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Integral inequalities of Hermite-Hadamard type for functions whose third derivatives are convex

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Abstract
In the paper, the authors establish some new inequalities of Hermite-Hadamard type for functions whose third derivatives are convex.
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1 Introduction

It is common knowledge in mathematical analysis that a function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex on an interval I if the inequality

$$f\bigl(\lambda x + (1-\lambda)y\bigr) \le \lambda f(x) + (1-\lambda)f(y) \tag{1.1}$$

is valid for all $x, y \in I$ and $\lambda \in [0,1]$. If $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a convex function on I and $a, b \in I$ with $a < b$, then the double inequality

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \le \frac{f(a)+f(b)}{2} \tag{1.2}$$

holds. This double inequality is known in the literature as Hermite-Hadamard's integral inequality for convex functions. The definition of convex functions and Hermite-Hadamard's integral inequality (1.2) have been generalized, refined, and extended by many mathematicians in a lot of references. Some of them may be recited as follows.

Theorem 1.1 ([1, Theorems 2.2 and 2.3]) *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable on I° and $a, b \in I^\circ$ with $a < b$.*

(1) *If $|f'(x)|$ is a convex function on $[a,b]$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \le \frac{b-a}{8}\bigl(\bigl|f'(a)\bigr| + \bigl|f'(b)\bigr|\bigr). \tag{1.3}$$

(2) *If* $|f'(x)|^{p/(p-1)}$ *for* $p > 1$ *is a convex function on* $[a, b]$, *then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \le \frac{b-a}{2(p+1)^{1/p}}\left[\frac{|f'(a)|^{p/(p-1)} + |f'(b)|^{p/(p-1)}}{2}\right]^{(p-1)/p}. \tag{1.4}$$

Theorem 1.2 ([2, Theorems 2.2 and 2.3]) *Let* $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ *be a differentiable mapping on* I° *and* $a, b \in I^\circ$ *with* $a < b$. *If* $|f|^{p/(p-1)}$ *for* $p > 1$ *is convex on* $[a, b]$, *then*

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \le \frac{b-a}{16}\left(\frac{4}{p+1}\right)^{1/p}\Big[\big(\left|f'(a)\right|^{p/(p-1)} + 3\left|f'(b)\right|^{p/(p-1)}\big)^{(p-1)/p} + \big(3\left|f'(a)\right|^{p/(p-1)} + \left|f'(b)\right|^{p/(p-1)}\big)^{(p-1)/p}\Big] \tag{1.5}$$

and

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \le \frac{b-a}{4}\left(\frac{4}{p+1}\right)^{1/p}\big[\left|f'(a)\right| + \left|f'(b)\right|\big]. \tag{1.6}$$

Definition 1.1 ([3]) A function $f : I \subseteq \mathbb{R} \to [0, \infty)$ is said to be quasi-convex if

$$f\big(\lambda x + (1-\lambda)y\big) \le \sup\big\{f(x), f(y)\big\} \tag{1.7}$$

holds for all $x, y \in I$ and $\lambda \in [0, 1]$.

Theorem 1.3 ([4, Theorem 2]) *Let* $f : I \subseteq \mathbb{R} \to \mathbb{R}$ *be differentiable on* I° *such that* $f''' \in L([a, b])$ *and* $a, b \in I^\circ$ *with* $a < b$. *If* $|f'''|$ *is quasi-convex on* $[a, b]$, *then*

$$\left|\int_a^b f(x)\,\mathrm{d}x - \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]\right| \le \frac{(b-a)^4}{1152}\left[\max\left\{\left|f'''(a)\right|, \left|f'''\left(\frac{a+b}{2}\right)\right|\right\} + \max\left\{\left|f'''\left(\frac{a+b}{2}\right)\right|, \left|f'''(b)\right|\right\}\right]. \tag{1.8}$$

Definition 1.2 ([5]) Let $s \in (0, 1]$. A function $f : \mathbb{R}_0 \to \mathbb{R}_0$ is said to be s-convex in the second sense if

$$f\big(\lambda x + (1-\lambda)y\big) \le \lambda^s f(x) + (1-\lambda)^s f(y) \tag{1.9}$$

for all $x, y \in I$ and $\lambda \in [0, 1]$.

