left(x^*, \bar{y}_i^*, 0\right) = 0; \qquad \nabla_p G\left(x^*, \bar{y}_i^*, 0\right) = \nabla g\left(x^*, \bar{y}_i^*\right), \quad i = 1, 2, \ldots, s,$$
$$H_j\left(x^*, 0\right) = 0; \qquad \nabla_p H_j\left(x^*, 0\right) = \nabla h_j\left(x^*\right), \quad j \in J.$$

*Then there exist $(s^*, t^*, \tilde{y}^*) \in S$ and $(x^*, \mu^*, \lambda^*, v^*, w^*, p^*) \in L(s^*, t^*, \tilde{y}^*)$ such that $(x^*, \mu^*, \lambda^*, v^*, w^*, s^*, t^*, \tilde{y}^*, p^* = 0)$ is a feasible solution of* (ND) *and the two objectives have the same values. Furthermore, if the assumptions of weak duality* (*Theorem* 3.1) *hold for all feasible solutions of* (NP) *and* (ND), *then $(x^*, \mu^*, \lambda^*, v^*, w^*, s^*, t^*, \tilde{y}^*, p^* = 0)$ is an optimal solution of* (ND).

*Proof* Since $x^*$ is an optimal solution of (NP) and $\nabla h_j(x^*)$, $j \in J(x^*)$ are linearly independent, by Theorem 2.1, there exist $(s^*, t^*, \tilde{y}^*) \in S$ and $(x^*, \mu^*, \lambda^*, v^*, w^*, p^*) \in L(s^*, t^*, \tilde{y}^*)$ such that $(x^*, \mu^*, \lambda^*, v^*, w^*, s^*, t^*, \tilde{y}^*, p^* = 0)$ is a feasible solution of (ND) and problems (NP) and (ND) have the same objectives values and

$$\lambda^* = \frac{f(x^*, \bar{y}_i^*) + (x^{*T} B x^*)^{1/2}}{g(x^*, \bar{y}_i^*) - (x^{*T} C x^*)^{1/2}}.$$

□

**Theorem 3.3** (Strict converse duality) *Let $x^*$ and $(z^*, \mu^*, \lambda^*, s^*, t^*, v^*, w^*, \tilde{y}^*, p^*)$ be the optimal solutions of* (NP) *and* (ND), *respectively. Suppose that*

$$\left[\sum_{i=1}^{s} t_i^* \{f(\cdot, \bar{y}_i^*) + (\cdot)^T Bw^* - \lambda(g(\cdot, \bar{y}_i^*) - (\cdot)^T Cv^*)\} + \sum_{j\in J_0} \mu_j^* h_j(\cdot), \right.$$
$$\left. \sum_{j\in j_\beta} \mu_j^* h_j(\cdot), \beta = 1, 2, \ldots, r\right]$$

*is higher-order $(\mathcal{F}, \alpha, \rho, d)$-strictly pseudoquasi-type I at $z^*$ with*

$$\frac{\rho_1^1}{\alpha^1(x^*, z^*)} + \sum_{\beta=1}^{r} \frac{\rho_\beta^2}{\alpha^2(x^*, z^*)} \geq 0,$$

*and that $\nabla h_j(x^*)$, $j \in J(x^*)$ are linearly independent. Then $z^* = x^*$; that is, $z^*$ is an optimal solution of* (NP).

*Proof* We assume that $z^* \neq x^*$ and reach a contradiction. From the strong duality theorem (Theorem 3.2), it follows that

$$\sup_{y\in Y} \frac{f(x^*, \tilde{y}^*) + (x^{*T} Bx^*)^{1/2}}{g(x^*, \tilde{y}^*) - (x^{*T} Cx^*)^{1/2}} = \lambda^*. \tag{3.10}$$

Now, proceeding as in Theorem 3.1, we get

$$\sum_{i=1}^{s} t_i^* [f(x^*, \bar{y}_i^*) + x^{*T} Bw^* - \lambda^* (g(x^*, \bar{y}_i^*) - x^{*T} Cv^*)] + \sum_{j\in J_0} \mu_j^* h_j(x^*) < 0. \tag{3.11}$$

The feasibility of $x^*$ for (NP), $\mu^* \geq 0$ and (3.3) imply

$$\sum_{j\in J_\beta} \mu_j^* h_j(x^*) \leq 0 \leq \sum_{j\in J_\beta} \mu_j^* [h_j(z^*) + H_j(z^*, p^*) - p^{*T} \nabla_p H_j(z^*, p^*)],$$

which along with the second part of higher-order $(\mathcal{F}, \alpha, \rho, d)$-strictly pseudoquasi-type I assumption on

$$\left[\sum_{i=1}^{s} t_i^* \{f(\cdot, \bar{y}_i^*) + (\cdot)^T Bw^* - \lambda(g(\cdot, \bar{y}_i^*) - (\cdot)^T Cv^*)\} + \sum_{j\in J_0} \mu_j^* h_j(\cdot), \right.$$
$$\left. \sum_{j\in j_\beta} \mu_j^* h_j(\cdot), \beta = 1, 2, \ldots, r\right]$$

at $z^*$ gives

$$\mathcal{F}\left(x^*, z^*; \alpha^2(x^*, z^*) \sum_{j\in j_\beta} \mu_j^* \nabla_p H_j(z^*, p^*)\right) < -\rho_\beta^2 d^2(x^*, z^*), \quad \beta = 1, 2, \ldots, r.$$

