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The connection between Hilbert and Hardy inequalities

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Abstract

In this paper we introduce some new forms of the Hilbert integral inequality, and we study the connection between the obtained inequalities with Hardy inequalities. The reverse form and some applications are also given.

MSC: 26D15

Keywords: Hilbert's inequality; Hölder's inequality; Hardy's inequality

1 Introduction

The famous Hardy-Hilbert inequality for positive functions f, g and two conjugate parameters p and q such that $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ is given as

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}\,dx\,dy < \frac{\pi}{\sin(\frac{\pi}{p})} \left\{ \int_0^\infty f^p(x)\,dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty g^q(x)\,dx \right\}^{\frac{1}{q}}, \tag{1.1}$$

provided that the integrals on the right-hand side are convergent. The constant $\frac{\pi}{\sin(\frac{\pi}{p})}$ is best possible [1]. In the last years, inequality (1.1) has been extended in different ways. In [2] the authors obtained the following extension of (1.1):

$$\begin{aligned} &\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda}\,dx\,dy \\ &\quad< B(1-pA_2, \lambda+pA_2-1) \left\{ \int_0^\infty x^{pqA_1-1} f^p(x)\,dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty y^{pqA_2-1} g^q(x)\,dx \right\}^{\frac{1}{q}}, \end{aligned} \tag{1.2}$$

where $B(1-pA_2, \lambda+pA_2-1)$ is the best possible constant ($B(x,y)$ is the beta function), $\lambda > 0$, $A_1 \in (\frac{1-\lambda}{q}, \frac{1}{q})$, $A_2 \in (\frac{1-\lambda}{p}, \frac{1}{p})$ and $pA_2 + qA_1 = 2 - \lambda$. For $0 < p < 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda > 0$, $A_1 \in (\frac{1}{q}, \frac{1-\lambda}{q})$, $A_2 \in (\frac{1-\lambda}{p}, \frac{1}{p})$ and $pA_2 + qA_1 = 2 - \lambda$, the reverse form of (1.2) is also valid with the same constant factor. In [3] the following extension was given:

$$\begin{aligned} &\int_a^b \int_a^b \frac{f(x)g(y)}{(au^2(x) + 2bu(x)v(y) + cv^2(y))^\lambda}\,dx\,dy \\ &\quad< L^* \left\{ \int_a^b \frac{u(x)^{pqA_1-1}}{u'(x)^{p-1}} f^p(x)\,dx \right\}^{\frac{1}{p}} \left\{ \int_a^b \frac{v(y)^{pqA_2-1}}{v'(y)^{q-1}} g^q(y)\,dy \right\}^{\frac{1}{q}}, \end{aligned} \tag{1.3}$$

where $L^* = a^{\frac{qA_1-1}{2}} c^{\frac{pA_2-1}{2}} B(1-pA_2, 2\lambda+pA_2-1) F(\frac{1-pA_2}{2}, \lambda - \frac{1-pA_2}{2}, \lambda + \frac{1}{2}; 1 - \frac{b^2}{ac})$ is best possible ($F(\alpha,\beta;\gamma;x)$ is the hypergeometric function), $\lambda > 0$, $A_1 \in (\frac{1-2\lambda}{q}, \frac{1}{q})$, $A_2 \in (\frac{1-2\lambda}{p}, \frac{1}{p})$ and

$pA_2 + qA_1 = 2 - 2\lambda$, $a, c > 0$, $b^2 < ac$, $\lambda > 0$, u and v are differentiable nonnegative strictly increasing functions on (a, b) $(-\infty \le a < b \le \infty)$, and they satisfy the following conditions: $\lim_{t\to a^+} u(t) = \lim_{t\to a^+} v(t) = 0$ and $\lim_{t\to b^-} u(t) = \lim_{t\to b^-} v(t) = \infty$. In particular, if we let $a = c = 1$, $b = 0$ and consider $\sqrt{u(x)}$, $\sqrt{v(y)}$ instead of $u(x)$ and $v(y)$ respectively in (1.3), we get

$$\begin{aligned}
&\int_a^b \int_a^b \frac{f(x)g(y)}{(u(x)+v(y))^\lambda}\,dx\,dy \\
&\quad < B(1-pA_2, \lambda + pA_2 - 1) \\
&\qquad \times \left\{\int_a^b \frac{u(x)^{pqA_1-1}}{u'(x)^{p-1}} f^p(x)\,dx\right\}^{\frac{1}{p}} \left\{\int_a^b \frac{v(y)^{pqA_2-1}}{v'(y)^{q-1}} g^q(y)\,dy\right\}^{\frac{1}{q}},
\end{aligned} \tag{1.4}$$

here $pA_2 + qA_1 = 2 - \lambda$ as in (1.2).