Theorem 1.4 ([6, Theorem 3.1]) *Let* $f : I \subseteq \mathbb{R}_0 \to \mathbb{R}$ *be differentiable on* I°, $a, b \in I^\circ$ *with* $a < b$, *and* $f''' \in L([a, b])$. *If* $q \ge 1$ *and* $|f'''|$ *is s-convex in the second sense on* $[a, b]$ *for* $s \in (0, 1]$, *then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x - \frac{b-a}{12}\big[f'(b) - f'(a)\big]\right| \le \frac{(b-a)^3}{192}\left[\frac{2^{2-s}(s+6+2^{s+2}s)}{(s+2)(s+3)(s+4)}\right]^{1/q}\big[\left|f'''(a)\right|^q + \left|f'''(b)\right|^q\big]^{1/q}. \tag{1.10}$$

For more information on Hermite-Hadamard type inequalities, please refer to [7–19], for example, and to monographs [20, 21] and related references therein.

In this paper, we will create some new integral inequalities of Hermite-Hadamard type for functions whose third derivatives are convex.

2 Lemma

For establishing some new integral inequalities of Hermite-Hadamard type for functions whose third derivatives are convex, we need an integral identity below.

Lemma 2.1 *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a three times differentiable mapping on I° and $a, b \in I^\circ$ with $a < b$. If $f''' \in L([a, b])$, then*

$$\begin{aligned} & f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x + \frac{b-a}{24}\left[f'(b) - f'(a)\right] \\ &\quad = \frac{(b-a)^3}{96}\left[\int_0^1 t(1-t)(2-t)f'''\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right)\mathrm{d}t \right. \\ &\qquad \left. - \int_0^1 t(1-t)(2-t)f'''\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)\mathrm{d}t\right]. \end{aligned} \tag{2.1}$$

Proof Integrating by part and changing variable of definite integral yield

$$\begin{aligned} & \int_0^1 t(1-t)(2-t)f'''\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right)\mathrm{d}t \\ &\quad = -\frac{2}{b-a}\int_0^1 \left(3t^2 - 6t + 2\right)f''\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right)\mathrm{d}t \\ &\quad = \frac{4}{(b-a)^2}\left[f'(b) + 2f'\left(\frac{a+b}{2}\right)\right] + \frac{48}{(b-a)^3}\int_0^1 (t-1)\,\mathrm{d}f\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right) \\ &\quad = \frac{4}{(b-a)^2}\left[f'(b) + 2f'\left(\frac{a+b}{2}\right)\right] + \frac{48}{(b-a)^3}f\left(\frac{a+b}{2}\right) \\ &\qquad - \frac{48}{(b-a)^3}\int_0^1 f\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right)\mathrm{d}t \end{aligned}$$

and

$$\begin{aligned} & \int_0^1 t(1-t)(2-t)f'''\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)\mathrm{d}t \\ &\quad = \frac{2}{b-a}\int_0^1 \left(3t^2 - 6t + 2\right)f''\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)\mathrm{d}t \\ &\quad = \frac{4}{(b-a)^2}\left[f'(a) + 2f'\left(\frac{a+b}{2}\right)\right] - \frac{48}{(b-a)^3}\int_0^1 (t-1)\,\mathrm{d}f\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right) \\ &\quad = \frac{4}{(b-a)^2}\left[f'(a) + 2f'\left(\frac{a+b}{2}\right)\right] - \frac{48}{(b-a)^3}f\left(\frac{a+b}{2}\right) \\ &\qquad + \frac{48}{(b-a)^3}\int_0^1 f\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)\mathrm{d}t. \end{aligned}$$

Lemma 2.1 is thus proved. □

3 Hermite-Hadamard type inequalities for convex functions

Basing on Lemma 2.1, we now start out to establish some new integral inequalities of Hermite-Hadamard type for functions whose third derivatives are convex.

