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Convergence rate of extremes from Maxwell sample

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Abstract
For the partial maximum from a sequence of independent and identically distributed random variables with Maxwell distribution, we establish the uniform convergence rate of its distribution to the extreme value distribution.
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1 Introduction

One interesting problem in extreme value theory is to consider the convergence rate of some extremes. For the uniform convergence rate of extremes under the second-order regular variation conditions, see Falk [1], Balkema and de Haan [2], de Haan and Resnick [3] and Cheng and Jiang [4]. For the extreme value distributions and their associated uniform convergence rates for given distributions, see Hall and Wellner [5], Hall [6], Peng *et al.* [7], Lin and Peng [8] and Lin *et al.* [9].

In this note, we discuss the uniform convergence rate of extremes from a sequence of independent and identically distributed (iid) random variables with Maxwell distribution (MD). The probability density function of MD is given by

$$f(x) = \sqrt{\frac{2}{\pi}}\frac{x^2}{\sigma^3}\exp\left(-\frac{x^2}{2\sigma^2}\right), \quad x > 0. \tag{1.1}$$

The MD and the convergence rate of extremes from Maxwell sample have been widely used in the field of physics. We establish the uniform convergence rate of its distribution to the extreme value distribution and give an improved proof for the pointwise convergence rate of MD.

Throughout this paper, let $(\xi_n, n \geq 1)$ be a sequence of iid random variables with common distribution $F(x) = \int_0^x f(t)\,dt$ with a probability density function $f(x)$ given by (1.1), and let $M_n = \max_{1\leq k\leq n} \xi_k$ be the partial maximum. Liu and Fu [10] proved that

$$\lim_{n\to\infty} P\left(\alpha_n^{-1}(M_n - \beta_n) \leq x\right) = \exp\left(-\exp(-x)\right) := \Lambda(x)$$

with the normalizing constants α_n and β_n given by

$$\alpha_n = \frac{\sigma}{(2\log n)^{\frac{1}{2}}}, \qquad \beta_n = \left(2\sigma^2 \log n\right)^{\frac{1}{2}} + \frac{\sigma\log(2\log n) + \sigma\log\frac{2}{\pi}}{2(2\log n)^{\frac{1}{2}}}. \tag{1.2}$$

By arguments similar to those of Hall [6], Peng *et al.* [7] and Lin *et al.* [9], the appropriate normalizing constants a_n and b_n can be given by the following equations:

$$a_n = \sigma^2 b_n^{-1} \tag{1.3}$$

and

$$\sqrt{\frac{\pi}{2}}\frac{b_n}{\sigma}\exp\left(\frac{b_n^2}{2\sigma^2}\right) = n. \tag{1.4}$$

By arguments similar to those of Example 2 of Resnick [11], we have

$$b_n = \left(2\sigma^2 \log n\right)^{\frac{1}{2}} + \frac{\sigma \log(2\log n) + \sigma \log \frac{2}{\pi}}{2(2\log n)^{\frac{1}{2}}} + o\left((\log n)^{-1/2}\right).$$

Hence

$$\alpha_n/a_n \to 1, \qquad (\beta_n - b_n)/a_n \to 0,$$

implying

$$\lim_{n\to\infty} P(M_n \le a_n x + b_n) = \lim_{n\to\infty} F^n(a_n x + b_n) = \Lambda(x),$$

cf. Leadbetter *et al.* [12] or Resnick [11].

This paper is organized as follows. Section 2 gives some auxiliary results. In Section 3, we present the main result. Related proofs are given in Section 4.

2 Auxiliary results

To establish the uniform convergence of $F^n(a_n x + b_n)$ to its extreme value distribution $\Lambda(x)$, we need some auxiliary results. The first result is the decomposition of $F(x)$, which is the following result.

Lemma 1 *Let $F(x)$ be the Maxwell distribution function. Then, for $x > 0$, we have*

$$1 - F(x) = \sqrt{\frac{2}{\pi}}\frac{x}{\sigma}\left(1 + \frac{\sigma^2}{x^2}\right)\exp\left(-\frac{x^2}{2\sigma^2}\right) - r(x) \tag{2.1}$$

with

$$0 < r(x) = \sqrt{\frac{2}{\pi}}\int_x^\infty \frac{\sigma}{y^2}\exp\left(-\frac{y^2}{2\sigma^2}\right)dy < \sqrt{\frac{2}{\pi}}\frac{\sigma^3}{x^3}\exp\left(-\frac{x^2}{2\sigma^2}\right). \tag{2.2}$$

For simplicity, throughout this paper, let C be a generic positive constant whose value may change from line to line, and let C_i, C_{ij} ($i \in N, j \in N$) be absolute positive constants.

