Başarır and Şahin *Journal of Inequalities and Applications* 2013, 2013:482
http://www.journalofinequalitiesandapplications.com/content/2013/1/482

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

On the strong and Δ-convergence of new multi-step and *S*-iteration processes in a CAT(0) space

Metin Başarır and Aynur Şahin*

*Correspondence:
ayuce@sakarya.edu.tr
Department of Mathematics,
Faculty of Sciences and Arts, Sakarya
University, Sakarya, 54187, Turkey

Abstract
In this paper, we introduce a new class of mappings and prove the demiclosedness principle for mappings of this type in a CAT(0) space. Also, we obtain the strong and Δ-convergence theorems of new multi-step and *S*-iteration processes in a CAT(0) space. Our results extend and improve the corresponding recent results announced by many authors in the literature.

MSC: 47H09; 47H10; 54E40; 58C30

Keywords: CAT(0) space; contractive-like mapping; strong convergence; Δ-convergence; new multi-step iteration; *S*-iteration; fixed point

1 Introduction

Contractive mappings and iteration processes are some of the main tools in the study of fixed point theory. There are many contractive mappings and iteration processes that have been introduced and developed by several authors to serve various purposes in the literature (see [1–6]).

Imoru and Olantiwo [7] gave the following contractive definition.

Definition 1 Let T be a self-mapping on a metric space X. The mapping T is called a contractive-like mapping if there exist a constant $\delta \in [0,1)$ and a strictly increasing and continuous function $\varphi : [0,\infty) \to [0,\infty)$ with $\varphi(0) = 0$ such that, for all $x, y \in X$,

$$d(Tx, Ty) \leq \delta d(x,y) + \varphi\big(d(x,Tx)\big). \tag{1.1}$$

This mapping is more general than those considered by Berinde [8, 9], Harder and Hicks [10], Zamfirescu [11], Osilike and Udomene [12].

By taking $\delta = 1$ in (1.1), we define a new class of mappings as follows.

Definition 2 The mapping T is called a generalized nonexpansive mapping if there exists a non-decreasing and continuous function $\varphi : [0,\infty) \to [0,\infty)$ with $\varphi(0) = 0$ such that, for all $x, y \in X$,

$$d(Tx, Ty) \leq d(x,y) + \varphi\big(d(x,Tx)\big). \tag{1.2}$$

Springer

©2013 Başarır and Şahin; licensee Springer. This is an Open Access article distributed under the terms of the Creative Commons Attribution License (http://creativecommons.org/licenses/by/2.0), which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Remark 1 For $x \in F(T)$ in (1.2), we have

$$d(x, Ty) = d(Tx, Ty) \leq d(x, y) + \varphi\bigl(d(x, Tx)\bigr) = d(x, y).$$

If X is an interval of $\mathbb{R}$, then $F(T)$ is convex. The same is also true in each space with unique geodesic for each pair of points (*e.g.*, metric trees or CAT(0) spaces).

In the case $\varphi(t) = 0$ for all $t \in [0, \infty)$, it is easy to show that every nonexpansive mapping satisfies (1.2), but the inverse is not necessarily true.

Example 1 Let $X = [0, 2]$, $d(x, y) = |x - y|$, $\varphi(t) = t$ and define T by

$$T(x) = \begin{cases} 0 & \text{if } x \neq 2, \\ 1 & \text{if } x = 2. \end{cases}$$

By taking $x = 2$ and $y = 1.5$, we have

$$d\bigl(T(2), T(1.5)\bigr) = 1 < 1.5 = d(2, 1.5) + \varphi\bigl(d\bigl(2, T(2)\bigr)\bigr)$$

but

$$d\bigl(T(2), T(1.5)\bigr) = 1 \nleq 0.5 = d(2, 1.5).$$

Therefore T is a generalized nonexpansive mapping, but T is not nonexpansive mapping.

Both a contractive-like mapping and a generalized nonexpansive mapping need not have a fixed point, even if X is complete. For example, let $X = [0, \infty)$, $d(x, y) = |x - y|$ and define T by

$$Tx = \begin{cases} 1 & \text{if } 0 \leq x \leq 0.8, \\ 0.6 & \text{if } 0.8 < x < +\infty. \end{cases}$$

It is proved in Gürsoy *et al.* [13] that T is a contractive-like mapping. Similarly, one can prove that T is a generalized nonexpansive mapping. But the mapping T has no fixed point.

By using (1.1), it is obvious that if a contractive-like mapping has a fixed point, then it is unique. However, if a generalized nonexpansive mapping has a fixed point, then it need not be unique. For example, let $\mathbb{R}$ be the real line with the usual norm $|\cdot|$, and let $K = [-1, 1]$. Define a mapping $T : K \to K$ by

$$Tx = \begin{cases} x & \text{if } x \in [0, 1], \\ -x & \text{if } x \in [-1, 0). \end{cases}$$

Now, we show that T is a nonexpansive mapping. In fact, if $x, y \in [0, 1]$ or $x, y \in [-1, 0)$, then we have

$$|Tx - Ty| = |x - y|.$$

If $x \in [0,1]$ and $y \in [-1,0)$ or $x \in [-1,0)$ and $y \in [0,1]$, then we have

$$|Tx - Ty| = |x + y| \leq |x - y|.$$

This implies that T is a nonexpansive mapping and so T is a generalized nonexpansive mapping with $\varphi(t) = 0$ for all $t \in [0,\infty)$. But $F(T) = \{x \in K; 0 \leq x \leq 1\}$.