Theorem 3.1 *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be three times differentiable on I° and $f''' \in L([a,b])$ for $a, b \in I^\circ$ with $a < b$. If $|f'''|^q$ for $q \geq 1$ is convex on $[a,b]$, then*

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x + \frac{b-a}{24}\left[f'(b) - f'(a)\right]\right| \leq \frac{(b-a)^3}{384}\left\{\left[\frac{4|f'''(a)|^q + 11|f'''(b)|^q}{15}\right]^{1/q} + \left[\frac{11|f'''(a)|^q + 4|f'''(b)|^q}{15}\right]^{1/q}\right\}. \tag{3.1}$$

Proof Since $|f'''|^q$ is convex on $[a,b]$, by Lemma 2.1 and Hölder's inequality, we have

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x + \frac{b-a}{24}\left[f'(b) - f'(a)\right]\right| \\
&\quad\leq \frac{(b-a)^3}{96}\left\{\int_0^1 t(1-t)(2-t)\left|f'''\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right)\right|\mathrm{d}t\right. \\
&\qquad\left. + \int_0^1 t(1-t)(2-t)\left|f'''\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)\right|\mathrm{d}t\right\} \\
&\quad\leq \frac{(b-a)^3}{96}\left[\int_0^1 t(1-t)(2-t)\,\mathrm{d}t\right]^{1-1/q} \\
&\qquad\times\left\{\left[\int_0^1 t(1-t)(2-t)\left|f'''\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right)\right|^q\mathrm{d}t\right]^{1/q}\right. \\
&\qquad\left. + \left[\int_0^1 t(1-t)(2-t)\left|f'''\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)\right|^q\mathrm{d}t\right]^{1/q}\right\} \\
&\quad\leq \frac{(b-a)^3}{96}\left(\frac{1}{4}\right)^{1-1/q}\left\{\left[\frac{1}{2}\int_0^1 t(1-t)^2(2-t)\left|f'''(a)\right|^q\mathrm{d}t\right.\right. \\
&\qquad\left. + \frac{1}{2}\int_0^1 t\left(1-t^2\right)(2-t)\left|f'''(b)\right|^q\mathrm{d}t\right]^{1/q} + \left[\frac{1}{2}\int_0^1 t\left(1-t^2\right)(2-t)\left|f'''(a)\right|^q\mathrm{d}t\right. \\
&\qquad\left.\left. + \frac{1}{2}\int_0^1 t(1-t)^2(2-t)\left|f'''(b)\right|^q\mathrm{d}t\right]^{1/q}\right\} \\
&\quad= \frac{(b-a)^3}{384}\left\{\left[\frac{4|f'''(a)|^q + 11|f'''(b)|^q}{15}\right]^{1/q} + \left[\frac{11|f'''(a)|^q + 4|f'''(b)|^q}{15}\right]^{1/q}\right\}.
\end{aligned}$$

The proof of Theorem 3.1 is complete. □

Corollary 3.1 *Under conditions of Theorem* 3.1, *if $q = 1$, we have*

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x + \frac{b-a}{24}\left[f'(b) - f'(a)\right]\right| \leq \frac{(b-a)^3}{384}\left[\left|f'''(a)\right| + \left|f'''(b)\right|\right]. \tag{3.2}$$

Theorem 3.2 *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be three times differentiable on I° and $f''' \in L([a,b])$ for $a, b \in I^\circ$ with $a < b$. If $|f'''|^q$ for $q > 1$ is convex on $[a,b]$ and if $q \geq r$ and $s \geq 0$, then*

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x + \frac{b-a}{24}\left[f'(b) - f'(a)\right] \right| \\
&\quad \leq \frac{(b-a)^3}{96}\left[2B\left(\frac{2q-r-1}{q-1}, \frac{2q-s-1}{q-1}\right) - B\left(\frac{3q-r-2}{q-1}, \frac{2q-s-1}{q-1}\right)\right]^{1-1/q} \\
&\qquad \times \left(\frac{1}{2}\right)^{1/q} \Big\{\Big(\big[2B(r+1,s+2) - B(r+2,s+2)\big]\big|f'''(a)\big|^q \\
&\qquad + \big[2B(r+1,s+1) + B(r+2,s+1) - B(r+3,s+1)\big]\big|f'''(b)\big|^q\Big)^{1/q} \\
&\qquad + \Big(\big[2B(r+1,s+1) + B(r+2,s+1) - B(r+3,s+1)\big]\big|f'''(a)\big|^q \\
&\qquad + \big[2B(r+1,s+2) - B(r+2,s+2)\big]\big|f'''(b)\big|^q\Big)^{1/q}\Big\},
\end{aligned}$$

where $B(x,y)$ is the classical Beta function, which may be defined for $\Re\mathfrak{e}(x) > 0$ and $\Re\mathfrak{e}(y) > 0$ by