For the normalizing constants a_n, b_n defined by (1.3) and (1.4), respectively, let

$$a_n^* = a_n r_n, \qquad b_n^* = b_n + a_n \delta_n, \tag{2.3}$$

where $r_n \to 1$, $\delta_n \to 0$, $n \to \infty$. So, $a_n^*/a_n \to 1$, $(b_n^* - b_n)/a_n \to 0$, implying $F^n(a_n^* x + b_n^*) \to \Lambda(x)$. For large n, we have the following result.

Lemma 2 *Let a_n^*, b_n^* be defined by* (2.3). *For fixed $x \in R$ and sufficiently large n, we have*

$$F^n(a_n^* x + b_n^*) - \Lambda(x) = \Lambda(x)e^{-x}\left(\left(\frac{x^2}{2} - x - 1\right)a_n b_n^{-1} + (r_n - 1)x + \delta_n + O\left[\left(a_n b_n^{-1}\right)^2 + (r_n - 1)^2 + \delta_n^2\right]\right). \tag{2.4}$$

Proof Note that $b_n \sim \sigma(2\log n)^{\frac{1}{2}}$, which means

$$a_n b_n^{-1} \sim \frac{1}{2\log n} \to 0.$$

For large n, we have

$$\begin{aligned}
&\sqrt{\frac{2}{\pi}}\frac{a_n^* x + b_n^*}{\sigma}\exp\left(-\frac{(a_n^* x + b_n^*)^2}{2\sigma^2}\right) \\
&\quad = \sqrt{\frac{2}{\pi}}\frac{b_n}{\sigma}\left(1 + a_n b_n^{-1}(r_n x + \delta_n)\right)\exp\left(-\frac{b_n^2}{2\sigma^2}\right) \\
&\qquad \times \exp\left(-\frac{a_n^2(r_n^2 x^2 + \delta_n^2 + 2r_n\delta_n x)}{2\sigma^2} - (r_n - 1)x - x - \delta_n\right) \\
&\quad = n^{-1}e^{-x}\left(1 - \left(\frac{x^2}{2} - x\right)a_n b_n^{-1} - (r_n - 1)x - \delta_n + O\left[\left(a_n b_n^{-1}\right)^2 + (r_n - 1)^2 + \delta_n^2\right]\right).
\end{aligned}$$

Since

$$\frac{\sigma^2}{(a_n^* x + b_n^*)^2} = a_n b_n^{-1} - 2x\left(a_n b_n^{-1}\right)^2 + O\left(\left(a_n b_n^{-1}\right)^3\right),$$

we have

$$\frac{\sigma^4}{(a_n^* x + b_n^*)^4} = \left(a_n b_n^{-1}\right)^2 + O\left(\left(a_n b_n^{-1}\right)^3\right).$$

Similarly,

$$\frac{\sigma^5}{(a_n^* x + b_n^*)^5}\exp\left(-\frac{(a_n^* x + b_n^*)^2}{2\sigma^2}\right) = O\left(n^{-1}\left(a_n b_n^{-1}\right)^2\right).$$

Hence,

$$1 - F\left(a_n^* x + b_n^*\right) = n^{-1}e^{-x}\left(1 - \left(\frac{x^2}{2} - x - 1\right)a_n b_n^{-1} - (r_n - 1)x - \delta_n + O\left[\left(a_n b_n^{-1}\right)^2 + (r_n - 1)^2 + \delta_n^2\right]\right). \tag{2.5}$$

So,

$$\begin{aligned}
&F^n\left(a_n^* x + b_n^*\right) - \Lambda(x) \\
&\quad = \left(1 - n^{-1}e^{-x}\left(1 - \left(\frac{x^2}{2} - x - 1\right)a_n b_n^{-1} - (r_n - 1)x\right.\right.
\end{aligned}$$

$$-\delta_n + O\big[\big(a_n b_n^{-1}\big)^2 + (r_n-1)^2 + \delta_n^2\big]\bigg)\bigg)^n - \Lambda(x)$$

$$= \Lambda(x)e^{-x}\left(\left(\frac{x^2}{2} - x - 1\right)a_n b_n^{-1} + (r_n-1)x + \delta_n + O\big[\big(a_n b_n^{-1}\big)^2 + (r_n-1)^2 + \delta_n^2\big]\right),$$

which is the desired result. □

3 Main results

In this section we present the pointwise convergence rate and the uniform convergence rate of $F^n(\cdot)$ to its extreme value distribution under different normalizing constants. The first result is the pointwise convergence of extremes under the normalizing constants given by (1.2).

