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New convex functions in linear spaces and Jensen's discrete inequality

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Abstract

In the present paper we introduce a new class of s-convex functions defined on a convex subset of a real linear space, establish some inequalities of Jensen's type for this class of functions. Our results in special cases yield some of the recent results on classic convex functions.

Keywords: s-Orlicz convex functions; s-convex functions; Jensen's inequalities

1 Introduction

The research on convexity and generalized convexity is one of the important subjects in mathematical programming, numerous generalizations of convex functions have been proved useful for developing suitable optimization problems (see [1–3]). s-convex functions defined on a space of real numbers was introduced by Orlicz in [4] and was used in the theory of Orlicz spaces. s-Orlicz convex sets and s-Orlicz convex mappings in linear spaces were introduced by Dragomir in [5]. Some properties of inequalities of Jensen's type for this class of mappings were discussed.

Definition 1.1 [5] Let X be a linear space and $s \in (0, \infty)$. The set $K \subset X$ is called s-Orlicz convex in X if the following condition is true:

$$x, y \in X \text{ and } \alpha, \beta \geq 0 \text{ with } \alpha^s + \beta^s = 1 \quad \text{imply} \quad \alpha x + \beta y \in X.$$

Remark If $s = 1$, then, by the above definition, we recapture the concept of convex sets in linear spaces.

Definition 1.2 [5] Let X be a linear space and $s \in (0, \infty)$. Let $K \subset X$ be an s-Orlicz convex set. The mapping $f : K \to R$ is called s-Orlicz convex on K if for all $x, y \in K$ and $\alpha, \beta \geq 0$ with $\alpha^s + \beta^s = 1$, one has the inequality

$$f(\alpha x + \beta y) \leq \alpha^s f(x) + \beta^s f(y). \tag{1.1}$$

Note that for $s = 1$ we recapture the class of convex functions.

In this paper we introduce another class of s-convex functions defined on convex sets in a linear space. Some discrete inequalities of Jensen's type are also obtained.

2 The relations among *s*-convex functions, *s*-Orlicz convex functions and convex functions in linear spaces

Let X be a linear space and $s \in (0, \infty)$ be a fixed positive number, let $K \subset X$ be a convex subset. It is natural to consider the following class of functions.

Definition 2.1 The mapping $f : K \to R$ is called *s*-convex on K if

$$f(\alpha x + \beta y) \le \alpha^s f(x) + \beta^s f(y) \qquad (2.1)$$

for all $x, y \in K$ and $\alpha, \beta \ge 0$ with $\alpha + \beta = 1$.

Remark 2.2 If $s = 1$, then, by the above definition, we recapture the concept of convex functions in linear spaces.

Remark 2.3 *s*-convex functions defined on a convex set in linear spaces are different from convex functions.

(1) There exist *s*-convex mappings in linear spaces which are not convex for some $s \in (0, \infty)|\{1\}$ (see the following Example 1).

(2) When $0 < s \le 1$, every non-negative convex function defined on a convex set in a linear space is also an *s*-convex function. When $s \ge 1$, every non-positive convex function defined on a convex set in a linear space is also an *s*-convex function.

Example 1 Let X be a normed linear space, let $K = X$ and $0 < s \le 1$, define $f(x) = \|x\|^s$ for all $x \in K$. For every $x, y \in K$ and $\alpha, \beta \ge 0$ with $\alpha + \beta = 1$, when $\|\alpha x\| = 0$, either $\alpha = 0$ or $x = 0$, therefore, $\alpha x = 0$, and

$$f(\alpha x + \beta y) = f(\beta y) = \|\beta y\|^s = \beta^s \|y\|^s = \beta^s f(y);$$

when $\|\alpha x\| \ne 0$, from the monotonous increasing of the function $g(t) = 1 + t^s - (1 + t)^s$ in the interval $[0, \infty)$, we have $g(t) = 1 + t^s - (1 + t)^s \ge g(0) = 0$, $\forall t > 0$, then

$$f(\alpha x + \beta y) = \|\alpha x + \beta y\|^s \le \left(\alpha \|x\| + \beta \|y\| \right)^s = \alpha^s \|x\|^s \left[1 + \frac{\beta \|y\|}{\alpha \|x\|} \right]^s$$

$$\le \alpha^s \|x\|^s \left[1 + \left(\frac{\beta \|y\|}{\alpha \|x\|} \right)^s \right] = \alpha^s \|x\|^s + \beta^s \|y\|^s = \alpha^s f(x) + \beta^s f(y),$$

hence f is an *s*-convex function on X; however, f is not a convex function on X with $0 < s < 1$.