The following inequalities are special cases of (1.4):

$$\begin{aligned}
&\int_{-\infty}^\infty \int_{-\infty}^\infty \frac{f(x)g(y)}{(e^x+e^y)^\lambda}\,dx\,dy \\
&\quad < B(1-pA_2, \lambda + pA_2 - 1)\left\{\int_{-\infty}^\infty e^{(pqA_1-p)x} f^p(x)\,dx\right\}^{\frac{1}{p}} \left\{\int_{-\infty}^\infty e^{(pqA_2-q)y} g^q(y)\,dy\right\}^{\frac{1}{q}},
\end{aligned} \tag{1.5}$$

$$\begin{aligned}
&\int_1^\infty \int_1^\infty \frac{f(x)g(y)}{(\ln x + \ln y)^\lambda}\,dx\,dy \\
&\quad < B(1-pA_2, \lambda + pA_2 - 1) \\
&\qquad \times \left\{\int_1^\infty \frac{[\ln x]^{pqA_1-1}}{x^{1-p}} f^p(x)\,dx\right\}^{\frac{1}{p}} \left\{\int_1^\infty \frac{[\ln y]^{pqA_2-1}}{y^{1-q}} g^q(y)\,dy\right\}^{\frac{1}{q}}.
\end{aligned} \tag{1.6}$$

Refinements of some Hilbert-type inequalities by virtue of various methods were obtained in [4, 5] and [6]. A survey of some recent results concerning Hilbert and Hilbert-type inequalities can be found in [7] and [8].

If $p > 1$, $f(x) > 0$, and $F(x) = \int_0^x f(t)\,dt$, then the well-known Hardy inequality [1] is given as

$$\int_0^\infty \left(\frac{F(x)}{x}\right)^p dx < \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)\,dx, \tag{1.7}$$

the constant $(\frac{p}{p-1})^p$ is best possible. A weighted form of (1.7) was given also by Hardy [1] as

$$\int_0^\infty x^a \left(\frac{F(x)}{x}\right)^p dx < \left(\frac{p}{p-1-a}\right)^p \int_0^\infty x^a f^p(x)\,dx, \tag{1.8}$$

where $p > 1$, $a < p - 1$ or $p < 0$, $a > p - 1$ and the constant $(\frac{p}{p-1-a})^p$ is best possible. For $0 < p < 1$ $(a < p - 1)$, inequality (1.8) holds in the reverse direction. Inequality (1.7) was discovered by Hardy while he was trying to introduce a simple proof of Hilbert's inequality. In the book [9], the following Hardy-type inequality is given:

$$\int_{-\infty}^\infty e^{kpx}\left(\int_{-\infty}^x f(t)\,dt\right)^p dx \le (-k)^{-p} \int_{-\infty}^\infty e^{kpx} f^p(x)\,dx, \tag{1.9}$$

where $k < 0$ and $p > 1$ or $p < 0$. If $0 < p < 1$, then the reverse form of (1.9) holds. The constant $(-k)^{-p}$ is best possible.

In [10](see also [9]), the following Hardy-type inequality is obtained for $p > 1$:

$$\int_1^\infty \frac{1}{x[\ln x]^p}\left(\int_1^x f(t)\,dt\right)^p dx \le \left(\frac{p}{p-1}\right)^p \int_1^\infty x^{p-1} f^p(x)\,dx. \tag{1.10}$$

For details about inequality (1.7) and its history and development, we refer the reader to the papers [11] and [12].

Recently, in [13], for $f, g > 0$, $f, g \in L(0,\infty)$, $F(x) = \int_0^x f(u)\,du$ and $G(x) = \int_0^x g(u)\,du$, $\lambda > 0$, the following form of (1.1) was obtained:

$$\begin{aligned}&\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda}\,dx\,dy\\&\quad\le \frac{\lambda^2}{pq}B\left(\frac{\lambda}{q},\frac{\lambda}{p}\right)\left(\int_0^\infty x^{-\lambda-1}F^p(x)\,dx\right)^{\frac{1}{p}}\left(\int_0^\infty y^{-\lambda-1}G^q(y)\,dy\right)^{\frac{1}{q}},\end{aligned} \tag{1.11}$$

the constant factor $\frac{\lambda^2}{pq}B(\frac{\lambda}{q},\frac{\lambda}{p})$ is best possible. For other Hilbert-type inequalities involving Hardy operators, see, for example, [14] and [15].