Theorem 1 *Let* $\{\xi_n, n \geq 1\}$ *be a sequence of independent identically distributed random variables with common distribution MD. Then*

$$F^n(\alpha_n x + \beta_n) - \Lambda(x) \sim \Lambda(x)e^{-x}\frac{(\log(2\log n))^2}{16\log n}, \tag{3.1}$$

for large n, where α_n, β_n *are defined in* (1.2).

Recently Liu and Fu [10] proved the result, we present an improved proof for the pointwise convergence rate in Section 4.

The following is the uniform convergence rate of extremes under the appropriate normalizing constants a_n and b_n given by (1.3) and (1.4), which shows that the optimal convergence rate is proportional to $1/\log n$.

Theorem 2 *Let* $(\xi_n, n \geq 1)$ *be a sequence of independent identically distributed random variables with common distribution MD. For large n, there exist absolute constants* $0 < d_1 < d_2$ *such that*

$$\frac{d_1}{\log n} < \sup_{x\in R}\big|F^n(a_n x + b_n) - \Lambda(x)\big| < \frac{d_2}{\log n}, \tag{3.2}$$

where a_n *and* b_n *are defined by* (1.3) *and* (1.4), *respectively.*

4 Proofs

Proof of Theorem 1 Firstly, we derive the following asymptotic expansions of b_n defined by (1.4)

$$b_n = \beta_n + o\big((\log n)^{-\frac{1}{2}}\big) \tag{4.1}$$

and

$$b_n = \beta_n - \frac{\sigma(\log(2\log n) + \log\frac{2}{\pi})^2}{16\sqrt{2}(\log n)^{\frac{3}{2}}} + \left(\sigma\log\frac{4\log n + \log(2\log n) + \log\frac{2}{\pi}}{4\log n}\right)\Big/(2\log n)^{\frac{1}{2}} + O\left(\frac{(\log(2\log n))^2}{(\log n)^{\frac{5}{2}}}\right). \tag{4.2}$$

Setting $b_n = \beta_n + \theta_n$ and substituting into (1.4), we obtain by taking logarithms that

$$\log\frac{\pi}{2} + \log\sigma - \log(\beta_n + \theta_n) + \frac{\beta_n^2}{2\sigma^2} + \frac{\beta_n\theta_n}{\sigma^2} + \frac{\theta_n^2}{\sigma^2} = \log n.$$

So,

$$\frac{(\log(2\log n) + \log\frac{2}{\pi})^2}{16\log n} - \log\frac{4\log n + \log(2\log n) + \log\frac{2}{\pi}}{4\log n} + \frac{\beta_n\theta_n}{\sigma^2} + \frac{\theta_n^2}{\sigma^2} - \log\left(1 + \frac{\theta_n}{\beta_n}\right) = 0, \tag{4.3}$$

therefore

$$\frac{\beta_n\theta_n}{\sigma^2} \sim -\frac{(\log(2\log n) + \log\frac{2}{\pi})^2}{16\log n} + \log\frac{4\log n + \log(2\log n) + \log\frac{2}{\pi}}{4\log n}, \tag{4.4}$$

which implies

$$\theta_n \sim -\frac{\sigma(\log(2\log n) + \log\frac{2}{\pi})^2}{16\sqrt{2}(\log n)^{\frac{3}{2}}} + \left(\sigma\log\frac{4\log n + \log(2\log n) + \log\frac{2}{\pi}}{4\log n}\right)\Big/(2\log n)^{\frac{1}{2}}. \tag{4.5}$$

Once again let

$$\theta_n = -\frac{\sigma(\log(2\log n) + \log\frac{2}{\pi})^2}{16\sqrt{2}(\log n)^{\frac{3}{2}}} + \left(\sigma\log\frac{4\log n + \log(2\log n) + \log\frac{2}{\pi}}{4\log n}\right)\Big/(2\log n)^{\frac{1}{2}} + \nu_n,$$

where $\nu_n = o(\frac{(\log(2\log n))^2}{(\log n)^{\frac{3}{2}}})$. By similar arguments, we can obtain (4.2).

