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Some Orlicz extended $\mathcal{I}$-convergent A-summable classes of sequences of fuzzy numbers

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Abstract

The article introduces some classes of sequences of fuzzy numbers extended by Orlicz functions by using the notions of $\mathcal{I}$-convergence and matrix transformation and investigates the classes for relationship between them as well as establishes some relevant properties. Further, the Hukuhara difference property is employed to derive a new kind of spaces and prove that such spaces can be equipped with a linear topological structure.

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1 Introduction

The concepts of fuzzy sets and fuzzy set operations were first introduced by Zadeh [1] and subsequently several authors have discussed various aspects of the theory and applications of fuzzy sets such as fuzzy topological spaces, similarity relations and fuzzy orderings, fuzzy measures of fuzzy events, fuzzy mathematical programming. Matloka [2] introduced bounded and convergent sequences of fuzzy numbers and studied some of their properties. Later on, the sequences of fuzzy numbers were discussed by Diamond and Kloeden [3], Nanda [4], Esi [5], Dutta [6–8] and many others.

A fuzzy number is a fuzzy set on the real axis, *i.e.*, a mapping $u : R \longrightarrow [0,1]$ which satisfies the following four conditions:

(i) u is normal, *i.e.*, there exists $x_0 \in R$ such that $u(x_0) = 1$.

(ii) u is fuzzy convex, *i.e.*, $u[\lambda x + (1-\lambda)y] \geq \min\{u(x), u(y)\}$ for all $x, y \in R$ and for all $\lambda \in [0,1]$.

(iii) u is upper semi-continuous.

(iv) The set $[u]_0 = \overline{\{x \in R : u(x) > 0\}}$ is compact, where $\overline{\{x \in R : u(x) > 0\}}$ denotes the closure of the set $\{x \in R : u(x) > 0\}$ in the usual topology of R.

We denote the set of all fuzzy numbers on R by E^1 and call it the space of fuzzy numbers. λ-level set $[u]_\lambda$ of $u \in E^1$ is defined by

$$[u]_\lambda = \begin{cases} \{t \in R : u(t) \geq \lambda\} & (0 < \lambda \leq 1), \\ \overline{\{t \in R : u(t) > \lambda\}} & (\lambda = 0). \end{cases}$$

The set $[u]_\lambda$ is a closed, bounded and non-empty interval for each $\lambda \in [0,1]$ which is defined by $[u]_\lambda = [u^-(\lambda), u^+(\lambda)]$. R can be embedded in E^1 since each $r \in R$ can be regarded as a fuzzy number

$$\overline{r}(t) = \begin{cases} 1, & t = r, \\ 0, & t \neq r. \end{cases}$$

Let W be the set of all closed bounded intervals A of real numbers such that $A = [A_1, A_2]$. Define the relation s on W as follows:

$$s(A,B) = \max\{|A_1 - B_1|, |A_2 - B_2|\}.$$

Then (W,s) is a complete metric space (see Diamond and Kloeden [9], Nanda [4]). Then Talo and Basar [10] defined the metric d on E^1 by means of Hausdorff metric s as

$$d(u,v) = \sup_{\lambda\in[0,1]} s([u]_\lambda, [v]_\lambda) = \sup_{\lambda\in[0,1]} \max s(|u^-(\lambda) - v^-(\lambda)|, |u^+(\lambda) - v^+(\lambda)|).$$

Lemma 1 (Talo and Basar [10]) *Let $u, v, w, z \in E^1$ and $k \in R$. Then*

(i) (E^1, d) *is a complete metric space.*
(ii) $d(ku, kv) = |k| d(u,v)$.
(iii) $d(u+v, w+v) = d(u,w)$.
(iv) $d(u+v, w+z) \leq d(u,w) + d(v,z)$.
(v) $|d(u,\overline{0}) - d(v,\overline{0})| \leq d(u,v) \leq d(u,\overline{0}) + d(v,\overline{0})$.

Lemma 2 (Talo and Basar [10]) *The following statements hold*:

(i) $d(uv,\overline{0}) \leq d(u,\overline{0}) d(v,\overline{0})$ *for all* $u, v \in E^1$.
(ii) *If* $u_k \longrightarrow u$ *as* $k \longrightarrow \infty$, *then* $d(u_k,\overline{0}) \longrightarrow d(u,\overline{0})$ *as* $k \longrightarrow \infty$.