In this paper, by estimating the double integral $\int_a^b \int_a^b \frac{f(x)g(y)}{(u(x)+v(y))^\lambda}\,dx\,dy$, we introduce an extension of (1.11) with the best constant factor. The reverse form is also obtained. Some applications are given. The connection between Hilbert and Hardy inequalities is also considered. As a consequence of Theorem 3.1, we obtain the following interesting inequality:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}\,dx\,dy \le \frac{\pi}{pq\sin\frac{\pi}{p}}\left(\int_0^\infty \left(\frac{F(x)}{x}\right)^p dx\right)^{\frac{1}{p}}\left(\int_0^\infty \left(\frac{G(y)}{y}\right)^q dy\right)^{\frac{1}{q}}.$$

2 Preliminaries and lemmas

Recall that the gamma function $\Gamma(\theta)$ and the beta function $B(\mu,\nu)$ are defined respectively by

$$\begin{aligned}\Gamma(\theta) &= \int_0^\infty t^{\theta-1}e^{-t}\,dt, \quad \theta > 0,\\ B(\mu,\nu) &= \int_0^\infty \frac{t^{\mu-1}}{(t+1)^{\mu+\nu}}\,dt, \quad \mu,\nu > 0.\end{aligned}$$

In this paper, we assume that u and v are defined as in inequality (1.3) from the introduction.

Lemma 2.1 *Let $r > 1$, $\frac{1}{r}+\frac{1}{s} = 1$, $\varphi > 0$, $\varphi \in L(a,b)$, $\Phi(x) = \int_a^x \varphi(u)\,du$, and let h be a differentiable nonnegative strictly increasing function on (a,b) such that $\lim_{x\to a^+} h(x) = 0$, $\lim_{x\to b^-} h(x) = \infty$. Then, for $t, \alpha > 0$, we have*

$$\int_a^b e^{-th(x)}\varphi(x)\,dx \le t^{\frac{1}{r}-\alpha}\Gamma(\alpha s+1)^{\frac{1}{s}}\left\{\int_a^b \left[h(x)\right]^{-\alpha r}h'(x)e^{-th(x)}\Phi^r(x)\,dx\right\}^{\frac{1}{r}}. \tag{2.1}$$

Proof Using integration by parts, we get

$$\int_a^b e^{-th(x)}\varphi(x)\,dx = t\int_a^b h'(x)e^{-th(x)}\Phi(x)\,dx. \tag{2.2}$$

Applying Hölder's inequality, we obtain

$$\begin{aligned}
&\int_a^b h'(x)e^{-th(x)}\Phi(x)\,dx\\
&\quad= \int_a^b \bigl(\bigl[h(x)\bigr]^{\alpha}\bigl[h'(x)\bigr]^{\frac{1}{s}}e^{-\frac{th(x)}{s}}\bigr)\bigl(\bigl[h(x)\bigr]^{-\alpha}\bigl[h'(x)\bigr]^{\frac{1}{r}}e^{-\frac{th(x)}{r}}\Phi(x)\bigr)\,dx\\
&\quad\le \biggl(\int_a^b \bigl[h(x)\bigr]^{\alpha s}h'(x)e^{-th(x)}\,dx\biggr)^{\frac{1}{s}}\biggl(\int_a^b \bigl[h(x)\bigr]^{-\alpha r}h'(x)e^{-th(x)}\Phi^r(x)\,dx\biggr)^{\frac{1}{r}}\\
&\quad= t^{\frac{-1}{s}-\alpha}\Gamma(\alpha s+1)^{\frac{1}{s}}\biggl(\int_a^b \bigl[h(x)\bigr]^{-\alpha r}h'(x)e^{-th(x)}\Phi^r(x)\,dx\biggr)^{\frac{1}{r}}.
\end{aligned}$$

Substituting the last inequality in (2.2), we get (2.1). □

Lemma 2.2 *Let* $0 < r < 1$, $\frac{1}{r}+\frac{1}{s}=1$, $\varphi > 0$, $\varphi \in L(a,b)$, $\Phi(x)=\int_a^x \varphi(u)\,du$, *and let* h *be as in Lemma* 2.1. *Then, for* $t > 0$ *and* $\beta \in \mathbb{R}$ $(\beta s+1>0)$, *we have*

$$\int_a^b e^{-th(x)}\varphi(x)\,dx \ge t^{\frac{1}{r}-\beta}\Gamma(\beta s+1)^{\frac{1}{s}}\biggl\{\int_a^b \bigl[h(x)\bigr]^{-\beta r}h'(x)e^{-th(x)}\Phi^r(x)\,dx\biggr\}^{\frac{1}{r}}. \tag{2.3}$$

Proof Integration by parts yields

$$\int_a^b e^{-th(x)}\varphi(x)\,dx = t\int_a^b h'(x)e^{-th(x)}\Phi(x)\,dx. \tag{2.4}$$