Agarwal *et al.* [1] introduced the S-iteration process which is independent of those of Mann [3] and Ishikawa [2] and converges faster than both of these. We apply this iteration process in a CAT(0) space as

$$\begin{cases} x_0 \in K, \\ x_{n+1} = (1-\alpha_n)Tx_n \oplus \alpha_n Ty_n, \\ y_n = (1-\beta_n)x_n \oplus \beta_n Tx_n, \quad n \geq 0. \end{cases} \tag{1.3}$$

Gürsoy *et al.* [13] introduced a new multi-step iteration process in a Banach space. We modify this iteration process in a CAT(0) space as follows.

For an arbitrary fixed order $k \geq 2$,

$$\begin{cases} x_0 \in K, \\ x_{n+1} = (1-\alpha_n)y_n^1 \oplus \alpha_n Ty_n^1, \\ y_n^1 = (1-\beta_n^1)y_n^2 \oplus \beta_n^1 Ty_n^2, \\ y_n^2 = (1-\beta_n^2)y_n^3 \oplus \beta_n^2 Ty_n^3, \\ \cdots \\ y_n^{k-2} = (1-\beta_n^{k-2})y_n^{k-1} \oplus \beta_n^{k-2} Ty_n^{k-1}, \\ y_n^{k-1} = (1-\beta_n^{k-1})x_n \oplus \beta_n^{k-1} Tx_n, \quad n \geq 0, \end{cases}$$

or, in short,

$$\begin{cases} x_0 \in K, \\ x_{n+1} = (1-\alpha_n)y_n^1 \oplus \alpha_n Ty_n^1, \\ y_n^i = (1-\beta_n^i)y_n^{i+1} \oplus \beta_n^i Ty_n^{i+1}, \quad i = 1,2,\ldots,k-2, \\ y_n^{k-1} = (1-\beta_n^{k-1})x_n \oplus \beta_n^{k-1} Tx_n, \quad n \geq 0. \end{cases} \tag{1.4}$$

By taking $k = 3$ and $k = 2$ in (1.4), we obtain the SP-iteration process of Phuengrattana and Suantai [4] and the two-step iteration process of Thianwan [6], respectively.

In this paper, motivated by the above results, we prove demiclosedness principle for a new class of mappings and the Δ-convergence theorems of the new multi-step iteration and the S-iteration processes for mappings of this type in a CAT(0) space. Also, we present the strong convergence theorems of these iteration processes for contractive-like mappings in a CAT(0) space.

2 Preliminaries on a CAT(0) space

A metric space X is a CAT(0) *space* if it is geodesically connected and if every geodesic triangle in X is at least as 'thin' as its comparison triangle in the Euclidean plane. Fixed point theory in a CAT(0) space was first studied by Kirk (see [14, 15]). He showed that every nonexpansive mapping defined on a bounded closed convex subset of a complete CAT(0) space always has a fixed point. Since then the fixed point theory in a CAT(0) space has been rapidly developed and many papers have appeared (see [14–20]). It is worth mentioning that the results in a CAT(0) space can be applied to any CAT(k) space with $k \leq 0$ since any CAT(k) space is a CAT(k') space for every $k' \geq k$ (see [21, p.165]).

Let (X, d) be a metric space. A *geodesic path* joining $x \in X$ to $y \in X$ (or more briefly, a *geodesic* from x to y) is a map c from a closed interval $[0, l] \subset R$ to X such that $c(0) = x$, $c(l) = y$ and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. In particular, c is an isometry and $d(x, y) = l$. The image of c is called a *geodesic* (or *metric*) *segment* joining x and y. When it is unique, this geodesic is denoted by $[x, y]$. The space (X, d) is said to be a *geodesic space* if every two points of X are joined by a geodesic and X is said to be a *uniquely geodesic* if there is exactly one geodesic joining x to y for each $x, y \in X$.

A *geodesic triangle* $\triangle(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points in X (the vertices of $\triangle$) and a geodesic segment between each pair of vertices (the edges of $\triangle$). A *comparison triangle* for the geodesic triangle $\triangle(x_1, x_2, x_3)$ in (X, d) is a triangle $\overline{\triangle}(x_1, x_2, x_3) = \triangle(\overline{x}_1, \overline{x}_2, \overline{x}_3)$ in the Euclidean plane $\mathbb{R}^2$ such that

$$d_{\mathbb{R}^2}(\overline{x}_i, \overline{x}_j) = d(x_i, x_j)$$

for $i, j \in \{1, 2, 3\}$. Such a triangle always exists (see [21]).

A geodesic metric space is said to be a CAT(0) space [21] if all geodesic triangles of appropriate size satisfy the following comparison axiom.