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,\mathrm{d}t. \tag{3.3}$$

Proof By Lemma 2.1, Hölder's inequality, and the convexity of $|f'''|^q$ on $[a,b]$, we have

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x + \frac{b-a}{24}\left[f'(b) - f'(a)\right] \right| \\
&\quad \leq \frac{(b-a)^3}{96}\left\{\int_0^1 t(1-t)(2-t)\left|f'''\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right)\right|\mathrm{d}t \right. \\
&\qquad \left. + \int_0^1 t(1-t)(2-t)\left|f'''\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)\right|\mathrm{d}t\right\} \\
&\quad \leq \frac{(b-a)^3}{96}\left(\int_0^1 t^{(q-r)/(q-1)}(1-t)^{(q-s)/(q-1)}(2-t)\,\mathrm{d}t\right)^{1-1/q} \\
&\qquad \times \left\{\left(\int_0^1 t^r(1-t)^s(2-t)\left|f'''\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right)\right|^q \mathrm{d}t\right)^{1/q} \right. \\
&\qquad \left. + \left(\int_0^1 t^r(1-t)^s(2-t)\left|f'''\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)\right|^q \mathrm{d}t\right)^{1/q}\right\} \\
&\quad \leq \frac{(b-a)^3}{96}\left[2B\left(\frac{2q-r-1}{q-1}, \frac{2q-s-1}{q-1}\right) - B\left(\frac{3q-r-2}{q-1}, \frac{2q-s-1}{q-1}\right)\right]^{1-1/q} \\
&\qquad \times \left\{\left[\frac{1}{2}\int_0^1 t^r(1-t)^{s+1}(2-t)\big|f'''(a)\big|^q\,\mathrm{d}t \right.\right. \\
&\qquad \left. + \frac{1}{2}\int_0^1 t^r(1+t)(1-t)^s(2-t)\big|f'''(b)\big|^q\,\mathrm{d}t\right]^{1/q} \\
&\qquad + \left[\frac{1}{2}\int_0^1 t^r(1+t)(1-t)^s(2-t)\big|f'''(a)\big|^q\,\mathrm{d}t \right. \\
&\qquad \left.\left. + \frac{1}{2}\int_0^1 t^r(1-t)^{s+1}(2-t)\big|f'''(b)\big|^q\,\mathrm{d}t\right]^{1/q}\right\}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{(b-a)^3}{96}\left[2B\left(\frac{2q-r-1}{q-1}, \frac{2q-s-1}{q-1}\right) - B\left(\frac{3q-r-2}{q-1}, \frac{2q-s-1}{q-1}\right)\right]^{1-1/q} \\
&\quad \times \left(\frac{1}{2}\right)^{1/q} \Big\{\big[\big[2B(r+1,s+2) - B(r+2,s+2)\big]\big|f'''(a)\big|^q \\
&\quad + \big[2B(r+1,s+1) + B(r+2,s+1) - B(r+3,s+1)\big]\big|f'''(b)\big|^q\big]^{1/q} \\
&\quad + \big[\big[2B(r+1,s+1) + B(r+2,s+1) - B(r+3,s+1)\big]\big|f'''(a)\big|^q \\
&\quad + \big[2B(r+1,s+2) - B(r+2,s+2)\big]\big|f'''(b)\big|^q\big]^{1/q}\Big\}.
\end{aligned}
$$