Using the reverse Hölder inequality, we obtain

$$\begin{aligned}
&\int_a^b h'(x)e^{-th(x)}\Phi(x)\,dx\\
&\quad= \int_a^b \bigl(\bigl[h(x)\bigr]^{\beta}\bigl[h'(x)\bigr]^{\frac{1}{s}}e^{-\frac{th(x)}{s}}\bigr)\bigl(\bigl[h(x)\bigr]^{-\beta}\bigl[h'(x)\bigr]^{\frac{1}{r}}e^{-\frac{th(x)}{r}}\Phi(x)\bigr)\,dx\\
&\quad\ge \biggl(\int_a^b \bigl[h(x)\bigr]^{\beta s}h'(x)e^{-th(x)}\,dx\biggr)^{\frac{1}{s}}\biggl(\int_a^b \bigl[h(x)\bigr]^{-\beta r}h'(x)e^{-th(x)}\Phi^r(x)\,dx\biggr)^{\frac{1}{r}}\\
&\quad= t^{\frac{-1}{s}-\beta}\Gamma(\beta s+1)^{\frac{1}{s}}\biggl(\int_a^b \bigl[h(x)\bigr]^{-\beta r}h'(x)e^{-th(x)}\Phi^r(x)\,dx\biggr)^{\frac{1}{r}}.
\end{aligned}$$

Substituting the last inequality in (2.4), we get (2.3).

By the definition of the gamma function above, we may write

$$\frac{1}{(x+y)^{\lambda}} = \frac{1}{\Gamma(\lambda)}\int_0^{\infty} t^{\lambda-1}e^{-(x+y)t}\,dt. \tag{2.5}$$

□

3 Main results

In this section, we introduce two main results in this paper. Theorem 3.1 gives an extended form of inequality (1.11) and it is connected to the famous Hardy inequality. In Theorem 3.2, we introduce the reverse form obtained in Theorem 3.1.

Theorem 3.1 *Let* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda > 0$, $\gamma \in (\frac{-\lambda}{p}, \frac{\lambda}{q})$, $f, g > 0$, $f, g \in L(a,b)$, *define* $F(x) = \int_a^x f(u)\,du$ *and* $G(x) = \int_a^x g(u)\,du$. *If* $\int_a^b [u(x)]^{-\lambda-1-p\gamma} u'(x)F^p(x)\,dx < \infty$ *and* $\int_a^b [v(y)]^{-\lambda-1+q\gamma} \times v'(y)G^q(y)\,dy < \infty$, *then*

$$\begin{aligned}
&\int_a^b \int_a^b \frac{f(x)g(y)}{(u(x)+v(y))^\lambda}\,dx\,dy \\
&\quad\le C\left(\int_a^b \bigl[u(x)\bigr]^{-\lambda-1-p\gamma} u'(x)F^p(x)\,dx\right)^{\frac{1}{p}} \\
&\qquad\times \left(\int_a^b \bigl[v(y)\bigr]^{-\lambda-1+q\gamma} v'(y)G^q(y)\,dy\right)^{\frac{1}{q}},
\end{aligned} \tag{3.1}$$

where the constant $C = (\frac{\lambda}{p} + \gamma)(\frac{\lambda}{q} - \gamma)B(\frac{\lambda}{p} + \gamma, \frac{\lambda}{q} - \gamma)$ *is best possible.*

Proof By using (2.5) and applying Hölder's inequality, we have

$$\begin{aligned}
I &=: \int_a^b \int_a^b \frac{f(x)g(y)}{(u(x)+v(y))^\lambda}\,dx\,dy \\
&= \frac{1}{\Gamma(\lambda)} \int_a^b \int_a^b f(x)g(y)\left(\int_0^\infty t^{\lambda-1}e^{-(u(x)+v(y))t}\,dt\right)dx\,dy \\
&= \frac{1}{\Gamma(\lambda)} \int_0^\infty \left(t^{\frac{\lambda-1}{p}+\gamma}\int_a^b e^{-u(x)t}f(x)\,dx\right)\left(t^{\frac{\lambda-1}{q}-\gamma}\int_a^b e^{-v(y)t}g(y)\,dy\right)dt \\
&\le \frac{1}{\Gamma(\lambda)}\left(\int_0^\infty t^{\lambda-1+p\gamma}\left(\int_a^b e^{-u(x)t}f(x)\,dx\right)^p dt\right)^{\frac{1}{p}} \\
&\quad\times \left(\int_0^\infty t^{\lambda-1-q\gamma}\left(\int_a^b e^{-v(y)t}g(y)\,dy\right)^q dt\right)^{\frac{1}{q}}.
\end{aligned} \tag{3.2}$$

By Lemma 2.1, for $r = p$, $s = q$, $\alpha = \frac{\lambda+p\gamma}{pq}$ and then for $r = q$, $s = p$, $\alpha = \frac{\lambda-q\gamma}{pq}$, we obtain, respectively,

$$\begin{aligned}
&\left(\int_a^b e^{-u(x)t}f(x)\,dx\right)^p \le t^{1-\frac{\lambda+p\gamma}{q}}\,\Gamma\left(\frac{\lambda}{p}+\gamma+1\right)^{\frac{p}{q}} \int_a^b \bigl[u(x)\bigr]^{-\frac{\lambda+p\gamma}{q}} u'(x)e^{-tu(x)}F^p(x)\,dx, \\
&\left(\int_a^b e^{-v(y)t}g(y)\,dy\right)^q \le t^{1-\frac{\lambda-q\gamma}{p}}\,\Gamma\left(\frac{\lambda}{q}-\gamma+1\right)^{\frac{q}{p}} \int_a^b \bigl[v(y)\bigr]^{-\frac{\lambda-q\gamma}{p}} v'(y)e^{-tv(y)}G^q(y)\,dy.
\end{aligned}$$