Note that $a_n = \frac{\sigma^2}{b_n}$, we have

$$r_n - 1 = \frac{\alpha_n}{a_n} - 1 \sim \frac{\log(2\log n)}{4\log n}, \qquad \delta_n = \frac{\beta_n - b_n}{a_n} \sim \frac{(\log(2\log n))^2}{16\log n}.$$

Noting $a_n b_n^{-1} \sim \frac{1}{2\log n}$, by Lemma 2, we have

$$F^n(\alpha_n x + \beta_n) - \Lambda(x) \sim \Lambda(x)e^{-x}\frac{(\log(2\log n))^2}{16\log n}.$$

The proof is complete. □

Proof of Theorem 2 Letting $r_n = 1$, $\delta_n = 0$ in (2.3) and noting $a_n b_n^{-1} \sim \frac{1}{2\log n}$, and by Lemma 2, there exists an absolute constant $d_1 > 0$ such that

$$\sup_{x\in R}\left|F^n(a_n x + b_n) - \Lambda(x)\right| > \frac{d_1}{\log n}. \tag{4.6}$$

Thus, in order to obtain the upper bound, we need to prove

$$\text{(a)} \quad \sup_{-c_n \le x < 0} \left| F^n(a_n x + b_n) - \Lambda(x) \right| < \mathbb{D}_1 a_n b_n^{-1}, \tag{4.7}$$

$$\text{(b)} \quad \sup_{0 \le x \le d_n} \left| F^n(a_n x + b_n) - \Lambda(x) \right| < \mathbb{D}_2 a_n b_n^{-1}, \tag{4.8}$$

$$\text{(c)} \quad \sup_{d_n \le x < \infty} \left| F^n(a_n x + b_n) - \Lambda(x) \right| < \mathbb{D}_3 a_n b_n^{-1}, \tag{4.9}$$

$$\text{(d)} \quad \sup_{-\infty < x \le -c_n} \left| F^n(a_n x + b_n) - \Lambda(x) \right| < \mathbb{D}_4 a_n b_n^{-1}, \tag{4.10}$$

where $\mathbb{D}_i > 0$ $(i = 1, 2, 3, 4)$, and

$$c_n =: \log\log \frac{b_n^2}{\sigma^2} > 0, \qquad d_n =: -\log\log \frac{b_n^2}{b_n^2 - \sigma^2} > 0.$$

Obviously,

$$\sigma (2\log n)^{\frac{1}{2}} < b_n < \sigma (2\log n)^{\frac{1}{2}} (1 + C_0)$$

and

$$b_n - a_n c_n = b_n \left(1 - \frac{\sigma^2}{b_n^2} c_n \right) = b_n \left(1 - \frac{\sigma^2}{b_n^2} \log\log \frac{b_n^2}{\sigma^2} \right) > 0.$$

Define $\Psi_n(x) = 1 - F(a_n x + b_n)$, then

$$\begin{aligned} n\log\bigl(1 - F(a_n x + b_n)\bigr) &= -n\Psi_n(x) + n\Psi_n(x) + n\log\bigl(1 - \Psi_n(x)\bigr) \\ &= -n\Psi_n(x) - R_n(x). \end{aligned} \tag{4.11}$$

By the following inequality

$$-x - \frac{x^2}{2(1-x)} < \log(1-x) < -x \quad (0 < x < 1),$$

we have

$$0 < R_n(x) = -\bigl(n\Psi_n(x) + n\log\bigl(1 - \Psi_n(x)\bigr)\bigr) < \frac{n\Psi_n^2(x)}{2(1 - \Psi_n(x))}.$$

First, suppose that $x \ge -c_n$. By (2.1), we have

$$\begin{aligned} \Psi_n(x) &\le \Psi_n(-c_n) = 1 - F(b_n - a_n c_n) \\ &< \sqrt{\frac{2}{\pi}} \frac{b_n - a_n c_n}{\sigma} \left(1 + \frac{\sigma^2}{(b_n - a_n c_n)^2} \right) \exp\left(-\frac{(b_n - a_n c_n)^2}{2\sigma^2} \right) \\ &< 2\sqrt{\frac{2}{\pi}} \frac{b_n}{\sigma} \bigl(1 - a_n b_n^{-1} c_n\bigr) \exp\left(-\frac{(b_n)^2}{2\sigma^2} + c_n - \frac{a_n b_n^{-1} c_n^2}{2} \right) \\ &< 2n^{-1} e^{c_n} = 2n^{-1} \log \frac{b_n^2}{\sigma^2} \\ &< \sup_{n \ge n_0} \frac{2\log(C_1 \log n)}{n} < \mathfrak{C}_{11} < 1 \end{aligned}$$