Remark 2.4 There exist *s*-Orlicz convex functions in linear spaces which are not *s*-convex functions for some $s \in (0, \infty)|\{1\}$.

Example 2 $0 < s \le 1$, $K = R^2$. Consider the set $K = \{(x, y) \in R^2 : |x|^s + |y|^s \le 1\} \subset R^2$. Let $X_1 = (x_1, y_1), X_2 = (x_2, y_2) \in K$, and let $\alpha, \beta \ge 0$ with $\alpha^s + \beta^s = 1$. Then

$$|x_1|^s + |y_1|^s \le 1, \qquad |x_2|^s + |y_2|^s \le 1,$$

which imply

$$|\alpha x_1 + \beta x_2|^s \le \left(\alpha|x_1| + \beta|x_2|\right)^s = \left\{\alpha|x_1|\left(1 + \frac{\beta|x_2|}{\alpha|x_1|}\right)\right\}^s = \left[\alpha|x_1|\right]^s\left(1 + \frac{\beta|x_2|}{\alpha|x_1|}\right)^s$$

$$\le \left[\alpha|x_1|\right]^s\left[1 + \left(\frac{\beta|x_2|}{\alpha|x_1|}\right)^s\right] = \left[\alpha|x_1|\right]^s + \left[\beta|x_2|\right]^s \le \alpha^s|x_1|^s + \beta^s|x_2|^s$$

and

$$|\alpha y_1 + \beta y_2|^s \le \alpha^s|y_1|^s + \beta^s|y_2|^s.$$

As

$$\alpha X_1 + \beta X_2 = (\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2) \quad \text{and}$$

$$|\alpha x_1 + \beta x_2|^s + |\alpha y_1 + \beta y_2|^s \le \alpha^s|x_1|^s + \beta^s|x_2|^s + \alpha^s|y_1|^s + \beta^s|y_2|^s$$

$$= \alpha^s\left(|x_1|^s + |y_1|^s\right) + \beta^s\left(|x_2|^s + |y_2|^s\right) = \alpha^s + \beta^s = 1,$$

we deduce that $\alpha X_1 + \beta X_2 \in K$, *i.e.*, K is s-Orlicz convex in R^2.

Define $f(u) = (x^2 + y^2)^{\frac{s}{2}}$, $u = (x, y) \in K$, we will show that f is an s-Orlicz convex function, f is not an s-convex function, for K is not a convex subset when $0 < s < 1$. Since $u_0 = (1, 0) \in K$, $v_0 = (0, 1) \in K$, but $\frac{1}{2}u_0 + \frac{1}{2}v_0 = (\frac{1}{2}, \frac{1}{2}) \notin K$ for $(\frac{1}{2})^s + (\frac{1}{2})^s = \frac{2}{2^s} > 1$ with $0 < s < 1$.

3 Properties of *s*-convex functions in linear spaces

We consider the following properties of s-convex functions in linear spaces.

Theorem 3.1 *Let $K \subset X$ be a convex set, $f_i : K \to R$ be s-convex functions, $i = 1, 2, \ldots, n$. Then*

(1) $h(x) = \max_{1 \le i \le n} f_i(x)$ *is an s-convex function on K.*

(2) $af_1(x) + bf_2(x)$ *is an s-convex function on K for all $a, b \ge 0$.*

The proof is omitted.

Proposition 3.2 *Let $f : K \to R$ be an s-convex function on K and $\gamma \in R$ so that $K_\gamma(f) = \{x \in K : f(x) \le \gamma\}$ is nonempty. Then $K_\gamma(f)$ is a convex subset of K when either one of the following two conditions is satisfied*:

(1) $\gamma \ge 0$ *and $s \ge 1$,*

(2) $\gamma \le 0$ *and $1 > s > 0$.*

Proof Let $x_1, x_2 \in K_\gamma(f)$ and $\alpha, \beta \ge 0$ so that $\alpha + \beta = 1$. Then $f(x_1) \le \gamma$ and $f(x_2) \le \gamma$ which imply that

$$f(\alpha x_1 + \beta x_2) \le \alpha^s f(x_1) + \beta^s f(x_2) \le \alpha^s \gamma + \beta^s \gamma = \left(\alpha^s + \beta^s\right)\gamma.$$

Then $f(\alpha x_1 + \beta x_2) \le (\alpha^s + \beta^s)\gamma \le \gamma$ when either one of (1) and (2) is satisfied, which shows that $K_\gamma(f)$ is a convex subset of K.   □

Theorem 3.3 *Let X be a linear space, let $K \subset X$ be a convex set, mapping $f : K \to R$, the following statements are equivalent*:

(1) f *is an s-convex function on* K.