CAT(0) Let $\triangle$ be a geodesic triangle in X, and let $\overline{\triangle}$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy the CAT(0) *inequality* if for all $x, y \in \triangle$ and all comparison points $\overline{x}, \overline{y} \in \overline{\triangle}$,

$$d(x, y) \leq d_{\mathbb{R}^2}(\overline{x}, \overline{y}).$$

We observe that if x, y_1, y_2 are points of a CAT(0) space and if y_0 is the midpoint of the segment $[y_1, y_2]$, then the CAT(0) inequality implies that

$$d(x, y_0)^2 \leq \frac{1}{2}d(x, y_1)^2 + \frac{1}{2}d(x, y_2)^2 - \frac{1}{4}d(y_1, y_2)^2. \tag{2.1}$$

The equality holds for the Euclidean metric. In fact (see [21, p.163]), a geodesic metric space is a CAT(0) space if and only if it satisfies the inequality (2.1) (which is known as the *CN* inequality of Bruhat and Tits [22]).

Let $x, y \in X$, by [18, Lemma 2.1(iv)] for each $t \in [0, 1]$, there exists a unique point $z \in [x, y]$ such that

$$d(x, z) = td(x, y), \qquad d(y, z) = (1 - t)d(x, y). \tag{2.2}$$

From now on, we will use the notation $(1 - t)x \oplus ty$ for the unique point z satisfying (2.2). By using this notation, Dhompongsa and Panyanak [18] obtained the following lemmas which will be used frequently in the proof of our main results.

Lemma 1 *Let X be a* CAT(0) *space. Then*

$$d\big((1-t)x \oplus ty, z\big) \leq (1-t)d(x,z) + td(y,z)$$

for all $t \in [0,1]$ and $x, y, z \in X$.

Lemma 2 *Let X be a* CAT(0) *space. Then*

$$d\big((1-t)x \oplus ty, z\big)^2 \leq (1-t)d(x,z)^2 + td(y,z)^2 - t(1-t)d(x,y)^2$$

for all $t \in [0,1]$ and $x, y, z \in X$.

3 Demiclosedness principle for a new class of mappings

In 1976 Lim [23] introduced the concept of convergence in a general metric space setting which is called $\triangle$-convergence. Later, Kirk and Panyanak [24] used the concept of $\triangle$-convergence introduced by Lim [23] to prove on a CAT(0) space analogs of some Banach space results which involve weak convergence. Also, Dhompongsa and Panyanak [18] obtained the $\triangle$-convergence theorems for the Picard, Mann and Ishikawa iterations in a CAT(0) space for nonexpansive mappings under some appropriate conditions.

Now, we recall some definitions.

Let $\{x_n\}$ be a bounded sequence in a CAT(0) space X. For $x \in X$, we set

$$r\big(x, \{x_n\}\big) = \limsup_{n\to\infty} d(x, x_n).$$

The *asymptotic radius* $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r\big(\{x_n\}\big) = \inf\big\{r\big(x, \{x_n\}\big) : x \in X\big\},$$

and the *asymptotic radius* $r_K(\{x_n\})$ of $\{x_n\}$ with respect to $K \subset X$ is given by

$$r_K\big(\{x_n\}\big) = \inf\big\{r\big(x, \{x_n\}\big) : x \in K\big\}.$$

The *asymptotic center* $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A\big(\{x_n\}\big) = \big\{x \in X : r\big(x, \{x_n\}\big) = r\big(\{x_n\}\big)\big\},$$

and the *asymptotic center* $A_K(\{x_n\})$ of $\{x_n\}$ with respect to $K \subset X$ is the set

$$A_K\big(\{x_n\}\big) = \big\{x \in K : r\big(x, \{x_n\}\big) = r_K\big(\{x_n\}\big)\big\}.$$

Proposition 1 ([25, Proposition 3.2]) *Let $\{x_n\}$ be a bounded sequence in a complete* CAT(0) *space X, and let K be a closed convex subset of X, then $A(\{x_n\})$ and $A_K(\{x_n\})$ are singletons.*

Definition 3 ([24, Definition 3.1]) A sequence $\{x_n\}$ in a CAT(0) space X is said to be $\triangle$-convergent to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we write $\triangle\text{-}\lim_n x_n = x$ and x is called the $\triangle$-limit of $\{x_n\}$.

Lemma 3

(i) *Every bounded sequence in a complete* CAT(0) *space always has a* $\triangle$*-convergent subsequence* (*see* [24, p.3690]).

(ii) *Let* K *be a nonempty closed convex subset of a complete* CAT(0) *space, and let* $\{x_n\}$ *be a bounded sequence in* K. *Then the asymptotic center of* $\{x_n\}$ *is in* K (*see* [17, Proposition 2.1]).

Lemma 4 ([18, Lemma 2.8]) *If* $\{x_n\}$ *is a bounded sequence in a complete* CAT(0) *space with* $A(\{x_n\}) = \{x\}$, $\{u_n\}$ *is a subsequence of* $\{x_n\}$ *with* $A(\{u_n\}) = \{u\}$ *and the sequence* $\{d(x_n, u)\}$ *converges, then* $x = u$.

Let $\{x_n\}$ be a bounded sequence in a CAT(0) space X, and let K be a closed convex subset of X which contains $\{x_n\}$. We denote the notation

$$\{x_n\} \rightharpoonup w \quad \Leftrightarrow \quad \Phi(w) = \inf_{x \in K} \Phi(x), \tag{3.1}$$

where $\Phi(x) = \limsup_{n\to\infty} d(x_n, x)$.

We note that $\{x_n\} \rightharpoonup w$ if and only if $A_K(\{x_n\}) = \{w\}$ (see [25]).

Nanjaras and Panyanak [25] gave a connection between the '$\rightharpoonup$' convergence and Δ-convergence.