with $C_1 = 2(1 + C_0)^2$, implying

$$\inf_{x \geq -c_n} \bigl(1 - \Psi_n(x)\bigr) > 1 - \mathfrak{C}_{11} > 0.$$

Therefore,

$$\begin{aligned} 0 < R_n(x) &\leq \frac{n\Psi_n^2(x)}{2(1-\mathfrak{C}_{11})} \leq \frac{n\Psi_n^2(-c_n)}{2(1-\mathfrak{C}_{11})} \\ &< \frac{n^{-1}(\log(C_1 \log n))^2}{2(1-\mathfrak{C}_{11})} = \frac{n^{-1}(\log(C_1 \log n))^2 a_n b_n^{-1}}{2(1-\mathfrak{C}_{11}) a_n b_n^{-1}} \\ &< \frac{n^{-1}(\log(C_1 \log n))^2}{4(1-\mathfrak{C}_{11}) \log n} a_n b_n^{-1} \\ &< C a_n b_n^{-1}. \end{aligned}$$

By $1 - e^{-x} < x$, $x > 0$, we have

$$\bigl|\exp\bigl(-R_n(x)\bigr) - 1\bigr| < R_n(x) < C a_n b_n^{-1}.$$

Setting $A_n(x) = \exp(-n\Psi_n(x) + e^{-x})$, $B_n(x) = \exp(-R_n(x))$, we obtain

$$\begin{aligned} \bigl|F^n(a_n x + b_n) - \Lambda(x)\bigr| &= \Lambda(x)\bigl|A_n(x)B_n(x) - 1\bigr| \\ &= \Lambda(x)\bigl|A_n(x)B_n(x) - B_n(x) + B_n(x) - 1\bigr| \\ &\leq \Lambda(x)\bigl|A_n(x) - 1\bigr| + \bigl|B_n(x) - 1\bigr| \\ &< \Lambda(x)\bigl|A_n(x) - 1\bigr| + C a_n b_n^{-1}. \end{aligned} \tag{4.12}$$

By (2.1) and (2.2), we have

$$\begin{aligned} -n\Psi_n(x) + e^{-x} &= -n\biggl[\sqrt{\frac{2}{\pi}} \frac{b_n + a_n x}{\sigma}\biggl(1 + \frac{\sigma^2}{(b_n + a_n x)^2}\biggr) \exp\biggl(-\frac{(b_n + a_n x)^2}{2\sigma^2}\biggr) \\ &\quad - r(a_n x + b_n)\biggr] + e^{-x} \\ &= \bigl(1 + a_n b_n^{-1} x\bigr) e^{-x} C_n(x), \end{aligned}$$

where

$$C_n(x) = \biggl(-1 - \frac{\sigma^2}{(b_n + a_n x)^2} + \frac{\sigma^4}{(b_n + a_n x)^4}\delta_n(a_n x + b_n)\biggr)\exp\biggl(-\frac{a_n b_n^{-1} x^2}{2}\biggr) + \bigl(1 + a_n b_n^{-1} x\bigr)^{-1}$$

with $0 < \delta_n(a_n x + b_n) < 1$. To prove (4.7), we consider the case of $-c_n \leq x < 0$. By $e^{-x} > 1 - x$, $x > 0$, we have

$$\begin{aligned} C_n(x) &< \biggl(1 - \frac{a_n b_n^{-1} x^2}{2}\biggr)\biggl(\biggl(-1 + \frac{\sigma^4}{(b_n + a_n x)^4}\biggr)\delta_n(a_n x + b_n)\biggr) + \bigl(1 + a_n b_n^{-1} x\bigr)^{-1} \\ &< \biggl(1 - \frac{a_n b_n^{-1} x^2}{2}\biggr)\bigl\{-1 + \bigl(a_n b_n^{-1}\bigr)^2\bigl(1 + a_n b_n^{-1} x\bigr)^{-4}\bigr\} + \bigl(1 + a_n b_n^{-1} x\bigr)^{-1} \\ &= \biggl(\bigl(1 + a_n b_n^{-1} x\bigr)^{-4} + \frac{x^2}{2} - x\bigl(1 + a_n b_n^{-1} x\bigr)^{-1}\biggr) a_n b_n^{-1} \end{aligned} \tag{4.13}$$

and

$$\begin{aligned}
C_n(x) &> \left(-1 - \frac{\sigma^2}{(b_n + a_n x)^2}\right)\exp\left(-\frac{a_n b_n^{-1} x^2}{2}\right) + \left(1 + a_n b_n^{-1} x\right)^{-1} \\
&> \left(-1 - \frac{\sigma^2}{(b_n + a_n x)^2}\right) + \left(1 + a_n b_n^{-1} x\right)^{-1} \\
&> \left(-\left(1 + a_n b_n^{-1} x\right)^{-2} - x\left(1 + a_n b_n^{-1} x\right)^{-1}\right) a_n b_n^{-1} \\
&> -\left(1 + a_n b_n^{-1} x\right)^{-2}.
\end{aligned} \tag{4.14}$$