As $\alpha^2(x^*,z^*) > 0$ and as $\mathcal{F}$ is sublinear, it follows that

$$\mathcal{F}\left(x^*,z^*;\sum_{j\in j_\beta}\mu_j^*\nabla_p H_j(z^*,p^*)\right) < -\frac{\rho_\beta^2}{\alpha^2(x^*,z^*)}d^2(x^*,z^*),\quad \beta = 1,2,\ldots,r. \tag{3.12}$$

From (3.1), (3.12) and the sublinearity of $\mathcal{F}$, we have

$$\begin{aligned}
&\mathcal{F}\Bigg(x^*,z^*;\sum_{i=1}^{s} t_i^*\big[\nabla_p\big(F(z^*,\bar{y}_i^*,p^*) - \lambda G(z^*,\bar{y}_i^*,p^*)\big)\big] + Bw^* + \lambda^* Cv^* \\
&\quad + \sum_{j\in j_0}\mu_j^*\nabla_p H_j(z^*,p^*)\Bigg) \geq \frac{\sum_{\beta=1}^{r}\rho_\beta^2}{\alpha^2(x^*,z^*)}d^2(x^*,z^*).
\end{aligned}$$

In view of $(\frac{\rho_1^1}{\alpha^1(x^*,z^*)} + \frac{\sum_{\beta=1}^{r}\rho_\beta^2}{\alpha^2(x^*,z^*)}) \geq 0$, $\alpha^1(x^*,z^*) > 0$ and the sublinearity of $\mathcal{F}$, the above inequality becomes

$$\begin{aligned}
&\mathcal{F}\Bigg(x^*,z^*;\alpha^1(x^*,z^*)\sum_{i=1}^{s} t_i^*\big[\nabla_p\big(F(z^*,\bar{y}_i^*,p^*) - \lambda G(z^*,\bar{y}_i^*,p^*)\big)\big] + Bw^* + \lambda^* Cv^* \\
&\quad + \sum_{j\in j_0}\mu_j^*\nabla_p H_j(z^*,p^*)\Bigg) \geq -\rho_1^1 d^2(x^*,z^*).
\end{aligned}$$

By using the first part of the said assumption imposed on

$$\begin{aligned}
&\Bigg[\sum_{i=1}^{s} t_i^*\big\{f(\cdot,\bar{y}_i^*) + (\cdot)^T Bw^* - \lambda^*\big(g(\cdot,\bar{y}_i^*) - (\cdot)^T Cv^*\big)\big\} + \sum_{j\in J_0}\mu_j^* h_j(\cdot), \\
&\quad \sum_{j\in j_\beta}\mu_j^* h_j(\cdot), \beta = 1,2,\ldots,r\Bigg]
\end{aligned}$$

at $z^*$, it follows that

$$\begin{aligned}
&\sum_{i=1}^{s} t_i^*\big[f(x^*,\bar{y}_i^*) + x^{*T}Bw^* - \lambda^*\big(g(x^*,\bar{y}_i^*) - x^{*T}Cv^*\big)\big] + \sum_{j\in J_0}\mu_j^* h_j(x^*) \\
&\quad > \sum_{i=1}^{s} t_i^*\Big[f(z^*,\bar{y}_i^*) + z^{*T}Bw^* - \lambda^*\big(g(z^*,\bar{y}_i^*) - z^{*T}Cv^*\big) + F(z^*,\bar{y}_i^*,p^*) \\
&\qquad - \lambda G(z^*,\bar{y}_i^*,p^*) + \sum_{j\in J_0}\mu_j^* h_j(z^*) - p^{*T}\nabla_p\big\{F(z^*,\bar{y}_i^*,p^*) - \lambda^* G(z^*,\bar{y}_i^*,p^*)\big\}\Big] \\
&\qquad + \sum_{j\in J_0}\mu_j^* H_j(z^*,p^*) - p^{*T}\sum_{j\in J_0}\mu_j^*\nabla_p H_j(z^*,p^*) \\
&\quad \geq 0 \quad \text{(by (3.2))},
\end{aligned}$$

which contradicts (3.11). Hence the result. □

## 4 Special cases

Let $J_0 = \varnothing$, $F(z,\bar{y}_i,p) = p^T\nabla f(z,\bar{y}_i) + \frac{1}{2}p^T\nabla^2 f(z,\bar{y}_i)$, $G(z,\bar{y}_i,p) = p^T\nabla g(z,\bar{y}_i) + \frac{1}{2}p^T\nabla^2 g(z,\bar{y}_i)$, $i = 1,2,\ldots,s$ and $H_j(z,p) = p^T\nabla h_j(z) + \frac{1}{2}p^T\nabla^2 h_j(z)p$, $j = 1,2,\ldots,m$. Then (ND) becomes the second-order dual studied in [17, 22]. If, in addition, $p = 0$, then we obtain the dual formulated by Ahmad *et al.* [20].

## 5 Conclusion

The notion of higher-order $(\mathcal{F},\alpha,\rho,d)$-pseudoquasi-type I is adopted, which includes many other generalized convexity concepts in mathematical programming as special cases. This concept is appropriate to discuss the weak, strong and strict converse duality theorems for a higher-order dual (ND) of a non-differentiable minimax fractional programming problem (NP). The results of this paper can be discussed by formulating a unified higher-order dual involving support functions on the lines of Ahmad [23].

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
All authors carried out the proof. All authors conceived of the study and participated in its design and coordination. All authors read and approved the final manuscript.

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