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Proof of an open inequality with double power-exponential functions

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Abstract
Cîrtoaje (J. Nonlinear Sci. Appl. 4(2):130-137, 2011) conjectured that the inequality $a^{(2b)^x} + b^{(2a)^x} \le 1$ with double power-exponential functions holds for all nonnegative real numbers a, b with $a + b = 1$ and all $x \ge 1$. In this paper, we shall prove the conjecture affirmatively.

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1 Introduction

The study of inequalities with power-exponential functions is one of the active areas of research in the mathematical analysis. Cîrtoaje [1, 2] studied inequalities with power-exponential functions and conjectured some open inequalities. He posed the open inequality as Conjecture 4.8 in [1],

$$a^{2b} + b^{2a} \le 1,$$

which holds for all nonnegative real numbers a, b with $a + b = 1$. He proved in [2] that this inequality holds. Moreover, he conjectured the more generalized inequality containing double power-exponential functions in [2]:

$$a^{(2b)^x} + b^{(2a)^x} \le 1 \tag{1.1}$$

holds for all nonnegative real numbers a, b with $a + b = 1$ and all $x \ge 1$, which is Conjecture 5.1 in [2] and still an open problem. Cîrtoaje's open inequality (1.1) is an interesting and new problem of great importance in the power exponential inequality theory. In this paper, we shall prove the conjecture affirmatively. The following is our main theorem.

Theorem 1.1 *For all nonnegative real numbers a, b with $a + b = 1$ and all $x \ge 1$, inequality* (1.1) *holds.*

We shall show this theorem by using differentiation mainly.

Let

$$b = 1 - a$$

and

$$F(x,a) = a^{(2b)^x} + b^{(2a)^x}.$$

Since $F(x,0) = F(x,\frac{1}{2}) = 1$ and $F(x,a) = F(x,1-a)$, it suffices to show that $F(x,a) \leq 1$ for $0 < a < \frac{1}{2}$ and $x \geq 1$. To prove Theorem 1.1, we shall investigate the sign of

$$F_x(x,a) = \frac{\partial}{\partial x}F(x,a) = b^{(2a)^x}(2a)^x \ln(2a)\ln b + a^{(2b)^x}(2b)^x \ln(2b)\ln a.$$

We set

$$G(x,a) = \ln\bigl[b^{(2a)^x}(2a)^x \ln(2a)\ln b\bigr] - \ln\bigl[a^{(2b)^x}(2b)^x \ln(2b)(-\ln a)\bigr].$$

Then $F_x(x,a)$ clearly has the same sign as $G(x,a)$.

Since $G(x,a)$ has the both signs, in order to get the sign of it, we need to investigate the signs of $G(1,a)$ and $G_x(1,a)$. We shall describe the results in Sections 2.1 and 2.2. Moreover, for fixed $a \in (0,\frac{1}{2})$, we show in Section 2.3 that $G_x = \partial G/\partial x$ is strictly increasing for $x \geq 1$; that is, $G(x,a)$ is convex on $[1,\infty)$, which is the main idea of our proof.

In Section 3, we consider the cases of $G_x(1,a) \geq 0$ and $G_x(1,a) < 0$ to prove Theorem 1.1. Using three propositions given in Section 2, we notice the following (1) and (2).

(1) From Proposition 2.8, for fixed a, if $G_x(1,a) \geq 0$, then $G_x(x,a) > 0$ for $x > 1$ and if $G_x(1,a) < 0$, then there exists uniquely a number $\tilde{x} > 1$ such that $G_x(\tilde{x},a) = 0$.

(2) From Propositions 2.5 and 2.6, we notice that $G(1,a) < 0$ when $G_x(1,a) < 0$.

(1) and (2) play an important role in the proof of Theorem 1.1.

We shall use the functions $F(x,a)$ and $G(x,a)$ defined here throughout this paper.

2 Preliminaries

2.1 The sign of $G(1,a)$

From the definition of $G(x,a)$, we have

$$G(x,a) = (2a)^x \ln b + x\ln(2a) - (2b)^x \ln a - x\ln(2b) + R(a),$$

where

$$\begin{aligned} R(a) &= \ln\bigl[\ln(2a)\ln b\bigr] - \ln\bigl[\ln(2b)(-\ln a)\bigr] \\ &= \ln\left(1 + \frac{\ln 2}{\ln a}\right) - \ln\left(-1 - \frac{\ln 2}{\ln b}\right). \end{aligned}$$

Then we have

$$G(1,a) = 2a\ln b + \ln(2a) - 2b\ln a - \ln(2b) + R(a), \quad b = 1-a.$$

In this subsection, we shall show that $G(1,a) < 0$ for $\frac{15}{100} \leq a < \frac{1}{2}$.

