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# On the stability of the heat equation with an initial condition

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**Abstract**

In this paper, we prove the generalized Hyers-Ulam stability of the heat equation with an initial condition

$$\begin{cases} \triangle u(x,t) = u_t(x,t) & \text{(for all } x \in \mathbb{R}^n \text{ and } t > 0), \\ u(x,0) = g(x) & \text{(for all } x \in \mathbb{R}^n) \end{cases}$$

in a class of twice continuously differentiable functions under certain conditions.

## 1 Introduction

Let $X$ be a normed space, and let $I$ be an open interval. If for any function $f : I \to X$ satisfying the differential inequality

$$\left\| a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_1(x)y'(x) + a_0(x)y(x) + h(x) \right\| \le \varepsilon$$

for all $x \in I$ and for some $\varepsilon \ge 0$ there exists a solution $f_0 : I \to X$ of the differential equation

$$a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_1(x)y'(x) + a_0(x)y(x) + h(x) = 0$$

such that $\|f(x) - f_0(x)\| \le K(\varepsilon)$ for any $x \in I$, where $K(\varepsilon)$ is an expression of $\varepsilon$ only, then we say that the above differential equation has the Hyers-Ulam stability.

If the above statement is also true when we replace $\varepsilon$ and $K(\varepsilon)$ by $\varphi(x)$ and $\Phi(x)$, where $\varphi, \Phi : I \to [0, \infty)$ are functions not depending on $f$ and $f_0$ explicitly, then we say that the corresponding differential equation has the generalized Hyers-Ulam stability. (This type of stability is sometimes called the Hyers-Ulam-Rassias stability.)

We may apply these terminologies to other differential equations and partial differential equations. For more detailed definitions of the Hyers-Ulam stability and the generalized Hyers-Ulam stability, refer to [1–8].

Obłoza seems to be the first author who has investigated the Hyers-Ulam stability of linear differential equations (see [9, 10]). Here, we introduce the result of Alsina and Ger (see [1]): If a differentiable function $f : I \to \mathbb{R}$ is a solution of the differential inequality $|y'(x) - y(x)| \le \varepsilon$, where $I$ is an open subinterval of $\mathbb{R}$, then there exists a solution $f_0 : I \to \mathbb{R}$ of the differential equation $y'(x) = y(x)$ such that $|f(x) - f_0(x)| \le 3\varepsilon$ for any $x \in I$. This result was generalized by Miura *et al.* (see [11, 12]).

In 2007, Jung and Lee [13] proved the Hyers-Ulam stability of the first-order linear partial differential equation

$$au_x(x,y) + bu_y(x,y) + cu(x,y) + d = 0,$$

where $a, b \in \mathbb{R}$ and $c, d \in \mathbb{C}$ are constants with $\Re(c) \neq 0$. It seems that the first paper dealing with the Hyers-Ulam stability of partial differential equations was written by Prastaro and Rassias [14]. For a recent result on this subject, refer to [15].

In this paper, using an idea from the papers [16, 17], we investigate the generalized Hyers-Ulam stability of the heat equation with an initial value condition

$$\begin{cases} \triangle u(x,t) = u_t(x,t) & \text{(for all } x \in \mathbb{R}^n \text{ and } t > 0\text{)}, \\ u(x,0) = g(x) & \text{(for all } x \in \mathbb{R}^n\text{)} \end{cases} \tag{1.1}$$

in the class of radially symmetric functions, where $\triangle$ denotes the Laplace operator. The heat equation plays an important role in a number of fields of science. It is strongly related to the Brownian motion in probability theory. The heat equation is also connected with chemical diffusion, and it is sometimes called the diffusion equation.

## 2 Main result

For a given integer $n \geq 2$, $x_i$ denotes the $i$th coordinate of any point $x$ in $\mathbb{R}^n$, *i.e.*, $x = (x_1, \ldots, x_i, \ldots, x_n)$. We assume that $t_1$ is a constant with $0 < t_1 \leq \infty$, and we define

$$T := \{t \in \mathbb{R} \mid 0 < t < t_1\} \quad \text{and} \quad |x| := \sqrt{x_1^2 + \cdots + x_n^2}.$$

Due to an idea from [18, Section 2.3.1], we may search for a solution of (1.1) of the form $u(x,t) = (1/t^{n/2})v(|x|/t^{1/2})$ for some function $v$. Based on this argument, we define

$$U := \left\{ u : \mathbb{R}^n \times T \to \mathbb{R} \mid u(x,t) = \frac{1}{t^{n/2}} w(r) \text{ for all } x \in \mathbb{R}^n, t \in T \right.$$

$$\text{and for some function } w : [0, \infty) \to \mathbb{R} \text{ with } r = \frac{|x|}{t^{1/2}}$$

$$\left. \text{and } \lim_{r \to \infty} r^n w(r) = \lim_{r \to \infty} r^{n-1} w'(r) = 0 \right\}.$$