The proof of Theorem 3.2 is completed. □

Corollary 3.2 *Under conditions of Theorem* 3.2,

(1) *if* $r = 0$, *we have*

$$
\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x + \frac{b-a}{24}\big[f'(b) - f'(a)\big]\right| \\
&\quad \le \frac{(b-a)^3}{96}\left[2B\left(\frac{2q-1}{q-1}, \frac{2q-s-1}{q-1}\right)\right. \\
&\quad \left. - B\left(\frac{3q-2}{q-1}, \frac{2q-s-1}{q-1}\right)\right]^{1-1/q}\left(\frac{1}{2(s+1)(s+2)(s+3)}\right)^{1/q} \\
&\quad \times \Big\{\big[(s+1)(2s+5)\big|f'''(a)\big|^q + \big(2s^2+11s+13\big)\big|f'''(b)\big|^q\big]^{1/q} \\
&\quad + \big[\big(2s^2+11s+13\big)\big|f'''(a)\big|^q + (s+1)(2s+5)\big|f'''(b)\big|^q\big]^{1/q}\Big\};
\end{aligned}
$$

(2) *if* $s = 0$, *we have*

$$
\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x + \frac{b-a}{24}\big[f'(b) - f'(a)\big]\right| \\
&\quad \le \frac{(b-a)^3}{96}\left[2B\left(\frac{2q-r-1}{q-1}, \frac{2q-1}{q-1}\right)\right. \\
&\quad \left. - B\left(\frac{3q-r-2}{q-1}, \frac{2q-1}{q-1}\right)\right]^{1-1/q}\left[\frac{1}{2(r+1)(r+2)(r+3)}\right]^{1/q} \\
&\quad \times \Big\{\big[(r+5)\big|f'''(a)\big|^q + \big(2r^2+11r+13\big)\big|f'''(b)\big|^q\big]^{1/q} \\
&\quad + \big[\big(2r^2+11r+13\big)\big|f'''(a)\big|^q + (r+5)\big|f'''(b)\big|^q\big]^{1/q}\Big\};
\end{aligned}
$$

(3) *if* $r = s = 0$, *we have*

$$
\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x + \frac{b-a}{24}\big[f'(b) - f'(a)\big]\right| \\
&\quad \le \frac{(b-a)^3}{96}\left[2B\left(\frac{2q-1}{q-1}, \frac{2q-1}{q-1}\right) - B\left(\frac{3q-2}{q-1}, \frac{2q-1}{q-1}\right)\right]^{1-1/q}\left(\frac{3}{2}\right)^{1/q} \\
&\quad \times \left\{\left[\frac{5|f'''(a)|^q + 13|f'''(b)|^q}{18}\right]^{1/q} + \left[\frac{13|f'''(a)|^q + 5|f'''(b)|^q}{18}\right]^{1/q}\right\};
\end{aligned}
$$

(4) *if* $r = q$, *we have*

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x + \frac{b-a}{24}\left[f'(b)-f'(a)\right]\right| \\
&\quad \le \frac{(b-a)^3}{96}\left[\frac{(5q-2s-3)(q-1)}{(q-s)^2+(5q-3s-2)(q-1)}\right]^{1-1/q}\left(\frac{1}{2}\right)^{1/q} \\
&\qquad \times \Big\{\Big[\big[2B(q+1,s+2)-B(q+2,s+2)\big]\big|f'''(a)\big|^q \\
&\qquad + \big[2B(q+1,s+1)+B(q+2,s+1)-B(q+3,s+1)\big]\big|f'''(b)\big|^q\Big]^{1/q} \\
&\qquad + \Big[\big[2B(q+1,s+1)+B(q+2,s+1)-B(q+3,s+1)\big]\big|f'''(a)\big|^q \\
&\qquad + \big[2B(q+1,s+2)-B(q+2,s+2)\big]\big|f'''(b)\big|^q\Big]^{1/q}\Big\};
\end{aligned}$$

(5) *if* $s = q$, *we have*

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x + \frac{b-a}{24}\left[f'(b)-f'(a)\right]\right| \\
&\quad \le \frac{(b-a)^3}{96}\left(\frac{(q-1)(4q-r-3)}{(2q-r-1)(3q-r-2)}\right)^{1-1/q}\left(\frac{1}{2}\right)^{1/q} \\
&\qquad \times \Big\{\Big[\big[2B(r+1,q+2)-B(r+2,q+2)\big]\big|f'''(a)\big|^q \\
&\qquad + \big[2B(r+1,q+1)+B(r+2,q+1)-B(r+3,q+1)\big]\big|f'''(b)\big|^q\Big]^{1/q} \\
&\qquad + \Big[\big[2B(r+1,q+1)+B(r+2,q+1)-B(r+3,q+1)\big]\big|f'''(a)\big|^q \\
&\qquad + \big[2B(r+1,q+2)-B(r+2,q+2)\big]\big|f'''(b)\big|^q\Big]^{1/q}\Big\};
\end{aligned}$$