Consider first the case $\frac{15}{100} \leq a \leq \frac{1}{4}$. We have

$$R'(a) = -\ln 2\left[\frac{1}{P(a)} + \frac{1}{P(b)}\right]$$

and

$$R''(a) = \ln 2\left[\frac{Q(a)}{P^2(a)} - \frac{Q(b)}{P^2(b)}\right],$$

where

$$P(x) = x\ln(2x)\ln x$$

and

$$Q(x) = (\ln x)^2 + (2 + \ln 2)\ln x + \ln 2.$$

Lemma 2.1 *If* $\frac{15}{100} \leq a \leq \frac{1}{4}$, *then*

$$G(1, a) < 0.$$

Proof First, we show that

$$R''(a) < 0$$

for $\frac{15}{100} \leq a \leq \frac{1}{4}$. Let λ_1 and λ_2 ($\lambda_1 < \lambda_2$) be the solutions of

$$x^2 + (2 + \ln 2)x + \ln 2 = 0,$$

then we have

$$\lambda_1 = \frac{-(2 + \ln 2) - \sqrt{(\ln 2)^2 + 4}}{2}, \qquad \lambda_2 = \frac{-(2 + \ln 2) + \sqrt{(\ln 2)^2 + 4}}{2}$$

and

$$e^{\lambda_1} \cong 0.0903, \qquad e^{\lambda_2} \cong 0.7495.$$

Since $Q(a) \leq 0$ for $e^{\lambda_1} \leq a \leq e^{\lambda_2}$, we have $Q(a) < 0$ for $\frac{15}{100} \leq a \leq \frac{1}{4}$. Since $Q(b) \geq 0$ for $b \geq e^{\lambda_2}$, that is, for $a \leq 1 - e^{\lambda_2} \cong 0.2505$, we have $Q(b) > 0$ for $\frac{15}{100} \leq a \leq \frac{1}{4}$. Therefore, from $Q(a) < 0$ and $Q(b) > 0$, we get $R''(a) < 0$ for $\frac{15}{100} \leq a \leq \frac{1}{4}$.

Next, we show that

$$R(a) < 5\left(a - \frac{1}{4}\right) - 1$$

for $\frac{15}{100} \leq a \leq \frac{1}{4}$. If we set

$$f(a) = R(a) - 5\left(a - \frac{1}{4}\right) + 1,$$

then

$$f'(a) = R'(a) - 5$$

and

$$f''(a) = R''(a).$$

Since $R''(a) < 0$ for $\frac{15}{100} \le a \le \frac{1}{4}$, f' is strictly decreasing on the interval $[\frac{15}{100}, \frac{1}{4}]$ and we have $f'(a) \ge f'(\frac{1}{4}) = R'(\frac{1}{4}) - 5(\cong 0.0377) > 0$. Therefore, f is strictly increasing on the interval $[\frac{15}{100}, \frac{1}{4}]$, so we have $f(a) \le f(\frac{1}{4}) = R(\frac{1}{4}) + 1(\cong -0.0363) < 0$. Thus, we get $R(a) < 5(a - \frac{1}{4}) - 1$ for $\frac{15}{100} \le a \le \frac{1}{4}$.

In order to complete the proof of this lemma, it suffices to show from the above inequality with respect to $R(a)$ that $g(a) < 0$ for $\frac{15}{100} \le a \le \frac{1}{4}$, where

$$g(a) = 2a \ln b + \ln(2a) - 2b \ln a - \ln(2b) + 5\left(a - \frac{1}{4}\right) - 1, \quad b = 1 - a.$$