**Theorem 2.1** *Let* $\varphi_1, \varphi_2 : [0, \infty) \to [0, \infty)$ *and* $\psi : T \to [0, \infty)$ *be functions such that*

$$D := \int_0^\infty \frac{e^{u^2/4}}{u^{n-1}} \int_u^\infty s^{n-1} \varphi_1(s) \, ds \, du < \infty, \tag{2.1}$$

$$c := \inf_{t \in T} t^{n/2+1} \psi(t) > 0. \tag{2.2}$$

*Assume that* $g : \mathbb{R}^n \to \mathbb{R}$ *is a function with* $g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. *If a twice continuously differentiable function* $u \in U$ *satisfies*

$$\begin{cases} |\triangle u(x,t) - u_t(x,t)| \leq \varphi_1(\frac{|x|}{t^{1/2}})\psi(t) & \text{(for all } x \in \mathbb{R}^n \text{ and } t \in T\text{)} \\ |u(x,0) - g(x)| \leq \varphi_2(|x|) & \text{(for all } x \in \mathbb{R}^n\text{)}, \end{cases} \tag{2.3}$$

*then there exist solutions $u_0, u_1 : \mathbb{R}^n \times T \to \mathbb{R}$ of the heat equation and a real number $\gamma$ such that*

$$\left| u(x,t) - \gamma u_0(x,t) \right| \leq \frac{cD}{t^{n/2}} e^{-|x|^2/4t}, \tag{2.4}$$

$$\left| \int_{\mathbb{R}^n} u(x-y,t)g(y)\,dy - \gamma u_1(x,t) \right| \leq (4\pi)^{n/2} cD \int_{\mathbb{R}^n} u_0(x-y,t)\left| g(y) \right| dy \tag{2.5}$$

*for all $x \in \mathbb{R}^n$ and $t \in T$ with $|x|/t^{1/2} > 0$, where*

$$u_0(x,t) := \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t} \quad and \quad u_1(x,t) := \int_{\mathbb{R}^n} u_0(x-y,t)g(y)\,dy. \tag{2.6}$$

*Proof* Since $u(x,t) \in U$, there exists a function $w : [0,\infty) \to \mathbb{R}$ such that

$$u(x,t) = \frac{1}{t^{n/2}} w(r)$$

for any $x \in \mathbb{R}^n$ and $t \in T$, where we set $r = |x|/t^{1/2}$. Using this notation, we calculate $u_t$ and $\triangle u$:

$$u_t(x,t) = -\frac{n}{2t^{n/2+1}} w(r) - \frac{1}{2t^{n/2+1}} rw'(r),$$

$$u_{x_i}(x,t) = \frac{1}{t^{n/2+1/2}} w'(r)\frac{x_i}{|x|},$$

$$u_{x_i x_i}(x,t) = \frac{1}{t^{n/2+1/2}} \left( \frac{1}{t^{1/2}} w''(r)\frac{x_i^2}{|x|^2} + w'(r)\left( \frac{1}{|x|} - \frac{x_i^2}{|x|^3} \right) \right).$$

So, we have

$$\triangle u(x,t) - u_t(x,t)$$

$$= \frac{1}{t^{n/2+1}} \left( w''(r) + \frac{n-1}{r} w'(r) + \frac{r}{2} w'(r) + \frac{n}{2} w(r) \right)$$

$$= \frac{1}{r^{n-1}} \frac{1}{t^{n/2+1}} \left( \left( r^{n-1} w''(r) + (n-1)r^{n-2} w'(r) \right) + \frac{1}{2} \left( r^n w'(r) + nr^{n-1} w(r) \right) \right)$$

$$= \frac{1}{r^{n-1}} \frac{1}{t^{n/2+1}} \left( r^{n-1} w'(r) + \frac{r^n}{2} w(r) \right)'$$

for all $x \in \mathbb{R}^n$, $t \in T$ with $r > 0$. Moreover, from the last equality and (2.3), it follows that

$$\left| \triangle u(x,t) - u_t(x,t) \right| = \frac{1}{r^{n-1}} \frac{1}{t^{n/2+1}} \left| \left( r^{n-1} w'(r) + \frac{r^n}{2} w(r) \right)' \right|$$

$$\leq \varphi_1(r)\psi(t)$$

or

$$\left| \left( r^{n-1} w'(r) + \frac{r^n}{2} w(r) \right)' \right| \leq r^{n-1} \varphi_1(r) t^{n/2+1} \psi(t)$$

for all $r > 0$ and $t \in T$. In view of (2.2), we have

$$-cr^{n-1}\varphi_1(r) \le \left(r^{n-1}w'(r) + \frac{r^n}{2}w(r)\right)' \le cr^{n-1}\varphi_1(r)$$

for any $r > 0$.