Proposition 2 ([25, Proposition 3.12]) *Let* $\{x_n\}$ *be a bounded sequence in a* CAT(0) *space* X, *and let* K *be a closed convex subset of* X *which contains* $\{x_n\}$. *Then* $\Delta\text{-}\lim_{n\to\infty} x_n = p$ *implies that* $\{x_n\} \rightharpoonup p$.

By using the convergence defined in (3.1), we obtain *the demiclosedness principle for the new class of mappings in a* CAT(0) *space*.

Theorem 1 *Let* K *be a nonempty closed convex subset of a complete* CAT(0) *space* X, *and let* $T : K \to K$ *be a generalized nonexpansive mapping with* $F(T) \neq \emptyset$. *Let* $\{x_n\}$ *be a bounded sequence in* K *such that* $\{x_n\} \rightharpoonup w$ *and* $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. *Then* $Tw = w$.

Proof By the hypothesis, $\{x_n\} \rightharpoonup w$. Then we have $A_K(\{x_n\}) = \{w\}$. By Lemma 3(ii), we obtain $A(\{x_n\}) = \{w\}$. Since $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, then we have

$$\Phi(x) = \limsup_{n\to\infty} d(x_n, x) = \limsup_{n\to\infty} d(Tx_n, x) \tag{3.2}$$

for all $x \in K$. By taking $x = Tw$ in (3.2), we have

$$\begin{aligned} \Phi(Tw) &= \limsup_{n\to\infty} d(Tx_n, Tw) \\ &\leq \limsup_{n\to\infty} \bigl\{ d(x_n, w) + \varphi\bigl(d(x_n, Tx_n)\bigr) \bigr\} \\ &\leq \limsup_{n\to\infty} d(x_n, w) + \varphi\Bigl(\limsup_{n\to\infty} d(x_n, Tx_n)\Bigr) \\ &= \limsup_{n\to\infty} d(x_n, w) \\ &= \Phi(w). \end{aligned}$$

The rest of the proof closely follows the pattern of Proposition 3.14 in Nanjaras and Panyanak [25]. Hence $Tw = w$ as desired. □

Now, we prove the Δ-convergence of the new multi-step iteration process for the new class of mappings in a CAT(0) space.

Theorem 2 *Let K be a nonempty closed convex subset of a complete* CAT(0) *space X, let $T : K \to K$ be a generalized nonexpansive mapping with $F(T) \neq \emptyset$, and let $\{x_n\}$ be a sequence defined by* (1.4) *such that $\{\alpha_n\}, \{\beta_n^i\} \subset [0,1]$, $i = 1,2,\dots,k-2$ and $\{\beta_n^{k-1}\} \subset [a,b]$ for some $a, b \in (0,1)$. Then the sequence $\{x_n\}$ Δ-converges to the fixed point of T.*

Proof Let $p \in F(T)$. From (1.2), (1.4) and Lemma 1, we have

$$\begin{aligned} d(x_{n+1},p) &= d\bigl((1-\alpha_n)y_n^1 \oplus \alpha_n T y_n^1, p\bigr) \\ &\le (1-\alpha_n) d\bigl(y_n^1,p\bigr) + \alpha_n d\bigl(T y_n^1,p\bigr) \\ &\le (1-\alpha_n) d\bigl(y_n^1,p\bigr) + \alpha_n \bigl\{d\bigl(y_n^1,p\bigr) + \varphi\bigl(d(p,Tp)\bigr)\bigr\} \\ &= d\bigl(y_n^1,p\bigr). \end{aligned}$$

Also, we obtain

$$\begin{aligned} d\bigl(y_n^1,p\bigr) &= d\bigl(\bigl(1-\beta_n^1\bigr)y_n^2 \oplus \beta_n^1 T y_n^2, p\bigr) \\ &\le \bigl(1-\beta_n^1\bigr) d\bigl(y_n^2,p\bigr) + \beta_n^1 d\bigl(T y_n^2,p\bigr) \\ &\le \bigl(1-\beta_n^1\bigr) d\bigl(y_n^2,p\bigr) + \beta_n^1 \bigl\{d\bigl(y_n^2,p\bigr) + \varphi\bigl(d(p,Tp)\bigr)\bigr\} \\ &= d\bigl(y_n^2,p\bigr). \end{aligned}$$

Continuing the above process, we have

$$d(x_{n+1},p) \le d\bigl(y_n^1,p\bigr) \le d\bigl(y_n^2,p\bigr) \le \cdots \le d\bigl(y_n^{k-1},p\bigr) \le d(x_n,p). \tag{3.3}$$