We have

$$g'(a) = 2 \ln b - \frac{2a}{b} + \frac{1}{a} + 2 \ln a - \frac{2b}{a} + \frac{1}{b} + 5$$

and

$$g''(a) = \frac{2(1-2a)}{ab} + \frac{1-2a}{a^2b^2}.$$

Since $g''(a) > 0$, g' is strictly increasing on the interval $[\frac{15}{100}, \frac{1}{4}]$. Since $g'(\frac{15}{100})(\cong -2.9624) < 0$ and $g'(\frac{1}{4})(\cong 0.3187) > 0$, there exists uniquely a number $c \in (\frac{15}{100}, \frac{1}{4})$ such that $g'(c) = 0$. Then we have $g'(a) < 0$ for $\frac{15}{100} < a < c$ and $g'(a) > 0$ for $c < a < \frac{1}{4}$. Hence, g is strictly decreasing on the interval $[\frac{15}{100}, c]$ and strictly increasing on the interval $[c, \frac{1}{4}]$. Therefore, $g(a) \le \max\{g(\frac{15}{100}), g(\frac{1}{4})\}$. Since $g(\frac{15}{100}) \cong -0.0582$ and $g(\frac{1}{4}) \cong -0.1630$, we can get $g(a) < 0$ for $\frac{15}{100} \le a \le \frac{1}{4}$. □

It still remains to show that $G(1, a) < 0$ for $\frac{1}{4} \le a < \frac{1}{2}$. Since

$$G(1, a) = 2a \ln b + \ln(2a) - 2b \ln a - \ln(2b) + \ln\left(\frac{-\ln(2a)}{\ln(2b)}\right) + \ln\left(\frac{\ln b}{\ln a}\right), \quad b = 1 - a,$$

using the substitution

$$a = \frac{1-t}{2},$$

we need to prove that $A(t) < 0$ for $0 < t \le \frac{1}{2}$, where

$$A(t) = (1-t) \ln\left(\frac{1+t}{2}\right) + \ln(1-t) - (1+t) \ln \frac{1-t}{2} - \ln(1+t) + \ln S_1(t) + \ln S_2(t),$$

$$S_1(t) = \frac{-\ln(1-t)}{\ln(1+t)}, \qquad S_2(t) = \frac{\ln 2 - \ln(1+t)}{\ln 2 - \ln(1-t)}.$$

Lemma 2.2 *If* $0 < t \le \frac{1}{2}$, *then*

$$S_1(t) < 1 + t + t^2.$$

Proof We need to prove that $f(t) > 0$ for $0 < t \leq \frac{1}{2}$, where

$$f(t) = \left(1 + t + t^2\right) \ln(1 + t) + \ln(1 - t).$$

We have

$$f'(t) = (1 + 2t) \ln(1 + t) + t + \frac{1}{1 + t} - \frac{1}{1 - t},$$
$$f''(t) = 2 \ln(1 + t) - \frac{1}{1 + t} - \frac{1}{(1 + t)^2} - \frac{1}{(1 - t)^2} + 3$$

and

$$f'''(t) = \frac{2}{1 + t} + \frac{1}{(1 + t)^2} + \frac{2}{(1 + t)^3} - \frac{2}{(1 - t)^3}.$$

If we set $g(t) = f'''(t) \times (1 + t)^3(1 - t)^3$, then we have

$$g(t) = -2t^5 + t^4 + 2t^3 - 4t^2 - 16t + 3,$$
$$g'(t) = -10t^4 + 4t^3 + 6t^2 - 8t - 16.$$

From

$$g'(t) < 4t^3 + 6t^2 - 16 < \frac{1}{2} + \frac{3}{2} - 16 < 0,$$

it follows that g is strictly decreasing on $(0, \frac{1}{2})$. Since $g(0) = 3$ and $g(\frac{1}{2}) = -\frac{23}{4}$, there exists uniquely a number $c_1 \in (0, \frac{1}{2})$ such that $g(c_1) = 0$. Since $g(t) > 0$ for $0 < t < c_1$ and $g(t) < 0$ for $c_1 < t < \frac{1}{2}$, we have $f'''(t) > 0$ for $0 < t < c_1$ and $f'''(t) < 0$ for $c_1 < t < \frac{1}{2}$. It follows that f'' is strictly increasing on the interval $(0, c_1)$ and strictly decreasing on the interval $(c_1, \frac{1}{2})$. Since $f''(0) = 0$ and $f''(\frac{1}{2}) = 2 \ln \frac{3}{2} - \frac{19}{9} (\cong -1.3001) < 0$, there exists uniquely a number $c_2 \in (0, \frac{1}{2})$ such that $f''(c_2) = 0$. Since $f''(t) > 0$ for $0 < t < c_2$ and $f''(t) < 0$ for $c_2 < t < \frac{1}{2}$, f' is strictly increasing on the interval $(0, c_2)$ and strictly decreasing on the interval $(c_2, \frac{1}{2})$. Since $f'(0) = 0$ and $f'(\frac{1}{2}) = 2 \ln \frac{3}{2} - \frac{5}{6} (\cong -0.0224) < 0$, there exists uniquely a number $c_3 \in (0, \frac{1}{2})$ such that $f'(c_3) = 0$. Hence, $f'(t) > 0$ for $0 < t < c_3$ and $f'(t) < 0$ for $c_3 < t < \frac{1}{2}$. Thus, f is strictly increasing on the interval $(0, c_3)$ and strictly decreasing on the interval $(c_3, \frac{1}{2})$. Since $f(0) = 0$ and $f(\frac{1}{2}) = \frac{7}{4} \ln \frac{3}{2} - \ln 2 \cong 0.0164$, we can get $f(t) > 0$ for $0 < t \leq \frac{1}{2}$. □