The notion of $\mathcal{I}$-convergence was initially introduced by Kostyrko *et al.* [11]. Later on, it was further investigated from the sequence space point of view and linked with the summability theory by Salat *et al.* [12, 13], Tripathy and Hazarika [14–16] and Kumar and Kumar [17] and many others. For some other related works, one may refer to Altinok *et al.* [18], Altin *et al.* [19–22], Çolak *et al.* [23], Güngör [24] and many others.

Let X be a non-empty set, then a family of sets $\mathcal{I} \subset 2^X$ (the class of all subsets of X) is called an *ideal* if and only if for each $A, B \in \mathcal{I}$, we have $A \cup B \in \mathcal{I}$ and for each $A \in \mathcal{I}$ and each $B \subset A$, we have $B \in \mathcal{I}$. A non-empty family of sets $F \subset 2^X$ is a *filter* on X if and only if $\phi \notin F$, for each $A, B \in F$, we have $A \cap B \in F$ and for each $A \in F$ and each $A \subset B$, we have $B \in F$. An ideal $\mathcal{I}$ is called *non-trivial ideal* if $\mathcal{I} \neq \phi$ and $X \notin \mathcal{I}$. Clearly, $\mathcal{I} \subset 2^X$ is a non-trivial ideal if and only if $F = F(\mathcal{I}) = \{X - A : A \in \mathcal{I}\}$ is a filter on X. A non-trivial ideal $\mathcal{I} \subset 2^X$ is called *admissible* if and only if $\{\{x\} : x \in X\} \subset \mathcal{I}$. A non-trivial ideal $\mathcal{I}$ is *maximal* if there cannot exist any non-trivial ideal $J \neq \mathcal{I}$ containing $\mathcal{I}$ as a subset. Further details on ideals of 2^X can be found in Kostyrko *et al.* [11].

Lemma 3 (Kostyrko *et al.* [11, Lemma 5.1]) *If* $\mathcal{I} \subset 2^N$ *is a maximal ideal, then for each* $A \subset N$, *we have either* $A \in \mathcal{I}$ *or* $N - A \in \mathcal{I}$.

Example 1 If we take $\mathcal{I} = \mathcal{I}_f = \{A \subseteq N : A$ is a finite subset$\}$, then $\mathcal{I}_f$ is a non-trivial admissible ideal of N and the corresponding convergence coincides with the usual convergence.

Example 2 If we take $\mathcal{I} = \mathcal{I}_\delta = \{A \subseteq N : \delta(A) = 0\}$, where $\delta(A)$ denotes the asymptotic density of the set A, then $\mathcal{I}_\delta$ is a non-trivial admissible ideal of N and the corresponding convergence coincides with the statistical convergence.

Recall in [25] that an *Orlicz function M* is a continuous, convex, nondecreasing function defined for $x > 0$ such that $M(0) = 0$ and $M(x) > 0$. If the convexity of an Orlicz function is replaced by $M(x + y) \le M(x) + M(y)$, then this function is called the *modulus function* and characterized by Ruckle [26]. The Orlicz function M is said to satisfy Δ_2-*condition* for all values of u if there exists $K > 0$ such that $M(2u) \le KM(u)$, $u \ge 0$.

Lemma 4 [27] *Let M be an Orlicz function which satisfies* Δ_2-*condition, and let* $0 < \delta < 1$. *Then, for each* $t \ge \delta$, *we have* $M(t) < K\delta^{-1}tM(2)$ *for some constant* $K > 0$.

Two Orlicz functions M_1 and M_2 are said to be *equivalent* if there exist positive constants α, β and x_0 such that

$$M_1(\alpha) \le M_2(x) \le M_1(\beta)$$

for all x with $0 \le x < x_0$.

Lindenstrauss and Tzafriri [28] studied some Orlicz-type sequence spaces defined as follows:

$$\ell_M = \left\{ (x_k) \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty \text{ for some } \rho > 0 \right\}.$$

The space ℓ_M with the norm

$$\|x\| = \inf\left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \le 1 \right\}$$

becomes a Banach space which is called an *Orlicz sequence space*. The space ℓ_M is closely related to the space ℓ_p which is an Orlicz sequence space with $M(t) = |t|^p$ for $1 \le p < \infty$.