This inequality guarantees that the sequence $\{d(x_n,p)\}$ is non-increasing and bounded below, and so $\lim_{n\to\infty} d(x_n,p)$ exists for all $p \in F(T)$. Let $\lim_{n\to\infty} d(x_n,p) = r$. By using (3.3), we get

$$\lim_{n\to\infty} d\bigl(y_n^{k-1},p\bigr) = r.$$

By Lemma 2, we also have

$$\begin{aligned} d\bigl(y_n^{k-1},p\bigr)^2 &= d\bigl(\bigl(1-\beta_n^{k-1}\bigr)x_n \oplus \beta_n^{k-1} T x_n, p\bigr)^2 \\ &\le \bigl(1-\beta_n^{k-1}\bigr) d(x_n,p)^2 + \beta_n^{k-1} d(Tx_n,p)^2 - \beta_n^{k-1}\bigl(1-\beta_n^{k-1}\bigr) d(x_n,Tx_n)^2 \\ &\le \bigl(1-\beta_n^{k-1}\bigr) d(x_n,p)^2 + \beta_n^{k-1} \bigl\{d(x_n,p) + \varphi\bigl(d(p,Tp)\bigr)\bigr\}^2 \\ &\quad - \beta_n^{k-1}\bigl(1-\beta_n^{k-1}\bigr) d(x_n,Tx_n)^2 \\ &= d(x_n,p)^2 - \beta_n^{k-1}\bigl(1-\beta_n^{k-1}\bigr) d(x_n,Tx_n)^2, \end{aligned}$$

which implies that

$$d(x_n, Tx_n)^2 \leq \frac{1}{a(1-b)}\left[d(x_n,p)^2 - d\left(y_n^{k-1},p\right)^2\right].$$

Thus $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. To show that the sequence $\{x_n\}$ $\triangle$-converges to a fixed point of T, we prove that

$$W_\triangle(x_n) = \bigcup_{\{u_n\}\subset\{x_n\}} A\left(\{u_n\}\right) \subseteq F(T)$$

and $W_\triangle(x_n)$ consists of exactly one point. Let $u \in W_\triangle(x_n)$. Then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. By Lemma 3, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\triangle\text{-}\lim_{n\to\infty} v_n = v \in K$. By Proposition 2 and Theorem 1, $v \in F(T)$. By Lemma 4, we have $u = v \in F(T)$. This shows that $W_\triangle(x_n) \subseteq F(T)$. Now, we prove that $W_\triangle(x_n)$ consists of exactly one point. Let $\{u_n\}$ be a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$, and let $A(\{x_n\}) = \{x\}$. We have already seen that $u = v$ and $v \in F(T)$. Finally, since $\{d(x_n, v)\}$ converges, by Lemma 4, $x = v \in F(T)$. This shows that $W_\triangle(x_n) = \{x\}$. This completes the proof. □

We give the following theorem related to the Δ-convergence of the S-iteration process for the new class of mappings in a CAT(0) space.

Theorem 3 *Let K be a nonempty closed convex subset of a complete* CAT(0) *space X, let $T : K \to K$ be a generalized nonexpansive mapping with $F(T) \neq \emptyset$, and let $\{x_n\}$ be a sequence defined by* (1.3) *such that $\{\alpha_n\}, \{\beta_n\} \subset [a, b]$ for some $a, b \in (0, 1)$. Then the sequence $\{x_n\}$ Δ-converges to the fixed point of T.*

Proof Let $p \in F(T)$. Using (1.2), (1.3) and Lemma 1, we have

$$\begin{aligned} d(x_{n+1},p) &= d\left((1-\alpha_n)Tx_n \oplus \alpha_n Ty_n, p\right) \\ &\leq (1-\alpha_n)d(Tx_n,p) + \alpha_n d(Ty_n,p) \\ &\leq (1-\alpha_n)\left\{d(x_n,p) + \varphi\left(d(p,Tp)\right)\right\} + \alpha_n\left\{d(y_n,p) + \varphi\left(d(p,Tp)\right)\right\} \\ &= (1-\alpha_n)d(x_n,p) + \alpha_n d(y_n,p). \end{aligned} \tag{3.4}$$

Also, we obtain

$$\begin{aligned} d(y_n,p) &= d\left((1-\beta_n)x_n \oplus \beta_n Tx_n, p\right) \\ &\leq (1-\beta_n)d(x_n,p) + \beta_n d(Tx_n,p) \\ &\leq (1-\beta_n)d(x_n,p) + \beta_n\left\{d(x_n,p) + \varphi\left(d(p,Tp)\right)\right\} \\ &= d(x_n,p). \end{aligned} \tag{3.5}$$

From (3.4) and (3.5), we have

$$d(x_{n+1},p) \leq d(x_n,p).$$

This inequality guarantees that the sequence $\{d(x_n,p)\}$ is non-increasing and bounded below, and so $\lim_{n\to\infty} d(x_n,p)$ exists for all $p \in F(T)$. Let

$$\lim_{n\to\infty} d(x_n,p) = r. \tag{3.6}$$

Now, we prove that $\lim_{n\to\infty} d(y_n,p) = r$. By (3.4), we have

$$d(x_{n+1},p) \le (1-\alpha_n)d(x_n,p) + \alpha_n d(y_n,p).$$

This gives that

$$\alpha_n d(x_n,p) \le d(x_n,p) + \alpha_n d(y_n,p) - d(x_{n+1},p)$$

or

$$\begin{aligned} d(x_n,p) &\le d(y_n,p) + \frac{1}{\alpha_n}\bigl[d(x_n,p) - d(x_{n+1},p)\bigr] \\ &\le d(y_n,p) + \frac{1}{a}\bigl[d(x_n,p) - d(x_{n+1},p)\bigr]. \end{aligned}$$