Lemma 2.3 *If* $0 < t \leq \frac{1}{2}$, *then*

$$\ln S_2(t) < \frac{-2}{\ln 2} t.$$

Proof We need to show that $f(t) < 0$, where

$$f(t) = \ln S_2(t) + \frac{2}{\ln 2} t.$$

We have

$$f'(t) = \left(\ln S_2(t)\right)' + \frac{2}{\ln 2}$$

and

$$f''(t) = \bigl(\ln S_2(t)\bigr)'',$$

where

$$\ln S_2(t) = \ln\bigl[\ln 2 - \ln(1+t)\bigr] - \ln\bigl[\ln 2 - \ln(1-t)\bigr],$$
$$\bigl(\ln S_2(t)\bigr)' = -\frac{1}{[\ln 2 - \ln(1+t)](1+t)} - \frac{1}{[\ln 2 - \ln(1-t)](1-t)},$$

and

$$\bigl(\ln S_2(t)\bigr)'' = \frac{-1 + \ln 2 - \ln(1+t)}{[\ln 2 - \ln(1+t)]^2(1+t)^2} + \frac{1 - \ln 2 + \ln(1-t)}{[\ln 2 - \ln(1-t)]^2(1-t)^2}.$$

We see that $(\ln S_2(t))''$ has the same sign as

$$\begin{aligned} B(t) &= \bigl(\ln S_2(t)\bigr)'' \times \bigl[\ln 2 - \ln(1+t)\bigr]^2(1+t)^2\bigl[\ln 2 - \ln(1-t)\bigr]^2(1-t)^2 \\ &= (1-\ln 2)B_1(t) + B_2(t) + B_3(t), \end{aligned}$$

where

$$B_1(t) = \bigl[\ln 2 - \ln(1+t)\bigr]^2(1+t)^2 - \bigl[\ln 2 - \ln(1-t)\bigr]^2(1-t)^2,$$
$$B_2(t) = -\ln(1+t)\bigl[\ln 2 - \ln(1-t)\bigr]^2(1-t)^2$$

and

$$B_3(t) = \ln(1-t)\bigl[\ln 2 - \ln(1+t)\bigr]^2(1+t)^2.$$

We have

$$B_1(t) = f_1(t)f_2(t),$$

where

$$f_1(t) = \bigl[\ln 2 - \ln(1+t)\bigr](1+t) + \bigl[\ln 2 - \ln(1-t)\bigr](1-t)$$

and

$$f_2(t) = \bigl[\ln 2 - \ln(1+t)\bigr](1+t) - \bigl[\ln 2 - \ln(1-t)\bigr](1-t).$$

Since $\ln 2 > \ln(1+t)$ and $\ln 2 > \ln(1-t)$ for $0 < t \le \frac{1}{2}$, we have $f_1(t) > 0$. Since $f_2'(t) = 2\ln 2 - 2 - \ln(1-t^2) \le 4\ln 2 - 2 - \ln 3 \cong -0.3260$ for $0 < t \le \frac{1}{2}$, f_2 is strictly decreasing on the interval $(0, \frac{1}{2}]$. Therefore, we can get $f_2(t) < \lim_{t\to 0} f_2(t) = 0$ for $0 < t \le \frac{1}{2}$. From $f_1(t) > 0$ and $f_2(t) < 0$, it follows that $B_1(t) < 0$. Since $B_1(t) < 0$, $B_2(t) < 0$ and $B_3(t) < 0$, we get $B(t) < 0$, hence $(\ln S_2(t))'' < 0$ for $0 < t \le \frac{1}{2}$. Thus, f' is strictly decreasing on the interval $(0, \frac{1}{2}]$ and we have $f'(t) < \lim_{t\to 0} f'(t) = 0$ for $0 < t \le \frac{1}{2}$. Since f is strictly decreasing on the interval $(0, \frac{1}{2}]$, we have $f(t) < \lim_{t\to 0} f(t) = 0$ for $0 < t \le \frac{1}{2}$. Therefore, we get $\ln S_2(t) < -\frac{2}{\ln 2}t$. □