(6) *if* $r = s = q$, *we have*

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x + \frac{b-a}{24}\left[f'(b)-f'(a)\right]\right| \\
&\quad \le \frac{3^{-1/q}(b-a)^3}{64}\Big\{\Big[\big[2B(q+1,q+2)-B(q+2,q+2)\big]\big|f'''(a)\big|^q \\
&\qquad + \big[2B(q+1,q+1)+B(q+2,q+1)-B(q+3,q+1)\big]\big|f'''(b)\big|^q\Big]^{1/q} \\
&\qquad + \Big[\big[2B(q+1,q+1)+B(q+2,q+1)-B(q+3,q+1)\big]\big|f'''(a)\big|^q \\
&\qquad + \big[2B(q+1,q+2)-B(q+2,q+2)\big]\big|f'''(b)\big|^q\Big]^{1/q}\Big\}.
\end{aligned}$$

Theorem 3.3 *Let* $f : I \subseteq \mathbb{R} \to \mathbb{R}$ *be three times differentiable on* I° *and* $f''' \in L([a,b])$ *for* $a, b \in I^\circ$ *with* $a < b$. *If* $|f'''|^q$ *is convex on* $[a,b]$ *for* $q > 1$ *and* $q \ge \ell \ge 0$, *then*

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x + \frac{b-a}{24}\left[f'(b)-f'(a)\right]\right| \\
&\quad \le \frac{(b-a)^3}{96}\left[\frac{(2^{\xi+2}+1)\xi-2^{\xi+2}+5}{\xi^3+6\xi^2+11\xi+6}\right]^{1-1/q}\left[\frac{1}{(\ell+1)(\ell+2)(\ell+3)(\ell+4)}\right]^{1/q} \\
&\qquad \times \Big\{\Big[\big(2^{\ell+1}\ell^2-\big(2^{\ell+1}+1\big)\ell+2^{\ell+3}-7\big)\big|f'''(a)\big|^q \\
&\qquad + \big(\big(2^{\ell+1}+1\big)\ell^2+2\big(7\times 2^\ell+5\big)\ell-3\times 2^{\ell+3}+27\big)\big|f'''(b)\big|^q\Big]^{1/q}
\end{aligned}$$

$$+ \big[\big((2^{\ell+1}+1)\ell^2 + 2(7\times 2^\ell + 5)\ell - 3\times 2^{\ell+3} + 27\big)\big|f'''(a)\big|^q + \big(2^{\ell+1}\ell^2 - (2^{\ell+1}+1)\ell + 2^{\ell+3} - 7\big)\big|f'''(b)\big|^q\big]^{1/q}\Big\},$$

where $\xi = \frac{q-\ell}{q-1}$.

Proof By Lemma 2.1, Hölder's inequality, and the convexity of $|f'''|^q$ on $[a,b]$, we have