Substituting these two inequalities in (3.2), we have

$$\begin{aligned}
I &\le \frac{\Gamma(\frac{\lambda}{p}+\gamma+1)^{\frac{1}{q}}\Gamma(\frac{\lambda}{q}-\gamma+1)^{\frac{1}{p}}}{\Gamma(\lambda)}\left(\int_a^b \bigl[u(x)\bigr]^{-\frac{\lambda+p\gamma}{q}} u'(x)F^p(x)\left(\int_0^\infty t^{\frac{\lambda}{p}+\gamma}e^{-u(x)t}\,dt\right)dx\right)^{\frac{1}{p}} \\
&\quad\times \left(\int_a^b \bigl[v(y)\bigr]^{-\frac{\lambda-q\gamma}{p}} v'(y)G^q(y)\left(\int_0^\infty t^{\frac{\lambda}{q}-\gamma}e^{-v(y)t}\,dt\right)dy\right)^{\frac{1}{q}}.
\end{aligned}$$

Since

$$\int_0^\infty t^{\frac{\lambda}{p}+\gamma} e^{-u(x)t}\,dt = \left[u(x)\right]^{\frac{-\lambda}{p}-\gamma-1}\Gamma\left(\frac{\lambda}{p}+\gamma+1\right)$$

and

$$\int_0^\infty t^{\frac{\lambda}{q}-\gamma} e^{-v(y)t}\,dt = \left[v(y)\right]^{-\frac{\lambda}{q}+\gamma-1}\Gamma\left(\frac{\lambda}{q}-\gamma+1\right),$$

we find

$$I \le C\left(\int_a^b \left[u(x)\right]^{-\lambda-1-p\gamma} u'(x)F^p(x)\,dx\right)^{\frac{1}{p}} \left(\int_a^b \left[v(y)\right]^{-\lambda-1+q\gamma} v'(y)G^q(y)\,dy\right)^{\frac{1}{q}}.$$

Now, since $\Gamma(u+1) = u\Gamma(u)$ and $\frac{\Gamma(\frac{\lambda}{p}+\gamma)\Gamma(\frac{\lambda}{q}-\gamma)}{\Gamma(\lambda)} = B(\frac{\lambda}{p}+\gamma, \frac{\lambda}{q}-\gamma)$, we get

$$C = \left(\frac{\lambda}{p}+\gamma\right)\left(\frac{\lambda}{q}-\gamma\right)B\left(\frac{\lambda}{p}+\gamma, \frac{\lambda}{q}-\gamma\right).$$

Inequality (3.1) is proved. We need to show that the constant factor C in (3.1) is best possible. For $0 < \varepsilon < \min\{\lambda - q\gamma, \lambda + p\gamma\}$, we define the functions $f_\varepsilon(x) = 0$, for $x \in (a, a_1)$ and $f_\varepsilon(x) = \frac{\lambda+p\gamma-\varepsilon}{p}[u(x)]^{\frac{\lambda+p\gamma-\varepsilon}{p}-1}u'(x)$ for $x \in [a_1, b)$ and $g_\varepsilon(y) = 0$ for $y \in (a, a_2)$ and $g_\varepsilon(y) = \frac{\lambda-q\gamma-\varepsilon}{q}[v(y)]^{\frac{\lambda-q\gamma-\varepsilon}{q}-1}v'(y)$ for $y \in [a_2, b)$, where a_1 and a_2 are such that $u(a_1) = v(a_2) = 1$. Then we get $F_\varepsilon(x) = ([u(x)]^{\frac{\lambda+p\gamma-\varepsilon}{p}} - 1)$ for $x \in [a_1, b)$ and $G_\varepsilon(y) = ([v(y)]^{\frac{\lambda-q\gamma-\varepsilon}{q}} - 1)$ for $y \in [a_2, b)$, $F_\varepsilon(x) = G_\varepsilon(y) = 0$ for $x \in (a, a_1)$, $y \in (a, a_2)$, respectively. Suppose that the constant $C = (\frac{\lambda}{p}+\gamma)(\frac{\lambda}{q}-\gamma)B(\frac{\lambda}{p}+\gamma, \frac{\lambda}{q}-\gamma)$ is not best possible, then there exists $0 < K < C$ such that