Lemma 2.4 *If* $\frac{1}{4} \le a \le \frac{1}{2}$, *then*

$$G(1,a) < 0.$$

Proof We need to show that $A(t) < 0$ for $0 < t \le \frac{1}{2}$. By Lemmas 2.2 and 2.3, it suffices to show that $f(t) < 0$, where

$$f(t) = (1-t)\ln\frac{1+t}{2} + \ln(1-t) - (1+t)\ln\frac{1-t}{2} - \ln(1+t) + \ln(1+t+t^2) - \frac{2}{\ln 2}t.$$

We have

$$f'(t) = -\ln(1+t) - \ln(1-t) + \frac{1}{1+t} + \frac{1}{1-t} + \frac{1+2t}{1+t+t^2} + 2\ln 2 - 2 - \frac{2}{\ln 2}$$

and

$$\begin{aligned} f''(t) &= \frac{2t}{(1+t)(1-t)} - \frac{2t(1+t)}{(1+t+t^2)^2} + \frac{4t}{(1+t)^2(1-t)^2} + \frac{1}{(1+t+t^2)^2} \\ &= \frac{2t^2(1+4t+3t^2+t^3)}{(1+t)(1-t)(1+t+t^2)^2} + \frac{4t}{(1+t)^2(1-t)^2} + \frac{1}{(1+t+t^2)^2}. \end{aligned}$$

Since $f''(t) > 0$ for $0 < t \le \frac{1}{2}$, f' is strictly increasing on the interval $(0, \frac{1}{2}]$. Since $\lim_{t\to 0} f'(t) = 1 + 2\ln 2 - \frac{2}{\ln 2} (\cong -0.4990) < 0$ and $f'(\frac{1}{2}) = 4\ln 2 + \frac{38}{21} - \ln 3 - \frac{2}{\ln 2} (\cong 0.5981) > 0$, there exists uniquely a number $c \in (0, \frac{1}{2})$ such that $f'(c) = 0$. Hence, f is strictly decreasing on the interval $(0, c)$ and strictly increasing on the interval $(c, \frac{1}{2}]$. Since $\lim_{t\to +0} f(t) = 0$ and $f(\frac{1}{2}) = -\frac{1}{2}\ln 3 + \ln 7 - \frac{1}{\ln 2} \cong -0.0460$, we get $f(t) < 0$. Therefore, $A(t) < f(t) < 0$ for $0 < t \le \frac{1}{2}$. □

From Lemmas 2.1 and 2.4, we get the following result.

Proposition 2.5 *If* $\frac{15}{100} \le a < \frac{1}{2}$, *then*

$$G(1,a) < 0.$$

We notice that $\lim_{a \to \frac{1}{2} - 0} G(1,a) = 0$.

2.2 The sign of $G_x(1,a)$

We have

$$G_x(x,a) = \frac{\partial}{\partial x} G(x,a) = (2a)^x \ln(2a)\ln b + \ln(2a) - (2b)^x \ln(2b)\ln a - \ln(2b),$$

hence

$$G_x(1,a) = (2a)(\ln 2 + \ln a)\ln b + \ln a - 2b(\ln 2 + \ln b)\ln a - \ln b,$$

where $b = 1 - a$.

Proposition 2.6 *There exists a number* $c \in (\frac{15}{100}, \frac{1}{2})$ *such that* $G_x(1,a) > 0$ *for* $0 < a < c$.