In the later stage, different classes of Orlicz sequence spaces were introduced and studied by Parashar and Choudhary [29], Esi and Et [30] and many others.

Throughout the article, N and R denote the set of positive integers and the set of real numbers, respectively. The zero sequence is denoted by θ.

Let $A = (a_{ki})$ be an infinite matrix of real numbers. We write $Ax = (A_k(x))$ if $A_k(x) = \sum_i a_{ik}x_k$ converges for each i.

Throughout the paper, w^F denotes the set of all sequences of fuzzy numbers.

Definition 1 A set $E^F \subset w^F$ is said to be solid if $(b_m) \in E^F$ whenever $d(b_m, \overline{0}) \le d(a_m, \overline{0})$ for all $m \in N$ and $(a_m) \in E^F$.

The following well-known inequality will be used throughout the article. Let $p = (p_k)$ be any sequence of positive real numbers with $0 \le p_k \le \sup_k p_k = G$, $D = \max\{1, 2^{G-1}\}$, then

$$|a_k + b_k|^{p_k} \le D\left(|a_k|^{p_k} + |b_k|^{p_k}\right)$$

for all $k \in N$ and $a_k, b_k \in C$. Also, $|a_k|^{p_k} \le \max\{1, |a|^G\}$ for all $a \in C$.

2 Some new sequence spaces

Let $\mathcal{I}$ be an admissible ideal of the non-empty set S, and let $p = (p_k)_{k=1}^{\infty}$ be a bounded sequence of positive real numbers. Let $\mathbf{M} = (M_k)_{k=1}^{\infty}$ be a sequence of Orlicz functions, let $A = (a_{ki})$ be an infinite matrix of real numbers, and let $x = (x_k)_{k=1}^{\infty}$ be a sequence of fuzzy numbers. Then we introduce the following sequence spaces:

$$W^{\mathcal{I}\mathcal{F}}(\mathbf{M}, A, p) = \left\{(x_k) \in w^F : \left\{n \in N : \frac{1}{n}\sum_{k=1}^{n}\left[M_k\left(\frac{d(A_k(x), L)}{\rho}\right)\right]^{p_k} \ge \varepsilon\right\} \in \mathcal{I} \right.$$
$$\left. \text{for some } \rho > 0 \text{ and } L\right\},$$

$$W_0^{\mathcal{I}\mathcal{F}}(\mathbf{M}, A, p) = \left\{(x_k) \in w^F : \left\{n \in N : \frac{1}{n}\sum_{k=1}^{n}\left[M_k\left(\frac{d(A_k(x), \bar{0})}{\rho}\right)\right]^{p_k} \ge \varepsilon\right\} \in \mathcal{I} \right.$$
$$\left. \text{for some } \rho > 0\right\},$$

$$W_\infty^{\mathcal{F}}(\mathbf{M}, A, p) = \left\{(x_k) \in w^F : \sup \frac{1}{n}\sum_{k=1}^{n}\left[M_k\left(\frac{d(A_k(x), \bar{0})}{\rho}\right)\right]^{p_k} < \infty \text{ for some } \rho > 0\right\}$$

and

$$W_\infty^{\mathcal{I}\mathcal{F}}(\mathbf{M}, A, p) = \left\{(x_k) \in w^F : \exists K > 0 \text{ such that} \right.$$
$$\left. \left\{n \in N : \frac{1}{n}\sum_{k=1}^{n}\left[M_k\left(\frac{d(A_k(x), \bar{0})}{\rho}\right)\right]^{p_k} \ge K\right\} \in \mathcal{I} \text{ for some } \rho > 0\right\}.$$

3 Main results

In this section we investigate the main results of this paper.