$$\begin{aligned}
I &\le K\left(\int_{a_1}^b \left[u(x)\right]^{-\lambda-1-p\gamma} u'(x)\left[\left[u(x)\right]^{\frac{\lambda+p\gamma-\varepsilon}{p}} - 1\right]^p dx\right)^{\frac{1}{p}} \\
&\quad \times \left(\int_{a_2}^b \left[v(y)\right]^{-\lambda-1+q\gamma} v'(y)\left[\left[v(y)\right]^{\frac{\lambda-q\gamma-\varepsilon}{q}} - 1\right]^q dy\right)^{\frac{1}{q}} \\
&< K\left(\int_{a_1}^b \left[u(x)\right]^{-\varepsilon-1}u'(x)\,dx\right)^{\frac{1}{p}} \left(\int_{a_2}^b \left[v(y)\right]^{-\varepsilon-1}v'(y)\,dy\right)^{\frac{1}{q}} \\
&= K\left(\int_1^\infty \delta^{-\varepsilon-1}\,d\delta\right)^{\frac{1}{p}} \left(\int_1^\infty \delta^{-\varepsilon-1}\,d\delta\right)^{\frac{1}{q}} = \frac{K}{\varepsilon}.
\end{aligned} \tag{3.3}$$

On the other hand, we have

$$\begin{aligned}
I &= \int_a^b \int_a^b \frac{f_\varepsilon(x)g_\varepsilon(y)}{(u(x)+v(y))^\lambda}\,dx\,dy \\
&= \frac{(\lambda+p\gamma-\varepsilon)(\lambda-q\gamma-\varepsilon)}{pq} \int_{a_1}^b \int_{a_2}^b \frac{[u(x)]^{\frac{\lambda+p\gamma-\varepsilon}{p}-1}u'(x)[v(y)]^{\frac{\lambda-q\gamma-\varepsilon}{q}-1}v'(y)}{(u(x)+v(y))^\lambda}\,dy\,dx \\
&= \frac{(\lambda+p\gamma-\varepsilon)(\lambda-q\gamma-\varepsilon)}{pq} \int_{a_1}^b \left[u(x)\right]^{-\varepsilon-1}u'(x)\left\{\int_{\frac{1}{u(x)}}^\infty \frac{\theta^{\frac{\lambda-q\gamma-\varepsilon}{q}-1}}{(\theta+1)^\lambda}\,d\theta\right\}dx
\end{aligned}$$

$$
\begin{aligned}
&= \frac{(\lambda + p\gamma - \varepsilon)(\lambda - q\gamma - \varepsilon)}{pq} \\
&\quad \times \left\{ \frac{1}{\varepsilon} \int_0^\infty \frac{\theta^{\frac{\lambda - q\gamma - \varepsilon}{q} - 1}}{(\theta + 1)^\lambda} \, d\theta \, du - \int_{a_1}^b \left[u(x)\right]^{-\varepsilon - 1} u'(x) \int_0^{\frac{1}{u(x)}} \frac{\theta^{\frac{\lambda - q\gamma - \varepsilon}{q} - 1}}{(\theta + 1)^\lambda} \, d\theta \, dx \right\} \\
&= \frac{(\lambda + p\gamma - \varepsilon)(\lambda - q\gamma - \varepsilon)}{pq} \frac{B(\frac{\lambda}{p} + \gamma, \frac{\lambda}{q} - \gamma) + o(1)}{\varepsilon} \\
&\quad - \frac{(\lambda + p\gamma - \varepsilon)(\lambda - q\gamma - \varepsilon)}{pq} \int_{a_1}^b \left[u(x)\right]^{-\varepsilon - 1} u'(x) \int_0^{\frac{1}{u(x)}} \frac{\theta^{\frac{\lambda - q\gamma - \varepsilon}{q} - 1}}{(\theta + 1)^\lambda} \, d\theta \, dx \\
&> \frac{(\lambda + p\gamma - \varepsilon)(\lambda - q\gamma - \varepsilon)}{pq} \frac{B(\frac{\lambda}{p} + \gamma, \frac{\lambda}{q} - \gamma) + o(1)}{\varepsilon} \\
&\quad - \frac{(\lambda + p\gamma - \varepsilon)(\lambda - q\gamma - \varepsilon)}{pq} \int_{a_1}^b \left[u(x)\right]^{-\varepsilon - 1} u'(x) \int_0^{\frac{1}{u(x)}} \theta^{\frac{\lambda - q\gamma - \varepsilon}{q} - 1} \, d\theta \, dx \\
&= \frac{(\lambda + p\gamma - \varepsilon)(\lambda - q\gamma - \varepsilon)}{pq} \frac{B(\frac{\lambda}{p} + \gamma, \frac{\lambda}{q} - \gamma) + o(1)}{\varepsilon} - O(1).
\end{aligned}
\tag{3.4}
$$