(2) *For every $x_1, x_2, \ldots, x_n \in K$ and non-negative real number $\alpha_1, \alpha_2, \ldots, \alpha_n$ with $\sum_{i=1}^{n} \alpha_i = 1$, we have that*

$$f\left(\sum_{i=1}^{n} \alpha_i x_i\right) \leq \sum_{i=1}^{n} \alpha_i^s f(x_i). \tag{3.1}$$

Proof $(2) \Rightarrow (1)$. This is obvious.

$(1) \Rightarrow (2)$. We will prove by induction over $n \in N$, $n \geq 2$. For $n = 2$, the inequality is obvious by Definition 2.1. Suppose that the above inequality is valid for all $n - 1$. For natural number n, let $x_1, x_2, \ldots, x_n \in K$ and $\alpha_1, \alpha_2, \ldots, \alpha_n \geq 0$ with $\sum_{i=1}^{n} \alpha_i = 1$.

If there is some $\alpha_i = 0$, then delete the number α_i and for the remaining $n - 1$ number, inequality (3.1) is obvious by using the inductive hypothesis.

Now suppose $\alpha_i \neq 0$, $\forall i = 1, \ldots, n$, let $\beta_1 = \sum_{i=1}^{n-1} \alpha_i$, $\beta_2 = \alpha_n$, $y_1 = \frac{1}{\beta_1} \sum_{i=1}^{n-1} \alpha_i x_i$, $y_2 = x_n$, since $\sum_{i=1}^{n-1} \frac{\alpha_i}{\beta_1} = 1$. Using the inductive hypothesis, we have

$$f(y_1) = f\left(\sum_{i=1}^{n-1} \frac{\alpha_i}{\beta_1} x_i\right) \leq \sum_{i=1}^{n-1} \left(\frac{\alpha_i}{\beta_1}\right)^s f(x_i) = \frac{1}{\beta_1^s} \sum_{i=1}^{n-1} \alpha_i^s f(x_i).$$

Then

$$f\left(\sum_{i=1}^{n} \alpha_i x_i\right) = f(\beta_1 y_1 + \beta_2 y_2) \leq \beta_1^s f(y_1) + \beta_2^s f(x_i)$$

$$\leq \sum_{i=1}^{n-1} \alpha_i^s f(x_i) + \alpha_n^s f(x_n) = \sum_{i=1}^{n} \alpha_i^s f(x_i),$$

and the theorem is proved. $\qquad\qquad\square$

The following corollaries are different formulations of the above inequalities of Jensen's type.

Corollary 3.4 *Let $f : K \to R$ be an s-convex function, $\alpha_1, \alpha_2, \ldots, \alpha_n$ be non-negative real numbers, $P_n = \sum_{i=1}^{n} \alpha_i > 0$. For every $x_1, x_2, \ldots, x_n \in K$, we have that*

$$f\left(\frac{1}{P_n} \sum_{i=1}^{n} \alpha_i x_i\right) \leq \frac{1}{P_n^s} \sum_{i=1}^{n} \alpha_i^s f(x_i).$$

Corollary 3.5 *Let $f : K \to R$ be an s-convex function, $x_1, x_2, \ldots, x_n \in K$, then we have*

$$f\left(\frac{1}{n} \sum_{i=1}^{n} x_i\right) \leq \frac{1}{n^s} \sum_{i=1}^{n} f(x_i).$$

Corollary 3.6 *Let $f : K \to R$ be an s-convex function, $x_1, x_2, \ldots, x_n \in K$. For every non-negative real number $\alpha_1, \alpha_2, \ldots, \alpha_n$, when $Q_n = \sum_{i=1}^{n} \alpha_i^{\frac{1}{s}} > 0$, then we have*

$$f\left(\frac{1}{Q_n} \sum_{i=1}^{n} \alpha_i^{\frac{1}{s}} x_i\right) \leq \frac{1}{Q_n^s} \sum_{i=1}^{n} \alpha_i f(x_i).$$

We consider the following function:

$$\theta(I, P, f, x) := \sum_{i \in I} \alpha_i^s f(x_i) - \left(\sum_{i \in I} \alpha_i\right)^s f\left(\frac{1}{\sum_{i \in I} \alpha_i} \sum_{i \in I} \alpha_i x_i\right),$$

where $\alpha_i > 0$, $i \in I$, f is an s-convex function on the convex set K, $x_i \in K$, $i \in I$, $I \in P(N)$, where $P(N)$ denotes the finite subsets of the natural number set N.