Hence, for $-c_n \le x < 0$, by combining (4.13) and (4.14) together, we have

$$\begin{aligned}
\left|C_n(x)\right| &< \left(\left(1 + a_n b_n^{-1} x\right)^{-4} + \frac{x^2}{2} - x\left(1 + a_n b_n^{-1} x\right)^{-1} + \left(1 + a_n b_n^{-1} x\right)^{-2}\right) a_n b_n^{-1} \\
&< \left(\left(1 - a_n b_n^{-1} c_n\right)^{-4} + \frac{c_n^2}{2} + c_n\left(1 - a_n b_n^{-1} c_n\right)^{-1} + \left(1 - a_n b_n^{-1} c_n\right)^{-2}\right) a_n b_n^{-1} \\
&< C_{21}.
\end{aligned}$$

Furthermore, for $-c_n \le x < 0$, we have

$$\begin{aligned}
\left|-n\Psi_n(x) + e^{-x}\right| &< \left(1 + a_n b_n^{-1} x\right) e^{-x} \left|C_n(x)\right| \\
&< \left(\left(1 + a_n b_n^{-1} x\right)^{-4} + \frac{x^2}{2} - x\left(1 + a_n b_n^{-1} x\right)^{-1} + \left(1 + a_n b_n^{-1} x\right)^{-2}\right) e^{-x} a_n b_n^{-1} \\
&< \left(\left(1 - a_n b_n^{-1} c_n\right)^{-4} + \frac{c_n^2}{2} + c_n\left(1 - a_n b_n^{-1} c_n\right)^{-1} \right.\\
&\quad \left. + \left(1 - a_n b_n^{-1} c_n\right)^{-2}\right) e^{c_n} a_n b_n^{-1} \\
&< C_{22}.
\end{aligned}$$

Noting that $0 < |e^x - 1| < |x|(e^x + 1)$, $x \in R$ and $e^{-x} > 1 - x + x^2/2$ for $-c_n \le x < 0$, we have

$$\begin{aligned}
\Lambda(x)\left|A_n(x) - 1\right| &= \Lambda(x)\left|\exp\left(-n\Psi_n(x) + e^{-x}\right) - 1\right| \\
&< \Lambda(x)\left|-n\Psi_n(x) + e^{-x}\right|\left(\exp\left(-n\Psi_n(x) + e^{-x}\right) + 1\right) \\
&< \left(e^{C_{22}} + 1\right)\left(\left(1 + a_n b_n^{-1} x\right)^{-4} + \frac{x^2}{2} - x\left(1 + a_n b_n^{-1} x\right)^{-1} + \left(1 + a_n b_n^{-1} x\right)^{-2}\right) \\
&\quad \times a_n b_n^{-1} \exp\left(-e^{-x} - x\right) \\
&< C_{23} a_n b_n^{-1}.
\end{aligned}$$

Together with (4.12), we establish (4.7).

Second, we prove (4.8). Note that

$$\begin{aligned}
C_n(x) &< \left(-1 + \frac{\sigma^4}{(b_n + a_n x)^4}\delta_n(a_n x + b_n)\right)\left(1 - \frac{a_n b_n^{-1} x^2}{2}\right) + \left(1 + a_n b_n^{-1} x\right)^{-1} \\
&< \left(a_n b_n^{-1}\right)^2\left(1 + a_n b_n^{-1} x\right)^{-4} + \frac{x^2}{2} a_n b_n^{-1} - a_n b_n^{-1} x\left(1 + a_n b_n^{-1} x\right)^{-1}
\end{aligned}$$

$$< (a_n b_n^{-1})^2 (1 + a_n b_n^{-1} x)^{-4} + \frac{x^2}{2} a_n b_n^{-1}$$
$$< \left(1 + \frac{x^2}{2}\right) a_n b_n^{-1} \tag{4.15}$$

and

$$C_n(x) > \left(-(1 + a_n b_n^{-1} x)^{-2} - x(1 + a_n b_n^{-1} x)^{-1}\right) a_n b_n^{-1}. \tag{4.16}$$