It is obvious that when $\varepsilon \to 0^+$ from (3.3) and (3.4), we obtain a contradiction. Thus, the proof of the theorem is completed. □

Theorem 3.2 *Let* $0 < p < 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda > 0$, $\gamma \in (\frac{-\lambda}{p}, \frac{\lambda}{q})$, $f, g > 0$, $f, g \in L(a,b)$, *define* $F(x) = \int_a^x f(u)\, du$ *and* $G(x) = \int_a^x g(u)\, du$. *If* $\int_a^b [u(x)]^{-\lambda-1-p\gamma} u'(x) F^p(x)\, dx < \infty$ *and* $\int_a^b [v(y)]^{-\lambda-1+q\gamma} v'(y) G^q(y)\, dy < \infty$, *then we obtain the reverse form of* (3.1) *as*

$$
\begin{aligned}
\int_a^b \int_a^b \frac{f(x)g(y)}{(u(x) + v(y))^\lambda} \, dx \, dy &\geq C \left(\int_a^b \left[u(x)\right]^{-\lambda - 1 - p\gamma} u'(x) F^p(x) \, dx \right)^{\frac{1}{p}} \\
&\quad \times \left(\int_a^b \left[v(y)\right]^{-\lambda - 1 + q\gamma} v'(y) G^q(y) \, dy \right)^{\frac{1}{q}},
\end{aligned}
\tag{3.5}
$$

where C *is as in Theorem* 3.1.

Proof If we use (2.5) and apply the reverse Hölder inequality, we have

$$
\begin{aligned}
I &=: \int_a^b \int_a^b \frac{f(x)g(y)}{(u(x) + v(y))^\lambda} \, dx \, dy \\
&= \frac{1}{\Gamma(\lambda)} \int_0^\infty \left(t^{\frac{\lambda - 1}{p} + \gamma} \int_a^b e^{-u(x)t} f(x) \, dx \right) \left(t^{\frac{\lambda - 1}{q} - \gamma} \int_a^b e^{-v(y)t} g(y) \, dy \right) dt \\
&\geq \frac{1}{\Gamma(\lambda)} \left(\int_0^\infty t^{\lambda - 1 + p\gamma} \left(\int_a^b e^{-u(x)t} f(x) \, dx \right)^p dt \right)^{\frac{1}{p}} \\
&\quad \times \left(\int_0^\infty t^{\lambda - 1 - q\gamma} \left(\int_a^b e^{-v(y)t} g(y) \, dy \right)^q dt \right)^{\frac{1}{q}}.
\end{aligned}
\tag{3.6}
$$

By Lemma 2.2, for $r = p$, $s = q$, $\beta = \frac{\lambda+p\gamma}{pq}$ and then for $r = q$, $s = p$, $\beta = \frac{\lambda-q\gamma}{pq}$, we obtain, respectively,

$$\left(\int_a^b e^{-u(x)t} f(x)\,dx\right)^p \geq t^{1-\frac{\lambda+p\gamma}{q}} \Gamma\left(\frac{\lambda}{p}+\gamma+1\right)^{\frac{p}{q}} \int_a^b \left[u(x)\right]^{-\frac{\lambda+p\gamma}{q}} u'(x) e^{-tu(x)} F^p(x)\,dx,$$

$$\left(\int_a^b e^{-v(y)t} g(y)\,dy\right)^q \leq t^{1-\frac{\lambda-q\gamma}{p}} \Gamma\left(\frac{\lambda}{q}-\gamma+1\right)^{\frac{q}{p}} \int_a^b \left[v(y)\right]^{-\frac{\lambda-q\gamma}{p}} v'(y) e^{-tv(y)} G^q(y)\,dy.$$

If we substitute these two inequalities in (3.6) and make some computations as we did in Theorem 3.1, we get inequality (3.5). □

4 Applications

In this section, we give some applications of Theorem 3.1 and Theorem 3.2. We consider some specific functions which satisfy the conditions of the functions u and v, and we see the connection between Hilbert and Hilbert-type inequalities with Hardy and Hardy-type inequalities from the introduction.

1. Let $u(x) = x$, $v(y) = y$, $x, y \in (0, \infty)$, then we find by (3.1)

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda}\,dx\,dy \leq C\left(\int_0^\infty x^{-\lambda-1-p\gamma} F^p(x)\,dx\right)^{\frac{1}{p}} \left(\int_0^\infty y^{-\lambda-1+q\gamma} G^q(y)\,dy\right)^{\frac{1}{q}}, \tag{4.1}$$

here $F(x) = \int_0^x f(t)\,dt$ and $G(y) = \int_0^y g(t)\,dt$. If we put $\gamma = 0$ in (4.1), we get (1.11). If we let $\lambda = 1$, $\gamma = \frac{p-2}{p}$, we obtain the following form:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}\,dx\,dy \leq \frac{\pi}{pq\sin\frac{\pi}{p}} \left(\int_0^\infty \left(\frac{F(x)}{x}\right)^p dx\right)^{\frac{1}{p}} \left(\int_0^\infty \left(\frac{G(y)}{y}\right)^q dy\right)^{\frac{1}{q}}. \tag{4.2}$$