Theorem 3.7 *Let $f : K \to R$ be an s-convex function, $\alpha_i > 0$, $x_i \in K$, $i \in N$, then*

(1) $\forall I, J \in P(N)$, $I \cap J = \Phi$, *we have that*

$$\theta(I \cup J, P, f, x) \geq \theta(I, P, f, x) + \theta(J, P, f, x) \geq 0; \tag{3.2}$$

(2) $\forall I, J \in P(N)$, $\Phi \neq J \subset I$, *we have*

$$\theta(I, P, f, x) \geq \theta(J, P, f, x) \geq 0, \tag{3.3}$$

that is, the mapping $\theta(I, P, f, x)$ is monotonic non-decreasing in the first variable on $P(N)$.

Proof (1) Let $\forall I, J \in P(N)$, $I \cap J = \Phi$. Then we have

$$
\begin{aligned}
\theta(I \cup J, P, f, x) &= \sum_{i \in I} \alpha_i^s f(x_i) + \sum_{i \in J} \alpha_i^s f(x_i) - \left(\sum_{i \in I \cup J} \alpha_i\right)^s f\left(\frac{\sum_{i \in I} \alpha_i x_i + \sum_{i \in J} \alpha_i x_i}{\sum_{i \in I \cup J} \alpha_i}\right) \\
&= \sum_{i \in I} \alpha_i^s f(x_i) + \sum_{i \in J} \alpha_i^s f(x_i) \\
&\quad - \left(\sum_{i \in I \cup J} \alpha_i\right)^s f\left(\frac{\sum_{i \in I} \alpha_i}{\sum_{i \in I \cup J} \alpha_i} \frac{\sum_{i \in I} \alpha_i x_i}{\sum_{i \in I} \alpha_i} + \frac{\sum_{i \in J} \alpha_i}{\sum_{i \in I \cup J} \alpha_i} \frac{\sum_{i \in J} \alpha_i x_i}{\sum_{i \in J} \alpha_i}\right) \\
&\geq \sum_{i \in I} \alpha_i^s f(x_i) - \left(\sum_{i \in I} \alpha_i\right)^s f\left(\frac{\sum_{i \in I} \alpha_i x_i}{\sum_{i \in I} \alpha_i}\right) \\
&\quad + \sum_{i \in J} \alpha_i^s f(x_i) - \left(\sum_{i \in J} \alpha_i\right)^s f\left(\frac{\sum_{i \in J} \alpha_i x_i}{\sum_{i \in J} \alpha_i}\right) \\
&= \theta(I, P, f, x) + \theta(J, P, f, x)
\end{aligned}
$$

and inequality (3.2) is proved.

(2) Suppose that $I, J \in P(N)$, with $\Phi \neq J \subset I$ and $J \neq I$.

With the above assumptions, consider the sequence

$$\theta(I, P, f, x) = \theta\left(J \cup (I \setminus J), P, f, x\right) \geq \theta(J, P, f, x) + \theta(I \setminus J, P, f, x)$$

whence we get

$$\theta(I, P, f, x) - \theta(J, P, f, x) \geq \theta(I|J, P, f, x) \geq 0$$

and inequality (3.3) is proved. □

With the above assumptions, consider the sequence

$$\theta_n := \sum_{i=1}^{n} \alpha_i^s f(x_i) - \left(\sum_{i=1}^{n} \alpha_i\right)^s f\left(\frac{1}{\sum_{i=1}^{n} \alpha_i} \sum_{i=1}^{n} \alpha_i x_i\right), \quad n > 2.$$

Corollary 3.8 *With the above assumptions,*

$$\theta_n \geq \theta_{n-1} \geq \cdots \geq \theta_2 \geq 0$$

i.e., the sequence θ_n is non-decreasing and one has the inequality

$$\theta_n \geq \max_{1 \leq i \leq j \leq n} \left[\alpha_i^s f(x_i) + \alpha_j^s f(x_j) - (\alpha_i + \alpha_j)^s f\left(\frac{\alpha_i x_i + \alpha_j x_j}{\alpha_i + \alpha_j}\right)\right] \geq 0.$$

Theorem 3.9 *Let $f : K \subset X \to R$ be an s-convex function, and $\alpha_i \geq 0$ so that $\sum_{i=1}^{n} \alpha_i = 1$. Let $x_{ij} \in X$, $0 \leq i, j \leq n$. Then one has the inequalities*

$$\sum_{i=1}^{n} \alpha_i^s \alpha_j^s f(x_{ij}) \geq \max\{A, B\} \geq \min\{A, B\} \geq f\left[\sum_{i=1}^{n} \alpha_i \alpha_j x_{ij}\right], \tag{3.4}$$

where $A := \sum_{i=1}^{n} \alpha_i^s f(\sum_{j=1}^{n} \alpha_j x_{ij})$, $B := \sum_{j=1}^{n} \alpha_j^s f(\sum_{i=1}^{n} \alpha_i x_{ij})$.