Theorem 1 *The spaces* $W^{\mathcal{I}\mathcal{F}}(\mathbf{M}, A, p)$, $W_0^{\mathcal{I}\mathcal{F}}(\mathbf{M}, A, p)$, $W_\infty^{\mathcal{I}\mathcal{F}}(\mathbf{M}, A, p)$ *and* $W_\infty^{\mathcal{F}}(\mathbf{M}, A, p)$ *are linear over the field of reals.*

Proof We give the proof for the space $W_0^{\mathcal{I}}(\mathbf{M}, A, p)$ only, and the others will follow similarly. Let $x = (x_k)$ and $y = (y_k)$ be two elements in $W_0^{\mathcal{I}\mathcal{F}}(\mathbf{M}, A, p)$. Then there exist $\rho_1 > 0$ and $\rho_2 > 0$ such that

$$A_{\frac{\varepsilon}{2}} = \left\{n \in N : \frac{1}{n}\sum_{k=1}^{n}\left[M_k\left(\frac{d(A_k(x), \bar{0})}{\rho_1}\right)\right]^{p_k} \ge \frac{\varepsilon}{2}\right\} \in \mathcal{I}$$

and

$$B_{\frac{\varepsilon}{2}} = \left\{ n \in N : \frac{1}{n} \sum_{k=1}^{n} \left[M_k \left(\frac{d(A_k(y), \bar{0})}{\rho_2} \right) \right]^{p_k} \geq \frac{\varepsilon}{2} \right\} \in \mathcal{I}.$$

Let α, β be two reals. By the continuity of the Orlicz functions (M_k)'s, we have the following inequality:

$$\begin{aligned} &\frac{1}{n} \sum_{k=1}^{n} \left[M_k \left(\frac{d(A_k(\alpha x + \beta y), \bar{0})}{|\alpha|\rho_1 + |\beta|\rho_2} \right) \right]^{p_k} \\ &\quad \leq D \frac{1}{n} \sum_{k=1}^{n} \left[M_k \left(\frac{d(A_k(x), \bar{0})}{\rho_1} \right) \right]^{p_k} + D \frac{1}{n} \sum_{k=1}^{n} \left[M_k \left(\frac{d(A_k(y), \bar{0})}{\rho_2} \right) \right]^{p_k} \\ &\quad \leq D \frac{1}{n} \sum_{k=1}^{n} \left[\frac{|\alpha|}{|\alpha|\rho_1 + |\beta|\rho_2} M_k \left(\frac{d(A_k(x), \bar{0})}{\rho_1} \right) \right]^{p_k} \\ &\qquad + D \frac{1}{n} \sum_{k=1}^{n} \left[\frac{|\beta|}{|\alpha|\rho_1 + |\beta|\rho_2} M_k \left(\frac{d(A_k(y), \bar{0})}{\rho_2} \right) \right]^{p_k}. \end{aligned}$$

Hence we have the following inclusion:

$$\begin{aligned} &\left\{ n \in N : \frac{1}{n} \sum_{k=1}^{n} \left[M_k \left(\frac{d(A_k(\alpha x + \beta y), \bar{0})}{|\alpha|\rho_1 + |\beta|\rho_2} \right) \right]^{p_k} \geq \varepsilon \right\} \\ &\quad \subseteq \left\{ n \in N : D \frac{1}{n} \sum_{k=1}^{n} \left[M_k \left(\frac{d(A_k(x), \bar{0})}{\rho_1} \right) \right]^{p_k} \geq \frac{\varepsilon}{2} \right\} \\ &\qquad \cup \left\{ n \in N : D \frac{1}{n} \sum_{k=1}^{n} \left[M_k \left(\frac{d(A_k(y), \bar{0})}{\rho_2} \right) \right]^{p_k} \geq \frac{\varepsilon}{2} \right\}. \end{aligned}$$

This completes the proof. □

It is not possible in general to find some fuzzy number $X - Y$ such that $X = Y + (X - Y)$ (called the Hukuhara difference when it exists). Since every real number is a fuzzy number, we can assume that $Sw^F \subset w^F$ is such a set of sequences of fuzzy numbers with the Hukuhara difference property.

For the next result, we consider $SW_\infty^{\mathcal{F}}(\mathbf{M}, A, p) \subset W_\infty^{\mathcal{F}}(\mathbf{M}, A, p)$ to be the space of sequences of fuzzy numbers with the Hukuhara difference property.

Theorem 2 *The space $SW_\infty^{\mathcal{F}}(\mathbf{M}, A, p)$ is a paranormed space (not totally paranormed) with the paranorm g defined by*

$$g(x) = \inf \left\{ \rho^{\frac{p_k}{H}} : \sup_k M_k \left(\frac{d(A_k(x), \bar{0})}{\rho} \right) \leq 1 \textit{ for some } \rho > 0 \right\},$$

where $H = \max\{1, \sup_k p_k\}$.