Proof Let us denote $G_x(1,a)$ by $f(a)$. We have

$$f'(a) = 2\ln(2a)\ln b + 2\ln b - \frac{2a}{b}\ln(2a) + \frac{1}{a} + 2\ln(2b)\ln a + 2\ln a - \frac{2b}{a}\ln(2b) + \frac{1}{b}$$

and

$$\begin{aligned} f''(a) = {} & \frac{2(b\ln b - a\ln a)}{ab} - \frac{2\ln(2a)}{b} - \frac{2\ln(2a)}{b^2} \\ & + \frac{2\ln(2b)}{a} + \frac{2\ln(2b)-1}{a^2} + \frac{4(1-2a)}{ab} + \frac{1}{b^2}. \end{aligned}$$

Since $0 < a < \frac{1}{2}$, we have $b\ln b - a\ln a > 0$, $\ln(2a) < 0$, $\ln(2b) > 0$ and $1-2a > 0$. Therefore, if $2\ln(2b) - 1 \geq 0$, then $f''(a) > 0$. The condition $2\ln(2b) - 1 \geq 0$ is true for $0 < a < a_0$, where

$$a_0 = 1 - \frac{1}{2}\sqrt{e} \cong 0.1756.$$

Consequently, f' is strictly increasing on $(0, a_0]$. Since $f'(a_0) \cong -2.5412$, it follows that $f'(a) < 0$ on $(0, a_0]$, and f is strictly decreasing on $(0, a_0]$. Since $\lim_{a\to+0} f(a) = \infty$ and $f(a_0) \cong -0.0413$, there exists uniquely a number $c \in (0, a_0)$ such that $f(c) = 0$. Then we have $f(a) > 0$ for $0 < a < c$ and $f(a) < 0$ for $c < a < a_0$. Since $\frac{15}{100} < a_0$ and $f(\frac{15}{100}) \cong 0.0354$, we can get $\frac{15}{100} < c$. □

2.3 The convexity of $G(x,a)$

In order to investigate the convexity of $G(x,a)$ with respect to x, we need the following lemma.

Lemma 2.7 *If* $0 < a < \frac{1}{2}$, *then*

$$(\ln 2 + \ln a)^2 \ln b > (\ln 2 + \ln b)^2 \ln a,$$

where $b = 1 - a$.

Proof We first show that the inequality

$$(\ln 2)^2 > \ln a \ln b$$

holds for $0 < a < \frac{1}{2}$. We denote

$$f(a) = \ln a \ln b.$$

Then we have

$$f'(a) = \frac{b\ln b - a\ln a}{ab}.$$

We set

$$g(a) = b\ln b - a\ln a,$$

then we have

$$g'(a) = -\ln b - \ln a - 2$$

and

$$g''(a) = \frac{2a-1}{ab} < 0.$$

Therefore, g' is strictly decreasing on the interval $(0, \frac{1}{2})$. Since $\lim_{a\to+0} g'(a) = \infty$ and $g'(\frac{1}{2}) = 2\ln 2 - 2 < 0$, there exists uniquely a number $c \in (0, \frac{1}{2})$ such that $g'(c) = 0$. Since $g'(a) > 0$ for $0 < a < c$ and $g'(a) < 0$ for $c < a < \frac{1}{2}$, g is strictly increasing on the interval $(0, c)$ and strictly decreasing on the interval $(c, \frac{1}{2})$. Since $\lim_{a\to+0} g(a) = 0$ and $g(\frac{1}{2}) = 0$, we get $g(a) > 0$ for $0 < a < \frac{1}{2}$. Therefore, we have $f'(a) > 0$ for any $a \in (0, \frac{1}{2})$. Since f is strictly increasing on the interval $(0, \frac{1}{2})$, we can get $f(a) < f(\frac{1}{2}) = (\ln 2)^2$. Hence, we have $(\ln 2)^2 > \ln a \ln b$ for $0 < a < \frac{1}{2}$. Also, the inequality

$$(\ln 2 + \ln a)^2 \ln b > (\ln 2 + \ln b)^2 \ln a$$

is equivalent to

$$(\ln b - \ln a)\big((\ln 2)^2 - \ln a \ln b\big) > 0.$$

From $\ln b - \ln a > 0$ and $(\ln 2)^2 - \ln a \ln b > 0$, it follows that $(\ln b - \ln a)((\ln 2)^2 - \ln a \ln b) > 0$. This completes the proof of the lemma. □

Proposition 2.8 *If $0 < a < \frac{1}{2}$ and $x \geq 1$, then G_x is strictly increasing with respect to x.*

Proof For fixed $a \in (0, \frac{1}{2})$, let us denote $f(x) = G_x(x, a)$; that is,

$$f(x) = (2a)^x \ln(2a) \ln b + \ln(2a) - (2b)^x \ln(2b) \ln a - \ln(2b), \quad b = 1 - a.$$