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x + \frac{b-a}{24}\big[f'(b)-f'(a)\big]\right| \\
&\le \frac{(b-a)^3}{96}\left\{\int_0^1 t(1-t)(2-t)\left|f'''\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right)\right|\mathrm{d}t \right.\\
&\quad \left. + \int_0^1 t(1-t)(2-t)\left|f'''\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)\right|\mathrm{d}t\right\} \\
&\le \frac{(b-a)^3}{96}\left(\int_0^1 t(1-t)(2-t)^{(q-\ell)/(q-1)}\,\mathrm{d}t\right)^{1-1/q} \\
&\quad \times \left\{\left(\int_0^1 t(1-t)(2-t)^\ell\left|f'''\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right)\right|^q\mathrm{d}t\right)^{1/q} \right.\\
&\quad \left. + \left(\int_0^1 t(1-t)(2-t)^\ell\left|f'''\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)\right|^q\mathrm{d}t\right)^{1/q}\right\} \\
&\le \frac{(b-a)^3}{96}\left(\int_0^1 t(1-t)(2-t)^{(q-\ell)/(q-1)}\,\mathrm{d}t\right)^{1-1/q} \\
&\quad \times \left\{\left[\frac{1}{2}\int_0^1 t(1-t)(2-t)^\ell\big[(1-t)\big|f'''(a)\big|^q + (1+t)\big|f'''(b)\big|^q\big]\,\mathrm{d}t\right]^{1/q}\right.\\
&\quad \left. + \left[\frac{1}{2}\int_0^1 t(1-t)(2-t)^\ell\big[(1+t)\big|f'''(a)\big|^q + (1-t)\big|f'''(b)\big|^q\big]\,\mathrm{d}t\right]^{1/q}\right\} \\
&= \frac{(b-a)^3}{96}\left[\frac{(2^{\xi+2}+1)\xi - 2^{\xi+2} + 5}{\xi^3 + 6\xi^2 + 11\xi + 6}\right]^{1-1/q}\left[\frac{1}{(\ell+1)(\ell+2)(\ell+3)(\ell+4)}\right]^{1/q} \\
&\quad \times \Big\{\big[\big(2^{\ell+1}\ell^2 - (2^{\ell+1}+1)\ell + 2^{\ell+3} - 7\big)\big|f'''(a)\big|^q \\
&\quad + \big((2^{\ell+1}+1)\ell^2 + 2(7\times 2^\ell + 5)\ell - 3\times 2^{\ell+3} + 27\big)\big|f'''(b)\big|^q\big]^{1/q} \\
&\quad + \big[\big((2^{\ell+1}+1)\ell^2 + 2(7\times 2^\ell + 5)\ell - 3\times 2^{\ell+3} + 27\big)\big|f'''(a)\big|^q \\
&\quad + \big(2^{\ell+1}\ell^2 - (2^{\ell+1}+1)\ell + 2^{\ell+3} - 7\big)\big|f'''(b)\big|^q\big]^{1/q}\Big\}.
\end{aligned}$$

The proof of Theorem 3.3 is complete. □

Corollary 3.3 *Under conditions of Theorem* 3.3.

(1) *if* $\ell = 0$, *we have*

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x + \frac{b-a}{24}\big[f'(b)-f'(a)\big]\right| \\
&\quad \le \frac{(b-a)^3}{96}\left(\frac{1}{6}\right)^{1/q}\left[\frac{(q-1)^2(6q + 2^{(3q-2)/(q-1)} - 5)}{(2q-1)(3q-2)(4q-3)}\right]^{1-1/q} \\
&\quad \times \left\{\left[\frac{|f'''(a)|^q + 3|f'''(b)|^q}{4}\right]^{1/q} + \left[\frac{3|f'''(a)|^q + |f'''(b)|^q}{4}\right]^{1/q}\right\}, \qquad (3.4)
\end{aligned}$$

(2) *if* $\ell = q$, *we have*

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x+\frac{b-a}{24}\left[f'(b)-f'(a)\right]\right| \\
&\quad\le \frac{(b-a)^3}{96}\left(\frac{1}{6}\right)^{1-1/q}\left[\frac{1}{(q+1)(q+2)(q+3)(q+4)}\right]^{1/q} \\
&\qquad\times\Big\{\big[\big(2^{q+1}q^2-\big(2^{q+1}+1\big)q+2^{q+3}-7\big)\big|f'''(a)\big|^q \\
&\qquad+\big(\big(2^{q+1}+1\big)q^2+2\big(7\times 2^q+5\big)q-3\times 2^{q+3}+27\big)\big|f'''(b)\big|^q\big]^{1/q} \\
&\qquad+\big[\big(\big(2^{q+1}+1\big)q^2+2\big(7\times 2^q+5\big)q-3\times 2^{q+3}+27\big)\big|f'''(a)\big|^q \\
&\qquad+\big(2^{q+1}q^2-\big(2^{q+1}+1\big)q+2^{q+3}-7\big)\big|f'''(b)\big|^q\big]^{1/q}\Big\}.
\end{aligned}$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript and read and approved the final manuscript.

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