Proof Clearly, $g(-x) = g(x)$ and $g(\theta) = 0$. Let $x = (x_k)$ and $y = (y_k)$ be two elements in $SW_\infty^{\mathcal{F}}(\mathbf{M}, A, p)$. Now, for $\rho_1, \rho_2 > 0$, we put

$$A_1 = \left\{\rho_1 > 0 : \sup_k M_k\left(\frac{d(A_k(x), \bar{0})}{\rho_1}\right) \le 1\right\}$$

and

$$A_2 = \left\{\rho_2 > 0 : \sup_k M_k\left(\frac{d(A_k(y), \bar{0})}{\rho_2}\right) \le 1\right\}.$$

Let us take $\rho = \rho_1 + \rho_2$. Then, using the convexity of Orlicz functions M_k's, we obtain

$$M_k\left(\frac{d(A_k(x+y), \bar{0})}{\rho}\right) \le \frac{\rho_1}{\rho_1 + \rho_2} M_k\left(\frac{d(A_k(x), \bar{0})}{\rho_1}\right) + \frac{\rho_2}{\rho_1 + \rho_2} M_k\left(\frac{d(A_k(y), \bar{0})}{\rho_2}\right),$$

which in turn gives us

$$\sup_k \left[M_k\left(\frac{d(A_k(x+y), \bar{0})}{\rho}\right)\right]^{p_k} \le 1$$

and

$$\begin{aligned} g(x+y) &= \inf\left\{(\rho_1 + \rho_2)^{\frac{p_k}{H}} : \rho_1 \in A_1, \rho_2 \in A_2\right\} \\ &\le \inf\left\{\rho_1^{\frac{p_k}{H}} : \rho_1 \in A_1\right\} + \inf\left\{\rho_2^{\frac{p_k}{H}} : \rho_2 \in A_2\right\} \\ &= g(x) + g(y). \end{aligned}$$

Let $t^m \to L$, where $t^m, L \in E^1$, and let $g(x^m - x) \to 0$ as $m \to \infty$. To prove that $g(t^m x^m - Lx) \to 0$ as $m \to \infty$, we put

$$A_3 = \left\{\rho_m > 0 : \sup_k \left[M_k\left(\frac{d(A_k(x^m), \bar{0})}{\rho_m}\right)\right]^{p_k} \le 1\right\}$$

and

$$A_4 = \left\{\rho_l > 0 : \sup_k \left[M_k\left(\frac{d(A_k(x^m - x), \bar{0})}{\rho_s}\right)\right]^{p_k} \le 1\right\}.$$

By the continuity of the sequence $\mathbf{M} = (M_k)$, we observe that

$$\begin{aligned} M_k\left(\frac{d(A_k(t^m x^m - Lx), \bar{0})}{|t^m - L|\rho_m + |L|\rho_s}\right) &\le M_k\left(\frac{d(A_k(t^m x^m - Lx^m), \bar{0})}{|t^m - L|\rho_m + |L|\rho_s}\right) \\ &\quad + M_k\left(\frac{d(A_k(Lx^m - Lx), \bar{0})}{|t^m - L|\rho_m + |L|\rho_s}\right) \\ &\le \frac{|t_k^m - L|\rho_m}{|t^m - L|\rho_m + |L|\rho_s} M_k\left(\frac{d(A_k(x^m), \bar{0})}{\rho_m}\right) \\ &\quad + \frac{|L|\rho_s}{|t^m - L|\rho_m + |L|\rho_s} M_k\left(\frac{d(A_k(x^m - x), \bar{0})}{\rho_s}\right). \end{aligned}$$

From the above inequality it follows that

$$\sup_k \left[M_k \left(\frac{d(A_k(t^m x^m - Lx), \bar{0})}{|t^m - L|\rho_m + |L|\rho_s} \right) \right]^{p_k} \le 1$$

and, consequently,

$$\begin{aligned} g(t^m x^m - Lx) &= \inf\left\{ \left(|t^m - L|\rho_m + |L|\rho_s \right)^{\frac{p_k}{H}} : \rho_m \in A_3, \rho_s \in A_4 \right\} \\ &\le |t^m - L|^{\frac{p_k}{H}} \inf\left\{ (\rho_m)^{\frac{p_k}{H}} : \rho_m \in A_3 \right\} \\ &\quad + |L|^{\frac{p_k}{H}} \inf\left\{ (\rho_s)^{\frac{p_k}{H}} : \rho_s \in A_4 \right\} \\ &\le \max\left\{ |t^m - L|, |t^m - L|^{\frac{p_k}{H}} \right\} g(x^m) \\ &\quad + \max\left\{ |L|, |L|^{\frac{p_k}{H}} \right\} g(x^m - x). \end{aligned} \tag{3.1}$$

