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On certain subclasses of multivalent functions defined by multiplier transformations

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Dedicated to Professor Hari M Srivastava

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Abstract
The purpose of the present paper is to introduce and investigate various properties of a certain class of multivalent functions in the open unit disk defined by a multiplier transformation. In particular, we obtain some inclusion relationships and integral preserving properties of this class of functions. Relevant connections of the results presented in this paper with various known results are also pointed out.
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1 Introduction

Let $\mathcal{A}_p$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k} \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. We write $\mathcal{A}_1 = \mathcal{A}$.

Suppose that f and g are analytic in $\mathbb{U}$. We say that the function f is subordinate to g in $\mathbb{U}$, and we write $f \prec g$ or $f(z) \prec g(z)$ $(z \in \mathbb{U})$, if there exists an analytic function ω in $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ for all $z \in \mathbb{U}$ such that $f(z) = g(\omega(z))$ in $\mathbb{U}$. If g is univalent in $\mathbb{U}$, then the following equivalence relationship holds true:

$$f(z) \prec g(z) \quad \Longleftrightarrow \quad f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

For functions f given by (1.1) and $g(z) = z^p + \sum_{k=1}^{\infty} b_{p+k} z^{p+k}$, the Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) = f(z) * g(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} b_{p+k} z^{p+k}.$$

For fixed parameters A, B $(-1 \leqq B < A \leqq 1)$, let $\mathcal{P}(A, B)$ be the class of functions of the form

$$\varphi(z) = 1 + c_1 z + c_2 z^2 + \cdots \tag{1.2}$$

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which are analytic in $\mathbb{U}$ and satisfy the condition

$$\varphi(z) \prec \frac{1+Az}{1+Bz} \quad (z \in \mathbb{U}).$$

The class $\mathcal{P}(A,B)$ was investigated in [1]. We denote by $\mathcal{S}_p^*(A,B)$ the class of functions $f \in \mathcal{A}_p$ such that $zf'/pf \in \mathcal{P}(A,B)$. Analogously, $\mathcal{K}(A,B)$ is the class of functions $f \in \mathcal{A}_p$ such that $(zf')'/pf' \in \mathcal{P}(A,B)$. It is easily seen that

$$\mathcal{S}_p^*\left(1-\frac{2\rho}{p},-1\right) = \mathcal{S}_p^*(\rho) \quad \text{and} \quad \mathcal{K}_p\left(1-\frac{2\rho}{p},-1\right) = \mathcal{K}_p(\rho) \quad (0 \leqq \rho < p),$$

the subclasses of $\mathcal{A}_p$, which are respectively, p-valently starlike of order ρ and p-valently convex of order ρ in $\mathbb{U}$. We also note that

$$\mathcal{S}_p^*(\rho) \subseteq \mathcal{S}_p^*(0) = \mathcal{S}_p^* \quad \text{and} \quad \mathcal{K}_p(\rho) \subseteq \mathcal{K}_p(0) = \mathcal{K}_p \quad (0 \leqq \rho < p),$$

where $\mathcal{S}_p^*$ and $\mathcal{K}_p$ are the subclasses of $\mathcal{A}_p$ consisting of functions that are p-valently starlike and p-valently convex in $\mathbb{U}$, respectively.

In the present investigation, we shall make use of the *Gauss hypergeometric function* ${}_2F_1$ defined in $\mathbb{U}$ by

$$ {}_2F_1(a,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!} \quad \bigl(a,b,c \in \mathbb{C}; c \notin \mathbb{Z}_0^- = \{0,-1,-2,\ldots\}\bigr), \tag{1.3}$$

where $(x)_n$ denotes the Pochhammer symbol (or shifted factorial) given by

$$(x)_n = \begin{cases} x(x+1)(x+2)\cdots(x+n-1) & (n \in \mathbb{N}), \\ 1 & (n = 0). \end{cases}$$

We note that the series defined by (1.3) converges absolutely for all $z \in \mathbb{U}$ and hence represents an analytic function in $\mathbb{U}$ [2, Chapter 14].

Motivated by the multiplier transformation introduced in [3] on $\mathcal{A}$, we introduce an operator $\phi_p(n,\lambda)$ on $\mathcal{A}_p$ by

$$\phi_p(n,\lambda)(z) = z^p + \sum_{k=1}^{\infty}\left(\frac{p+k+\lambda}{p+\lambda}\right) z^{p+k} \quad \bigl(\lambda > -p, n \in \mathbb{Z} = \{0,\pm 1,\pm 2,\ldots\}; z \in \mathbb{U}\bigr).$$

The operator $\phi_p(n,\lambda)$ is related to the multiplier transformation studied in [4].