This gives

$$r \le \liminf_{n\to\infty} d(y_n,p).$$

By (3.5) and (3.6), we obtain

$$\limsup_{n\to\infty} d(y_n,p) \le r.$$

Then we get

$$\lim_{n\to\infty} d(y_n,p) = r.$$

By Lemma 2, we also have

$$\begin{aligned} d(y_n,p)^2 &= d\bigl((1-\beta_n)x_n \oplus \beta_n Tx_n, p\bigr)^2 \\ &\le (1-\beta_n)d(x_n,p)^2 + \beta_n d(Tx_n,p)^2 - \beta_n(1-\beta_n)d(x_n,Tx_n)^2 \\ &\le (1-\beta_n)d(x_n,p)^2 + \beta_n\bigl\{d(x_n,p) + \varphi\bigl(d(p,Tp)\bigr)\bigr\}^2 - \beta_n(1-\beta_n)d(x_n,Tx_n)^2 \\ &= d(x_n,p)^2 - \beta_n(1-\beta_n)d(x_n,Tx_n)^2, \end{aligned}$$

which implies that

$$d(x_n,Tx_n)^2 \le \frac{1}{a(1-b)}\bigl[d(x_n,p)^2 - d(y_n,p)^2\bigr].$$

Thus $\lim_{n\to\infty} d(x_n,Tx_n) = 0$. The rest of the proof follows the pattern of the above theorem and is therefore omitted. □

4 Strong convergence theorems for a contractive-like mapping

Now, we prove the strong convergence of the new multi-step iteration process for a contractive-like mapping in a CAT(0) space.

Theorem 4 *Let K be a nonempty closed convex subset of a complete* CAT(0) *space X, let $T : K \to K$ be a contractive-like mapping with $F(T) \neq \emptyset$, and let $\{x_n\}$ be a sequence defined by* (1.4) *such that $\{\alpha_n\} \subset [0,1)$, $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\{\beta_n^i\} \subset [0,1)$, $i = 1,2,\ldots,k-1$. Then the sequence $\{x_n\}$ converges strongly to the unique fixed point of T.*

Proof Let p be the unique fixed point of T. From (1.1), (1.4) and Lemma 1, we have

$$\begin{aligned} d(x_{n+1},p) &= d\bigl((1-\alpha_n)y_n^1 \oplus \alpha_n Ty_n^1, p\bigr) \\ &\leq (1-\alpha_n)d\bigl(y_n^1,p\bigr) + \alpha_n d\bigl(Ty_n^1,p\bigr) \\ &\leq (1-\alpha_n)d\bigl(y_n^1,p\bigr) + \alpha_n\bigl\{\delta d\bigl(y_n^1,p\bigr) + \varphi\bigl(d(p,Tp)\bigr)\bigr\} \\ &= (1-\alpha_n)d\bigl(y_n^1,p\bigr) + \alpha_n\delta d\bigl(y_n^1,p\bigr) \\ &= \bigl[1-\alpha_n(1-\delta)\bigr]d\bigl(y_n^1,p\bigr). \end{aligned}$$

Also, we obtain

$$\begin{aligned} d\bigl(y_n^1,p\bigr) &= d\bigl(\bigl(1-\beta_n^1\bigr)y_n^2 \oplus \beta_n^1 Ty_n^2, p\bigr) \\ &\leq \bigl(1-\beta_n^1\bigr)d\bigl(y_n^2,p\bigr) + \beta_n^1 d\bigl(Ty_n^2,p\bigr) \\ &\leq \bigl(1-\beta_n^1\bigr)d\bigl(y_n^2,p\bigr) + \beta_n^1\bigl\{\delta d\bigl(y_n^2,p\bigr) + \varphi\bigl(d(p,Tp)\bigr)\bigr\} \\ &= \bigl(1-\beta_n^1\bigr)d\bigl(y_n^2,p\bigr) + \beta_n^1\delta d\bigl(y_n^2,p\bigr) \\ &= \bigl[1-\beta_n^1(1-\delta)\bigr]d\bigl(y_n^2,p\bigr). \end{aligned}$$

In a similar fashion, we can get

$$d\bigl(y_n^2,p\bigr) \leq \bigl[1-\beta_n^2(1-\delta)\bigr]d\bigl(y_n^3,p\bigr).$$

Continuing the above process, we have

$$\begin{aligned} d(x_{n+1},p) \leq{}& \bigl[1-\alpha_n(1-\delta)\bigr]\bigl[1-\beta_n^1(1-\delta)\bigr]\bigl[1-\beta_n^2(1-\delta)\bigr]\cdots \\ & \bigl[1-\beta_n^{k-2}(1-\delta)\bigr]d\bigl(y_n^{k-1},p\bigr). \end{aligned} \tag{4.1}$$

In addition, we obtain

$$\begin{aligned} d\bigl(y_n^{k-1},p\bigr) &= d\bigl(\bigl(1-\beta_n^{k-1}\bigr)x_n \oplus \beta_n^{k-1} Tx_n, p\bigr) \\ &\leq \bigl(1-\beta_n^{k-1}\bigr)d(x_n,p) + \beta_n^{k-1} d(Tx_n,p) \\ &\leq \bigl(1-\beta_n^{k-1}\bigr)d(x_n,p) + \beta_n^{k-1}\bigl\{\delta d(x_n,p) + \varphi\bigl(d(p,Tp)\bigr)\bigr\} \\ &= \bigl(1-\beta_n^{k-1}\bigr)d(x_n,p) + \beta_n^{k-1}\delta d(x_n,p) \\ &= \bigl[1-\beta_n^{k-1}(1-\delta)\bigr]d(x_n,p). \end{aligned} \tag{4.2}$$