Note that $g(x^m) \le g(x) + g(x^m - x)$ for all $m \in N$. Hence, by our assumption, the right-hand side of relation (3.1) tends to 0 as $m \to \infty$ and the result follows. This completes the proof. □

Theorem 3 *Let* $\mathbf{M} = (M_k)$ *and* $\mathbf{S} = (S_k)$ *be two sequences of Orlicz functions. Then the following statements hold*:

(i) $W^{\mathcal{IF}}(\mathbf{S}, A, p) \subseteq W^{\mathcal{IF}}(\mathbf{M} \circ \mathbf{S}, A, p)$ *provided* $p = (p_k)$ *is such that* $G_0 = \inf p_k > 0$.

(ii) $W^{\mathcal{IF}}(\mathbf{M}, A, p) \cap W^{\mathcal{IF}}(\mathbf{S}, A, p) \subseteq W^{\mathcal{IF}}(\mathbf{M} + \mathbf{S}, A, p)$.

Proof (i) Let $\varepsilon > 0$ be given. Choose $\varepsilon_1 > 0$ such that $\max\{\varepsilon_1^G, \varepsilon_1^{G_0}\} < \varepsilon$. Choose $0 < \delta < 1$ such that $0 < t < \delta$ implies that $M_k(t) < \varepsilon_1$ for each $k \in N$. Let $x = (x_k) \in W^{\mathcal{IF}}(\mathbf{S}, A, p)$ be any element. Put

$$A_\delta = \left\{ n \in N : \frac{1}{n} \sum_{k=1}^{n} \left[S_k \left(\frac{d(A_k(x), L)}{\rho} \right) \right]^{p_k} \ge \delta^G \right\}.$$

Then, by the definition of ideal, we have $A_\delta \in \mathcal{I}$. If $n \notin A_\delta$, we have

$$\begin{aligned} &\frac{1}{n} \sum_{k=1}^{n} \left[S_k \left(\frac{d(A_k(x), L)}{\rho} \right) \right]^{p_k} < \delta^G \\ &\quad \Rightarrow \quad \sum_{k=1}^{n} \left[S_k \left(\frac{d(A_k(x), L)}{\rho} \right) \right]^{p_k} < n\delta^G \\ &\quad \Rightarrow \quad \left[S_k \left(\frac{d(A_k(x), L)}{\rho} \right) \right]^{p_k} < \delta^G \quad \text{for } k = 1, 2, 3, \ldots, n \\ &\quad \Rightarrow \quad S_k \left(\frac{d(A_k(x), L)}{\rho} \right) < \delta^G. \end{aligned} \tag{3.2}$$

Using the continuity of the sequence $\mathbf{M} = (M_k)$ from relation (3.2), we have

$$M_k \left(S_k \left(\frac{d(A_k(x), L)}{\rho} \right) \right) < \varepsilon_1 \quad \text{for } k = 1, 2, 3, \ldots, n.$$

Consequently, we get

$$\sum_{k=1}^{n}\left[M_k\left(S_k\left(\frac{d(A_k(x),L)}{\rho}\right)\right)\right]^{p_k} < n\cdot\max\left\{\varepsilon_1^{G},\varepsilon_1^{G_0}\right\} < n\varepsilon$$

$$\Rightarrow\quad \frac{1}{n}\sum_{k=1}^{n}\left[M_k\left(S_k\left(\frac{d(A_k(x),L)}{\rho}\right)\right)\right]^{p_k} < \varepsilon.$$

This implies that

$$\left\{n\in N:\frac{1}{n}\sum_{k=1}^{n}\left[M_k\left(S_k\left(\frac{d(A_k(x),L)}{\rho}\right)\right)\right]^{p_k}\geq\varepsilon\right\}\subseteq A_\delta\in\mathcal{I}.$$

This completes the proof.