Corresponding to the function $\phi_p(n,\lambda)$, we define a new function $\phi_p^{(\dagger)}(n,\lambda)$ in terms of the Hadamard product by

$$\phi_p(n,\lambda)(z) * \phi_p^{(\dagger)}(n,\lambda)(z) = \frac{z^p}{(1-z)^{p+\mu}} \quad (\mu > -p; z \in \mathbb{U}). \tag{1.4}$$

We now introduce the operator $I_p^n(\lambda,\mu) : \mathcal{A}_p \longrightarrow \mathcal{A}_p$ by

$$I_p^n(\lambda,\mu)f(z) = \phi_p^{(\dagger)}(n,\lambda)(z) * f(z) \quad (n \in \mathbb{Z}; \lambda,\mu > -p). \tag{1.5}$$

If the function f is given by (1.1), then from (1.4) and (1.5) we deduce that

$$I_p^n(\lambda,\mu)f(z) = z^p + \sum_{k=1}^{\infty} \frac{(p+\mu)_k}{(1)_k}\left(\frac{p+\lambda}{p+k+\lambda}\right)^n a_{p+k} z^{p+k} \quad (z \in \mathbb{U}).$$

In view of (1.5), it follows that

$$z\big(I_p^n(\lambda,\mu)f\big)'(z) = (p+\lambda)I_p^{n-1}(\lambda,\mu)f(z) - \lambda I_p^n(\lambda,\mu)f(z) \quad (f \in \mathcal{A}_p; z \in \mathbb{U}). \tag{1.6}$$

In particular, we note that for $z \in \mathbb{U}$,

$$\begin{aligned}
&I_p^0(0,1-p)f(z) = f(z),\\
&I_p^1(\delta,1-p)f(z) = \left((p+\delta)\int_0^z t^{\delta-1}f(t)\,dt\right)\Big/ z^\delta \quad (\delta > -p) \;[\textit{cf.}\text{ Eqn. (3.10)}],\\
&I_p^{-1}(\lambda,1-p)f(z) = \big(zf'(z) + \lambda f(z)\big)/(p+\lambda),\\
&I_p^{-2}(0,1-p)f(z) = \big(z^2f''(z) + zf'(z)\big)/p^2 \quad \text{and}\\
&I_p^{-3}(0,1-p)f(z) = \big(z^3f'''(z) + 3z^2f''(z) + zf'(z)\big)/p^3.
\end{aligned}$$

The operator $I_p^n(\lambda,1-p)$ $(n \in \mathbb{Z}_0^-)$ is closely related to the Sălăgean derivative operator [5]. The operator $I_\lambda^n = I_1^n(\lambda,0)$ was recently studied in [3, 6, 7]. For any $n \in \mathbb{Z}$, the operator $I_n = I_1^n(1,0)$ was studied in [8].

By using the operator $I_p^n(\lambda,\mu)$, we introduce the subclass of $\mathcal{A}_p$ as follows.

Definition For fixed parameters A, B $(-1 \leqq B < A \leqq 1)$, $n \in \mathbb{Z}$, $\lambda, \mu > -p$ and $\alpha \geqq 0$, we say that a function $f \in \mathcal{A}_p$ is in the class $\mathcal{S}_{p,\lambda,\mu}^n(\alpha; A, B)$ if

$$(1-\alpha)\frac{I_p^n(\lambda,\mu)f(z)}{I_p^{n+1}(\lambda,\mu)f(z)} + \alpha\frac{I_p^{n-1}(\lambda,\mu)f(z)}{I_p^n(\lambda,\mu)f(z)} \prec \frac{1+Az}{1+Bz} \quad (z \in \mathbb{U}).$$

It is readily seen that

$$\mathcal{S}_{p,0,1-p}^0(1;A,B) = \mathcal{S}_p^*(A,B) \quad \text{and} \quad \mathcal{S}_{p,0,1-p}^{-1}(1;A,B) = \mathcal{K}_p(A,B).$$

For the sake of convenience, we write

$$\mathcal{S}_{p,\lambda,\mu}^n(A,B) = \mathcal{S}_{p,\lambda,\mu}^n(1;A,B) = \left\{f \in \mathcal{A}_p : \frac{I_p^n(\lambda,\mu)f(z)}{I_p^{n+1}(\lambda,\mu)f(z)} \prec \frac{1+Az}{1+Bz}, z \in \mathbb{U}\right\}.$$

The object of the present paper is to investigate some inclusion properties of the class $\mathcal{S}_{p,\lambda,\mu}^n(\alpha;A,B)$. Integral-preserving and convolution properties in connection with the operator $I_p^n(\lambda,\mu)$ are also considered. Relevant connections of the results presented here with those obtained in the earlier works are pointed out.