From (4.1) and (4.2), we have

$$\begin{aligned}
d(x_{n+1},p) &\leq \bigl[1-\alpha_n(1-\delta)\bigr]\bigl[1-\beta_n^1(1-\delta)\bigr]\bigl[1-\beta_n^2(1-\delta)\bigr]\cdots \\
&\qquad \bigl[1-\beta_n^{k-2}(1-\delta)\bigr]\bigl[1-\beta_n^{k-1}(1-\delta)\bigr]d(x_n,p) \\
&\leq \bigl[1-\alpha_n(1-\delta)\bigr]d(x_n,p) \\
&\leq \prod_{j=0}^{n}\bigl[1-\alpha_j(1-\delta)\bigr]d(x_0,p) \\
&\leq e^{-(1-\delta)\sum_{j=0}^{n}\alpha_j}d(x_0,p).
\end{aligned} \tag{4.3}$$

Using the fact that $0 \leq \delta < 1$, $\alpha_j \in [0,1]$ and $\sum_{n=0}^{\infty}\alpha_n = \infty$, we get that

$$\lim_{n\to\infty} e^{-(1-\delta)\sum_{j=0}^{n}\alpha_j} = 0.$$

This together with (4.3) implies that

$$\lim_{n\to\infty} d(x_{n+1},p) = 0.$$

Consequently, $x_n \to p \in F(T)$ and this completes the proof. □

Remark 2 In Theorem 4 the condition $\sum_{n=0}^{\infty}\alpha_n = \infty$ may be replaced with $\sum_{n=0}^{\infty}\beta_n^i = \infty$ for a fixed $i = 1,2,\ldots,k-1$.

Finally, we give the strong convergence theorem of the S-iteration process for a contractive-like mapping in a CAT(0) space as follows.

Theorem 5 *Let K be a nonempty closed convex subset of a complete* CAT(0) *space X, let $T: K \to K$ be a contractive-like mapping with $F(T) \neq \emptyset$, and let $\{x_n\}$ be a sequence defined by* (1.3) *such that $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$. Then the sequence $\{x_n\}$ converges strongly to the unique fixed point of T.*

Proof Let p be the unique fixed point of T. From (1.1), (1.3) and Lemma 1, we have

$$\begin{aligned}
d(x_{n+1},p) &= d\bigl((1-\alpha_n)Tx_n \oplus \alpha_n Ty_n, p\bigr) \\
&\leq (1-\alpha_n)d(Tx_n,p) + \alpha_n d(Ty_n,p) \\
&\leq (1-\alpha_n)\bigl\{\delta d(x_n,p) + \varphi\bigl(d(p,Tp)\bigr)\bigr\} + \alpha_n\bigl\{\delta d(y_n,p) + \varphi\bigl(d(p,Tp)\bigr)\bigr\} \\
&= (1-\alpha_n)\delta d(x_n,p) + \alpha_n \delta d(y_n,p).
\end{aligned} \tag{4.4}$$

Similarly, we obtain

$$\begin{aligned}
d(y_n,p) &= d\bigl((1-\beta_n)x_n \oplus \beta_n Tx_n, p\bigr) \\
&\leq (1-\beta_n)d(x_n,p) + \beta_n d(Tx_n,p) \\
&\leq (1-\beta_n)d(x_n,p) + \beta_n\bigl\{\delta d(x_n,p) + \varphi\bigl(d(p,Tp)\bigr)\bigr\}
\end{aligned}$$

$$\begin{aligned}
&= (1-\beta_n)d(x_n,p) + \beta_n\delta d(x_n,p) \\
&= \bigl(1-\beta_n(1-\delta)\bigr)d(x_n,p) \\
&\leq d(x_n,p). \qquad (4.5)
\end{aligned}$$

Then, from (4.4) and (4.5), we get that

$$\begin{aligned}
d(x_{n+1},p) &\leq (1-\alpha_n)\delta d(x_n,p) + \alpha_n\delta d(y_n,p) \\
&\leq (1-\alpha_n)\delta d(x_n,p) + \alpha_n\delta d(x_n,p) \\
&\leq \delta d(x_n,p) \\
&\vdots \\
&\leq \delta^{n+1} d(x_0,p).
\end{aligned}$$

If $\delta \in (0,1)$, we obtain

$$\lim_{n\to\infty} d(x_{n+1},p) = 0.$$

Thus we have $x_n \to p \in F(T)$. If $\delta = 0$, the result is clear. This completes the proof. □

5 Conclusions

The new multi-step iteration reduces to the two-step iteration and the SP-iteration processes. Also, the class of generalized nonexpansive mappings includes nonexpansive mappings. Then these results presented in this paper extend and generalize some works for a CAT(0) space in the literature.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors contributed equally and significantly in writing this article. Both authors read and approved the final manuscript.

Acknowledgements
The authors are grateful to the referee for his/her careful reading and valuable comments and suggestions which led to the present form of the paper. This paper has been presented in International Conference 'Anatolian Communications in Nonlinear Analysis (ANCNA 2013)' in Abant Izzet Baysal University, Bolu, Turkey, July 03-06, 2013. This paper was supported by Sakarya University Scientific Research Foundation (Project number: 2013-02-00-003).