(ii) Let $x=(x_k)\in W^{\mathcal{I}\mathcal{F}}(\mathbf{M},A,p)\cap W^{\mathcal{I}\mathcal{F}}(\mathbf{S},A,p)$. Then the result follows from the following inequality:

$$\frac{1}{n}\sum_{k=1}^{n}\left[(M_k+S_k)\left(\frac{d(A_k(x),L)}{\rho}\right)\right]^{p_k}\leq D\frac{1}{n}\sum_{k=1}^{n}\left[M_k\left(\frac{d(A_k(x),L)}{\rho}\right)\right]^{p_k} + D\frac{1}{n}\sum_{k=1}^{n}\left[S_k\left(\frac{d(A_k(x),L)}{\rho}\right)\right]^{p_k}.$$ □

Taking $L=\bar{0}$ in the proof of the above theorem, we have the following corollary.

Corollary 1 *Let $\mathbf{M}=(M_k)$ and $\mathbf{S}=(S_k)$ be two sequences of Orlicz functions. Then the following statements hold*:

(i) $W_0^{\mathcal{I}\mathcal{F}}(\mathbf{S},A,p)\subseteq W_0^{\mathcal{I}\mathcal{F}}(\mathbf{M}\circ\mathbf{S},A,p)$ *provided* $p=(p_k)$ *is such that* $G_0=\inf p_k>0$.
(ii) $W_0^{\mathcal{I}\mathcal{F}}(\mathbf{M},A,p)\cap W_0^{\mathcal{I}\mathcal{F}}(\mathbf{S},A,p)\subseteq W_0^{\mathcal{I}\mathcal{F}}(\mathbf{M}+\mathbf{S},A,p)$.

The proofs of the following two theorems are easy and so they are omitted.

Theorem 4 *Let $0<p_k\leq q_k$ and $(\frac{q_k}{p_k})$ be bounded, then*

$$W_0^{\mathcal{I}\mathcal{F}}(\mathbf{M},A,q)\subseteq W_0^{\mathcal{I}\mathcal{F}}(\mathbf{M},A,p).$$

Theorem 5 *For any two sequences of positive real numbers $p=(p_k)$ and $q=(q_k)$, the following statement holds*:

$$Z(\mathbf{M},A,p)\cap Z(\mathbf{M},A,q)\neq\phi\quad \textit{for } Z=W^{\mathcal{I}\mathcal{F}},W_0^{\mathcal{I}\mathcal{F}},W_\infty^{\mathcal{I}\mathcal{F}}\textit{ and } W_\infty^{\mathcal{F}}.$$

Proposition 1 *The sequence spaces $Z(\mathbf{M},A,p)$ are solid for $Z=W_0^{\mathcal{I}\mathcal{F}}$ and $W_\infty^{\mathcal{I}\mathcal{F}}$.*

Proof We give the proof of the proposition for $W_0^{\mathcal{I}\mathcal{F}}(\mathbf{M},A,p)$ only. Let $x=(x_k)\in W_0^{\mathcal{I}\mathcal{F}}(\mathbf{M},A,p)$ and $y=(y_k)$ be such that $d(y_k,\bar{0})\leq d(x_k,\bar{0})$ for all $k\in N$. Then, for given $\varepsilon>0$, we have

$$B=\left\{n\in N:\frac{1}{n}\sum_{k=1}^{n}\left[M_k\left(\frac{d(A_k(x),\bar{0})}{\rho}\right)\right]^{p_k}\geq\varepsilon\right\}\in\mathcal{I}.$$

Again the set

$$E = \left\{ n \in N : \frac{1}{n} \sum_{k=1}^{n} \left[M_k \left(\frac{d(A_k(y), \bar{0})}{\rho} \right) \right]^{p_k} \geq \varepsilon \right\} \subseteq B.$$

Hence $E \in \mathcal{I}$ and so $y = (y_k) \in W_0^{\mathcal{I}\mathcal{F}}(\mathbf{M}, A, p)$. Thus the space $W_0^{\mathcal{I}\mathcal{F}}(\mathbf{M}, A, p)$ is solid. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
AE proposed the research content. HD formulated the structure of the paper. ABK collected necessary literature for review and study. AE and ABK constructed the new spaces. HD equipped the spaces with a linear topological structure. The proofs of the results were completed through combined efforts of all the authors. All authors read and approved the final manuscript.

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