By (4.15) and (4.16), for $0 \le x < d_n$, we have

$$\begin{aligned} |C_n(x)| &< \left(1 + \frac{x^2}{2} + (1 + a_n b_n^{-1} x)^{-2} + x(1 + a_n b_n^{-1} x)^{-1}\right) a_n b_n^{-1} \\ &< \left(2 + x + \frac{x^2}{2}\right) a_n b_n^{-1}. \end{aligned}$$

Hence,

$$\begin{aligned} |-n\Psi_n(x) + e^{-x}| &< (1 + a_n b_n^{-1} x) e^{-x} |C_n(x)| \\ &< (1 + a_n b_n^{-1} x) e^{-x} \left(2 + x + \frac{x^2}{2}\right) a_n b_n^{-1} \\ &< C_{31} a_n b_n^{-1} < C_{32}. \end{aligned}$$

Therefore

$$\begin{aligned} \Lambda(x)|A_n(x) - 1| &< \Lambda(x)|-n\Psi_n(x) + e^{-x}|\left(\exp(-n\Psi_n(x) + e^{-x}) + 1\right) \\ &< C_{31}(e^{C_{32}} + 1)\Lambda(d_n) a_n b_n^{-1} \\ &< C_{33} a_n b_n^{-1}. \end{aligned} \tag{4.17}$$

Combining (4.12) and (4.17) together, we can derive that

$$\sup_{0 \le x \le d_n} |F^n(a_n x + b_n) - \Lambda(x)| < (C_{12} + C_{33}) a_n b_n^{-1} =: \mathbb{D}_2 a_n b_n^{-1}.$$

Hence (4.8) is proved.

Third, for $x \ge d_n$, we have

$$\sup_{x \ge d_n} (1 - \Lambda(x)) \le 1 - \Lambda(d_n) = a_n b_n^{-1}. \tag{4.18}$$

By $1 - e^x < -x$, $x \in R$, we have

$$\begin{aligned} 1 - F^n(a_n d_n + b_n) &= 1 - \exp\left(n \log F(a_n d_n + b_n)\right) \\ &< -n \log F(a_n d_n + b_n) \\ &= n\Psi_n(d_n) + R_n(d_n). \end{aligned} \tag{4.19}$$

By (2.1) and $\log(1+x) < x$, $0 < x < 1$, we have

$$\begin{aligned}
n\Psi_n(d_n) &= n\bigl(1 - F(a_n d_n + b_n)\bigr) \\
&< \bigl(1 + a_n b_n^{-1} d_n\bigr) e^{-d_n} \bigl(1 + a_n b_n^{-1}\bigl(1 + a_n b_n^{-1} d_n\bigr)^{-2}\bigr) \\
&< 2\bigl(1 + a_n b_n^{-1} d_n\bigr) \log \frac{b_n^2}{b_n^2 - \sigma^2} \\
&< 2\bigl(d_n + a_n^{-1} b_n\bigr) \frac{\sigma^2}{b_n^2 - \sigma^2} a_n b_n^{-1} \\
&= \left(1 + \frac{\sigma^2}{b_n^2 - \sigma^2} - \frac{\sigma^2}{b_n^2 - \sigma^2} \log\log\left(\frac{\sigma^2}{b_n^2 - \sigma^2}\right)\right) a_n b_n^{-1} \\
&< C_{41} a_n b_n^{-1}.
\end{aligned} \tag{4.20}$$

Noting that $R_n(d_n) < C_{12} a_n b_n^{-1}$, and combining (4.18), (4.19), (4.20) and (4.14) together, we have

$$\begin{aligned}
\sup_{x \ge d_n} \bigl|F^n(a_n x + b_n) - \Lambda(x)\bigr| &< \sup_{x \ge d_n} \bigl(1 - F^n(a_n x + b_n)\bigr) + \sup_{x \ge d_n} \bigl(1 - \Lambda(x)\bigr) \\
&< n\Psi_n(d_n) + R_n(d_n) + a_n b_n^{-1} \\
&< (C_{41} + C_{12} + 1) a_n b_n^{-1} =: \mathbb{D}_3 a_n b_n^{-1},
\end{aligned}$$

which is (4.9).