Clearly, we need to show that $f'(x) > 0$ for $x \geq 1$. From

$$f'(x) = (2a)^x (\ln 2 + \ln a)^2 \ln b - (2b)^x (\ln 2 + \ln b)^2 \ln a,$$

we can write the inequality $f'(x) > 0$ in the form

$$\left(\frac{a}{b}\right)^x (\ln 2 + \ln a)^2 < (\ln 2 + \ln b)^2 \frac{\ln a}{\ln b}.$$

Since $(\frac{a}{b})^x < 1$, it is enough to show that

$$(\ln 2 + \ln a)^2 < (\ln 2 + \ln b)^2 \frac{\ln a}{\ln b},$$

which follows immediately from Lemma 2.7. □

3 Proof of Theorem 1.1

In this section, we shall prove the main theorem.

Proof We shall show that if $0 \leq a \leq \frac{1}{2}$ and $x \geq 1$, then $F(x,a) \leq 1$. The inequality $F(1,a) \leq 1$ is proved in Cîrtoaje [2]. Since $F(x,0) = F(x,\frac{1}{2}) = 1$, we may show this inequality for $0 < a < \frac{1}{2}$. We note that $G_x(1,a)$ has the both signs, so we consider the cases of $G_x(1,a) \geq 0$ and $G_x(1,a) < 0$.

(1) We first assume that $G_x(1,a) \geq 0$. We have $G_x(x,a) > G_x(1,a)$ for $x > 1$ from Proposition 2.8. Since we have $G_x(x,a) > 0$ on the assumption, G is strictly increasing with respect to x. Therefore, $G(x,a) > G(1,a)$ for $x > 1$. We note that $G(1,a)$ has the both signs, so we consider the cases of $G(1,a) \geq 0$ and $G(1,a) < 0$.

(i) If $G(1,a) \geq 0$, then $G(x,a) > 0$, so we have $F_x(x,a) > 0$ for $x > 1$. Hence, F is strictly increasing with respect to x, and we have $F(x,a) \leq \lim_{x\to\infty} F(x,a) = 1$.

(ii) If $G(1,a) < 0$, then since $\lim_{x\to\infty} G(x,a) = \infty$, there exists uniquely a number $x_1 > 1$ satisfying $G(x_1,a) = 0$. Since $G(x,a) < 0$ for $1 < x < x_1$ and $G(x,a) > 0$ for $x > x_1$, we have $F_x(x,a) < 0$ for $1 < x < x_1$ and $F_x(x,a) > 0$ for $x > x_1$. Hence, F is strictly decreasing for $1 < x < x_1$ and strictly increasing for $x > x_1$. Therefore, we get

$$F(x,a) \leq \max\left\{F(1,a), \lim_{x\to\infty} F(x,a)\right\} = 1.$$

(2) We next assume that $G_x(1,a) < 0$. Since G_x is strictly increasing with respect to x from Proposition 2.8 and $\lim_{x\to\infty} G_x(x,a) = \infty$, there exists uniquely a number $x_2 > 1$ satisfying $G_x(x_2,a) = 0$. Since $G_x(x,a) < 0$ for $1 < x < x_2$ and $G_x(x,a) > 0$ for $x > x_2$, G is strictly decreasing for $1 < x < x_2$ and strictly increasing for $x > x_2$. By the assumption $G_x(1,a) < 0$ and Proposition 2.6, it follows that $a > \frac{15}{100}$. Hence, $G(1,a) < 0$ by Proposition 2.5. From $\lim_{x\to\infty} G(x,a) = \infty$, there exists uniquely a number $x_3 > x_2$ satisfying $G(x_3,a) = 0$. If $1 < x < x_3$, then $G(x,a) < 0$, so $F_x(x,a) < 0$. If $x > x_3$, then $G(x,a) > 0$, so $F_x(x,a) > 0$. Hence F is strictly decreasing for $1 < x < x_3$ and strictly increasing for $x > x_3$. So, we get

$$F(x,a) \leq \max\left\{F(1,a), \lim_{x\to\infty} F(x,a)\right\} = 1.$$

This completes the proof of Theorem 1.1. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
MM had the first drift of the paper and YN added some contents and re-organized the paper. All authors read and approved the final manuscript.

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