Received: 1 August 2013 Accepted: 19 September 2013 Published: 07 Nov 2013

References
1. Agarwal, RP, O'Regan, D, Sahu, DR: Iterative construction of fixed points of nearly asymptotically nonexpansive mappings. J. Nonlinear Convex Anal. **8**(1), 61-79 (2007)
2. Ishikawa, S: Fixed points by a new iteration method. Proc. Am. Math. Soc. **44**, 147-150 (1974)
3. Mann, WR: Mean value methods in iterations. Proc. Am. Math. Soc. **4**, 506-510 (1953)
4. Phuengrattana, W, Suantai, S: On the rate of convergence of Mann, Ishikawa, Noor and SP iterations for continuous functions on an arbitrary interval. J. Comput. Appl. Math. **235**(9), 3006-3014 (2011)
5. Rhoades, BE: A comparison of various definitions of contractive mappings. Trans. Am. Math. Soc. **226**, 257-290 (1977)
6. Thianwan, S: Common fixed points of new iterations for two asymptotically nonexpansive nonself-mappings in a Banach space. J. Comput. Appl. Math. **224**, 688-695 (2009)
7. Imoru, CO, Olantiwo, MO: On the stability of Picard and Mann iteration processes. Carpath. J. Math. **19**(2), 155-160 (2003)
8. Berinde, V: On the stability of some fixed point procedures. Bul. ştiinţ. - Univ. Baia Mare, Ser. B Fasc. Mat.-Inform. **18**(1), 7-14 (2002)

9. Berinde, V: On the convergence of the Ishikawa iteration in the class of quasi contractive operators. Acta Math. Univ. Comen. **73**(1), 119-126 (2004)
10. Harder, AM, Hicks, TL: Stability results for fixed point iteration procedures. Math. Jpn. **33**(5), 693-706 (1988)
11. Zamfirescu, T: Fix point theorems in metric spaces. Arch. Math. **23**(1), 292-298 (1972)
12. Osilike, MO, Udomene, A: Short proofs of stability results for fixed point iteration procedures for a class of contractive-type mappings. Indian J. Pure Appl. Math. **30**(12), 1229-1234 (1999)
13. Gürsoy, F, Karakaya, V, Rhoades, BE: Data dependence results of new multi-step and *S*-iterative schemes for contractive-like operators. Fixed Point Theory Appl. **2013**, Article ID 76 (2013). doi:10.1186/1687-1812-2013-76
14. Kirk, WA: Geodesic geometry and fixed point theory. In: Seminar of Mathematical Analysis (Malaga/Seville, 2002/2003). Colecc. Abierta, vol. 64, pp. 195-225. Universidad de Sevilla Secr. Publ., Sevilla (2003)
15. Kirk, WA: Geodesic geometry and fixed point theory II. In: International Conference on Fixed Point Theory and Applications, pp. 113-142. Yokohama Publ., Yokohama (2004)
16. Dhompongsa, S, Kirk, WA, Sims, B: Fixed points of uniformly Lipschitzian mappings. Nonlinear Anal., Theory Methods Appl. **65**(4), 762-772 (2006)
17. Dhompongsa, S, Kirk, WA, Panyanak, B: Nonexpansive set-valued mappings in metric and Banach spaces. J. Nonlinear Convex Anal. **8**(1), 35-45 (2007)
18. Dhompongsa, S, Panyanak, B: On Δ-convergence theorems in CAT(0) spaces. Comput. Math. Appl. **56**(10), 2572-2579 (2008)
19. Şahin, A, Başarır, M: On the strong convergence of a modified *S*-iteration process for asymptotically quasi-nonexpansive mappings in a CAT(0) space. Fixed Point Theory Appl. **2013**, Article ID 12 (2013). doi:10.1186/1687-1812-2013-12
20. Şahin, A, Başarır, M: On the strong and Δ-convergence of *SP*-iteration on a CAT(0) space. J. Inequal. Appl. **2013**, Article ID 311 (2013). doi:10.1186/1029-242X-2013-311
21. Bridson, M, Haefliger, A: Metric Spaces of Non-Positive Curvature. Grundlehren der Mathematischen Wissenschaften, vol. 319. Springer, Berlin (1999)
22. Bruhat, F, Tits, J: Groupes réductifs sur un corps local. Publ. Math. Inst. Hautes Études Sci. **41**, 5-251 (1972)
23. Lim, TC: Remarks on some fixed point theorems. Proc. Am. Math. Soc. **60**, 179-182 (1976)
24. Kirk, WA, Panyanak, B: A concept of convergence in geodesic spaces. Nonlinear Anal., Theory Methods Appl. **68**(12), 3689-3696 (2008)
25. Nanjaras, B, Panyanak, B: Demiclosed principle for asymptotically nonexpansive mappings in CAT(0) spaces. Fixed Point Theory Appl. **2010**, Article ID 268780 (2010). doi:10.1155/2010/268780

10.1186/1029-242X-2013-482
Cite this article as: Başarır and Şahin: **On the strong and Δ-convergence of new multi-step and *S*-iteration processes in a CAT(0) space.** *Journal of Inequalities and Applications* 2013, 2013:482

Submit your manuscript to a SpringerOpen journal and benefit from:

- Convenient online submission
- Rigorous peer review
- Immediate publication on acceptance
- Open access: articles freely available online
- High visibility within the field
- Retaining the copyright to your article

Submit your next manuscript at ► springeropen.com