Finally, consider the case of $-\infty < x < -c_n$. If $a_n x + b_n \le 0$, then $F^n(a_n x + b_n) = 0$. By $\Lambda(-x) < \frac{1}{x}$, $x > 1$, we have

$$\sup_{x \le -b_n/a_n} \bigl|F^n(a_n x + b_n) - \Lambda(x)\bigr| = \sup_{x \le -b_n/a_n} \Lambda(x) \le \Lambda\bigl(-b_n^2/\sigma^2\bigr) < \frac{\sigma^2}{b_n^2} = a_n b_n^{-1}.$$

So, we only need to consider the case of $a_n x + b_n > 0$. By using the monotonicity of $\Lambda(x)$, we have

$$\sup_{x \le -c_n} \Lambda(x) \le \Lambda(-c_n) = a_n b_n^{-1}. \tag{4.21}$$

Noting $\log(1-x) < -x$, $0 < x < 1$ and $e^{-x} > 1 - x$, $x \in R$, and combining (2.1) and (2.2) together, we have

$$\begin{aligned}
&\sup_{x \le -c_n} F^n(a_n x + b_n) \\
&\quad \le F^n(b_n - a_n c_n) \\
&\quad < \left(1 - n^{-1}\bigl(1 - a_n b_n^{-1} c_n\bigr)\bigl(1 - \bigl(a_n b_n^{-1}\bigr)^2\bigl(1 - a_n b_n^{-1} c_n\bigr)^{-4}\bigr) \exp\left(c_n - \frac{a_n b_n^{-1} c_n^2}{2}\right)\right)^n \\
&\quad < \exp\left(-e^{c_n}\bigl(1 - a_n b_n^{-1} c_n\bigr)\bigl(1 - \bigl(a_n b_n^{-1}\bigr)^2\bigl(1 - a_n b_n^{-1} c_n\bigr)^{-4}\bigr) \exp\left(-\frac{a_n b_n^{-1} c_n^2}{2}\right)\right) \\
&\quad < \exp\left(-e^{c_n}\left(1 - \left(a_n b_n^{-1} c_n + \bigl(a_n b_n^{-1}\bigr)^2\bigl(1 - a_n b_n^{-1} c_n\bigr)^{-3} + \frac{a_n b_n^{-1} c_n^2}{2}\right)\right)\right)
\end{aligned}$$

$$< \exp(-e^{c_n}) \exp\left(\left(a_n b_n^{-1} c_n + (a_n b_n^{-1})^2 (1 - a_n b_n^{-1} c_n)^{-3} + \frac{a_n b_n^{-1} c_n^2}{2}\right) e^{c_n}\right)$$
$$< C_{51} a_n b_n^{-1}.$$

Together with (4.21), we have

$$\begin{aligned}\sup_{-\infty<x\le -c_n} \left|F^n(a_n x + b_n) - \Lambda(x)\right| &\le \sup_{-\infty<x<-c_n} F^n(a_n x + b_n) + \sup_{-\infty<x<-c_n} \Lambda(x) \\ &\le F^n(b_n - a_n c_n) + \Lambda(-c_n) \\ &< (C_{12} + 1) a_n b_n^{-1} =: \mathbb{D}_4 a_n b_n^{-1}.\end{aligned}$$

This is (4.10). The proof is complete. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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References
1. Falk, A: Rate of uniform convergence of extreme order statistics. Ann. Inst. Stat. Math. **38**(2), 245-262 (1986)
2. Balkema, AA, de Haan, L: A convergence rate in extreme-value theory. J. Appl. Probab. **27**, 577-585 (1990)
3. de Haan, L, Resnick, SI: Second order regular variation and rates of convergence in extreme value theory. Ann. Probab. **24**, 97-124 (1996)
4. Cheng, S, Jiang, C: The Edgeworth expansion for distributions of extreme values. Sci. China Ser. A **4**, 427-437 (2001)
5. Hall, WJ, Wellner, JA: The rate of convergence in law of the maximum of an exponential sample. Stat. Neerl. **33**, 151-154 (1979)
6. Hall, P: On the rate of convergence of normal extremes. J. Appl. Probab. **16**, 433-439 (1979)
7. Peng, Z, Nadarajah, S, Lin, F: Convergence rate of extremes for the general error distribution. J. Appl. Probab. **47**, 668-679 (2010)
8. Lin, F, Peng, Z: Tail behavior and extremes of short-tailed symmetric distribution. Commun. Stat., Theory Methods **39**, 2811-2817 (2010)
9. Lin, F, Zhang, X, Peng, Z, Jiang, Y: On the rate of convergence of STSD extremes. Commun. Stat., Theory Methods **40**, 1795-1806 (2011)
10. Liu, B, Fu, Y: The pointwise rate of extremes for Maxwell distribution. J. Southwest Univ., Nat. Sci. **5**, 86-99 (2013)
11. Resnick, SI: Extreme Values, Regular Variation and Point Processes. Springer, New York (1987)
12. Leadbetter, MR, Lindgren, G, Rootzen, H: Extremes and Related Properties of Random Sequences and Processes. Springer, New York (1983)

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