We integrate each term of the last inequality from $r$ to $\infty$ and take account of the definition of $U$ to get

$$-c\int_r^\infty s^{n-1}\varphi_1(s)\,ds \le -r^{n-1}w'(r) - \frac{r^n}{2}w(r) \le c\int_r^\infty s^{n-1}\varphi_1(s)\,ds$$

or

$$\left|w'(r) + \frac{r}{2}w(r)\right| \le \frac{c}{r^{n-1}}\int_r^\infty s^{n-1}\varphi_1(s)\,ds$$

for all $r > 0$.

According to [19, Theorem 1], together with (2.1), there exists a (unique) constant $\gamma \in \mathbb{R}$ such that

$$\left|w(r) - \frac{\gamma}{(4\pi)^{n/2}}e^{-r^2/4}\right| \le ce^{-r^2/4}\int_r^\infty \frac{e^{u^2/4}}{u^{n-1}}\int_u^\infty s^{n-1}\varphi_1(s)\,ds\,du$$

for all $r > 0$, or equivalently

$$\left|u(x,t) - \frac{\gamma}{(4\pi t)^{n/2}}e^{-|x|^2/4t}\right| \le \frac{c}{t^{n/2}}e^{-|x|^2/4t}\int_{|x|/t^{1/2}}^\infty \frac{e^{u^2/4}}{u^{n-1}}\int_u^\infty s^{n-1}\varphi_1(s)\,ds\,du$$

for all $x \in \mathbb{R}^n$ and $t \in T$ with $|x|/t^{1/2} > 0$, which proves the validity of inequality (2.4), and in view of (2.6), it is not difficult to prove that $u_0(x,t)$ is a solution of the heat equation, i.e., $\triangle u_0(x,t) - \frac{\partial}{\partial t}u_0(x,t) = 0$.

If we replace $x$ with $x - y$ and multiply each term by $|g(y)|$, and if we integrate all terms in the last inequality over $\mathbb{R}^n$, then we obtain

$$-(4\pi)^{n/2}cD\int_{\mathbb{R}^n} u_0(x-y,t)|g(y)|\,dy$$

$$\le \int_{\mathbb{R}^n} u(x-y,t)g(y)\,dy - \gamma\int_{\mathbb{R}^n} u_0(x-y,t)g(y)\,dy$$

$$\le (4\pi)^{n/2}cD\int_{\mathbb{R}^n} u_0(x-y,t)|g(y)|\,dy$$

for all $x, y \in \mathbb{R}^n$ and $t \in T$. If we define $u_1(x,t)$ as in (2.6) for any $x \in \mathbb{R}^n$ and $t \in T$, following the proof of [18, Theorem 1 in Section 2.3], we can then easily prove that

$$\triangle u_1(x,t) - \frac{\partial}{\partial t}u_1(x,t) = 0$$

for all $x \in \mathbb{R}^n$ and $t \in T$ with $|x|/t^{1/2} > 0$, which shows the validity of inequality (2.5).  □

**Remark 2.2**

   (i) The linearity of solutions of heat equation (1.1) implies that $\gamma u_0(x,t)$ and $\gamma u_1(x,t)$ are also solutions of the heat equation.

(ii) As in the proof of [18, Theorem 1 in Section 2.3], we can show that

$$\lim_{\substack{(x,t)\to(x_0,0)\\x\in\mathbb{R}^n,t>0}} u_1(x,t) = g(x_0)$$

for each $x_0 \in \mathbb{R}^n$.

(iii) If a function $\varphi_1 : [0,\infty) \to [0,\infty)$ satisfies

$$\int_u^\infty s^{n-1}\varphi_1(s)\,ds = O\big(u^n e^{-\alpha u^2}\big)$$

for all $u \geq 0$ and for some $\alpha > 1/4$, then there exists a positive number $A$ such that

$$\begin{aligned}
D &= \int_0^\infty \frac{e^{u^2/4}}{u^{n-1}} \int_u^\infty s^{n-1}\varphi_1(s)\,ds\,du\\
&\leq \int_0^\infty Au e^{(1/4-\alpha)u^2}\,du\\
&= \frac{2A}{4\alpha-1} < \infty,
\end{aligned}$$

*i.e.*, $\varphi_1$ satisfies condition (2.1).

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