Applying Hardy's inequality (1.7) to the right-hand side of (4.2), we get Hilbert's inequality (1.1). If we apply the weighted Hardy inequality (1.8) to (4.1), we get

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda}\,dx\,dy < C_1\left(\int_0^\infty x^{p-\lambda-1-p\gamma} f^p(x)\,dx\right)^{\frac{1}{p}} \left(\int_0^\infty y^{q-\lambda-1+q\gamma} g^q(y)\,dy\right)^{\frac{1}{q}}, \tag{4.3}$$

where $C_1 = B(\frac{\lambda}{p}+\gamma, \frac{\lambda}{q}-\gamma)$. Inequality (4.3) is equivalent to inequality (1.2) if we set $\gamma = \frac{p-\lambda-pqA_1}{p}$ $(-\frac{\lambda}{p} < \gamma < \frac{\lambda}{q})$ under the condition $pA_2 + qA_1 = 2 - \lambda$. By Theorem 3.2, we have the reverse form of (4.1)

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda}\,dx\,dy \geq C\left(\int_0^\infty x^{-\lambda-1-p\gamma} F^p(x)\,dx\right)^{\frac{1}{p}} \left(\int_0^\infty y^{-\lambda-1+q\gamma} G^q(y)\,dy\right)^{\frac{1}{q}}. \tag{4.4}$$

If we apply the reverse inequality of (1.8) to the first integral on the right-hand side in (4.4) and inequality (1.8) to the second integral ($q < 0$), we get

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda}\,dx\,dy > C_1\left(\int_0^\infty x^{p-\lambda-1-p\gamma} f^p(x)\,dx\right)^{\frac{1}{p}} \left(\int_0^\infty y^{q-\lambda-1+q\gamma} g^q(y)\,dy\right)^{\frac{1}{q}}. \tag{4.5}$$

Inequality (4.5) is equivalent to the reverse form of (1.2) if we set $\gamma = \frac{p-\lambda-pqA_1}{p}$ under the condition $pA_2 + qA_1 = 2 - \lambda$.

2. If $u(x) = e^x$, $v(y) = e^y$, $x, y \in (-\infty, \infty)$, we obtain by (3.1)

$$\int_{-\infty}^\infty \int_{-\infty}^\infty \frac{f(x)g(y)}{(e^x+e^y)^\lambda}\,dx\,dy \le C\left(\int_{-\infty}^\infty e^{-(\lambda+p\gamma)x} F^p(x)\,dx\right)^{\frac{1}{p}} \left(\int_{-\infty}^\infty e^{-(\lambda-q\gamma)y} G^q(y)\,dy\right)^{\frac{1}{q}}, \tag{4.6}$$

here $F(x) = \int_{-\infty}^x f(t)\,dt$ and $G(y) = \int_{-\infty}^y g(t)\,dt$. If we apply (1.9) to the integrals on the right-hand side of (4.6) and set $\gamma = \frac{p-\lambda-pqA_1}{p}$, we obtain (1.5). The reverse form of (4.6) is also valid, and we may obtain a reverse inequality of (1.5) if we use (1.9) and its reverse form.

3. If $u(x) = \ln x$, $v(y) = \ln y$, $x, y \in (1, \infty)$, then we have

$$\int_1^\infty \int_1^\infty \frac{f(x)g(y)}{(\ln x + \ln y)^\lambda}\,dx\,dy \le C\left(\int_1^\infty \frac{F^p(x)}{x[\ln x]^{\lambda+p\gamma+1}}\,dx\right)^{\frac{1}{p}} \left(\int_1^\infty \frac{G^q(y)}{y[\ln y]^{\lambda-q\gamma+1}}\,dy\right)^{\frac{1}{q}}, \tag{4.7}$$

here $F(x) = \int_1^x f(t)\,dt$ and $G(y) = \int_1^y g(t)\,dt$. In particular, for $\lambda = 1$, $\gamma = \frac{p-2}{p}$, we get

$$\int_1^\infty \int_1^\infty \frac{f(x)g(y)}{\ln x + \ln y}\,dx\,dy \le \frac{\pi}{pq \sin\frac{\pi}{p}} \left(\int_1^\infty \frac{F^p(x)}{x[\ln x]^p}\,dx\right)^{\frac{1}{p}} \left(\int_1^\infty \frac{G^q(y)}{y[\ln y]^q}\,dy\right)^{\frac{1}{q}};$$

if we apply (1.10), we get Hilbert-type inequality (1.6).

Competing interests
The author declares that he has no competing interests.

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