Theorem 3.10 *Let $f : K \subset X \to R$ be an s-convex function, and $\alpha_i \geq 0$ so that $\sum_{i=1}^{n} \alpha_i = 1$. Then, for every $\alpha, \beta \geq 0$, with $\alpha + \beta = 1$, we have*

(1) $\left(\sum_{j=1}^{n} \alpha_j^s\right)(\alpha^s + \beta^s) \sum_{i=1}^{n} \alpha_i^s f(x_i) \geq \sum_{i,j=1}^{n} \alpha_i^s \alpha_j^s f(\alpha x_i + \beta x_j),$

(2) $\sum_{i,j=1}^{n} \alpha_i^s \alpha_j^s f(\alpha x_i + \beta x_j) \geq 2^{s-1} \sum_{i,j=1}^{n} \alpha_i^s \alpha_j^s f\left(\frac{x_i + x_j}{2}\right),$

(3) $\sum_{i,j=1}^{n} \alpha_i^s \alpha_j^s f\left(\frac{x_i + x_j}{2}\right) \geq f\left(\sum_{i=1}^{n} \alpha_i x_i\right),$

(4) $(\alpha^s + \beta^s) \sum_{i=1}^{n} \alpha_i f(x_i) \geq \sum_{i,j=1}^{n} \alpha_i \alpha_j f(\alpha x_i + \beta x_j).$

Proof By the *s*-convexity of f, we can state

$$f(\alpha x_i + \beta x_j) \leq \alpha^s f(x_i) + \beta^s f(x_j)$$

$$\Rightarrow \quad \sum_{i,j=1}^{n} \alpha_i^s \alpha_j^s f(\alpha x_i + \beta x_j) \leq \alpha^s \sum_{i,j=1}^{n} \alpha_i^s \alpha_j^s f(x_i) + \beta^s \sum_{i,j=1}^{n} \alpha_i^s \alpha_j^s f(x_j)$$

$$= \alpha^s \sum_{j=1}^{n} \alpha_j^s \sum_{i=1}^{n} \alpha_i^s f(x_i) + \beta^s \sum_{i=1}^{n} \alpha_i^s \sum_{j=1}^{n} \alpha_j^s f(x_j)$$

$$= \left(\sum_{j=1}^{n} \alpha_j^s \right) (\alpha^s + \beta^s) \sum_{i=1}^{n} \alpha_i^s f(x_i).$$

We get the first inequality (1) in Theorem 3.10.

By the following inequality

$$f\left(\frac{x_i + x_j}{2} \right) = f\left(\frac{\alpha x_i + \beta x_j + \alpha x_j + \beta x_i}{2} \right) \leq \frac{1}{2^s} \left[f(\alpha x_i + \beta x_j) + f(\alpha x_j + \beta x_i) \right],$$

we have

$$\sum_{i,j=1}^{n} \alpha_i^s \alpha_j^s f\left(\frac{x_i + x_j}{2} \right) \leq \frac{1}{2^s} \left[\sum_{i,j=1}^{n} \alpha_i^s \alpha_j^s f(\alpha x_i + \beta x_j) + \sum_{i,j=1}^{n} \alpha_i^s \alpha_j^s f(\alpha x_j + \beta x_i) \right]$$

$$= \frac{1}{2^{s-1}} \left[\sum_{i,j=1}^{n} \alpha_i^s \alpha_j^s f(\alpha x_i + \beta x_j) \right].$$

We get the second inequality (2) in Theorem 3.10.

Let $x_{ij} = \frac{x_i + x_j}{2}$, by the inequality in Theorem 3.9, we have

$$\sum_{i,j=1}^{n} \alpha_i^s \alpha_j^s f\left(\frac{x_i + x_j}{2} \right) \geq f\left(\sum_{i,j=1}^{n} \alpha_i \alpha_j \frac{x_i + x_j}{2} \right) = f\left(\sum_{i=1}^{n} \alpha_i x_i \right),$$

i.e., inequality (3) is obtained.

$$\sum_{i,j=1}^{n} \alpha_i \alpha_j f(\alpha x_i + \beta x_j) \leq \sum_{i,j=1}^{n} \alpha_i \alpha_j \alpha^s f(x_i) + \sum_{i,j=1}^{n} \alpha_i \alpha_j \beta^s f(x_j)$$

$$= (\alpha^s + \beta^s) \sum_{i=1}^{n} \alpha_i f(x_i)$$

i.e., inequality (4) is proved. □

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