2 Preliminaries

We denote by $\mathcal{H}$ the class of all analytic functions in $\mathbb{U}$ and by $\mathcal{B}$ the class of functions $\Omega \in \mathcal{H}$ such that $\omega(0) = 0$ and $|\Omega(z)| < 1$ for $z \in \mathbb{U}$.

We shall need the following lemmas to prove our results.

Lemma 1 ([9], see also [10, p.71]) *Let h be analytic and convex (univalent) in $\mathbb{U}$ with $h(0)=1$. Suppose also that the function φ defined by* (1.2) *is analytic in $\mathbb{U}$. If*

$$\varphi(z)+\frac{z\varphi'(z)}{\kappa}\prec h(z)\quad\bigl(\kappa\neq 0,\Re(\kappa)\geqq 0;z\in\mathbb{U}\bigr),$$

then

$$\varphi(z)\prec\psi(z)=\frac{\kappa}{z^{\kappa}}\int_0^z t^{\kappa-1}h(t)\,dt\prec h(z)\quad(z\in\mathbb{U})\tag{2.1}$$

and ψ is the best dominant of (2.1).

Lemma 2 [10, p.35] *Suppose that the function $\Psi:\mathbb{C}^2\times\mathbb{U}\longrightarrow\mathbb{C}$ satisfies the condition*

$$\Psi(ix,y;z)\leqq\varepsilon$$

for $\varepsilon>0$, real $x,y\leqq-(1+x^2)/2$ and all $z\in\mathbb{U}$. If the function φ, given by (1.2) *is analytic in $\mathbb{U}$ and*

$$\mathrm{Re}\bigl(\Psi\bigl(\varphi(z),z\varphi'(z);z\bigr)\bigr)>\varepsilon,$$

then $\mathrm{Re}(\varphi(z))>0$ *in* $\mathbb{U}$.

Lemma 3 [11] *Let $0<\gamma_1<\gamma<1$ and $\Theta\in\mathcal{H}$ satisfy*

$$\Theta(z)\prec 1+\gamma_1 z,\qquad\Theta(0)=1.$$

(i) *If $\varphi\in\mathcal{H}$, $\varphi(0)=1$ and satisfies*

$$\Theta(z)\bigl(\beta+(1-\beta)\varphi(z)\bigr)\prec 1+\gamma z\quad(z\in\mathbb{U}),$$

where

$$\beta=\begin{cases}\frac{1-\gamma}{1+\gamma_1} & (0<\gamma_1+\gamma\leqq 1),\\ \frac{1-(\gamma_1^2+\gamma^2)}{2(1-\gamma_1^2)} & (\gamma_1^2+\gamma^2\leqq 1\leqq\gamma_1+\gamma),\end{cases}\tag{2.2}$$

then $\mathrm{Re}(\varphi(z))>0$ *in* $\mathbb{U}$.

(ii) *If $\omega\in\mathcal{H}$ with $\omega(0)=0$ satisfies*

$$\Theta(z)\bigl(1+\omega(z)\bigr)\prec 1+\gamma z\quad(z\in\mathbb{U}),$$

then, for $0<2\gamma_1+\gamma\leqq 1$, we have

$$\bigl|\omega(z)\bigr|\leqq\frac{\gamma_1+\gamma}{1-\gamma_1}\quad(z\in\mathbb{U}).\tag{2.3}$$

The value of β in (2.2) *and the bound in* (2.3) *are best possible.*

Lemma 4 [11] *If* $\omega \in \mathcal{B}$ *and*

$$\varphi(z) = \frac{1 + \gamma\omega(z)}{1 + \gamma\delta' \int_0^1 t^{\delta'-1}\omega(tz)\,dt} \quad \bigl(0 < \gamma < 1, \delta' > 0, z \in \mathbb{U}\bigr),$$

then $\operatorname{Re}(\varphi(z)) > \beta$ $(0 \leqq \beta < 1)$ *in* $\mathbb{U}$, *where*

$$\beta = \begin{cases} \frac{1-\gamma}{1+\gamma\delta_1} & (0 < \gamma \leqq \frac{1}{1+\delta_1}), \\ \frac{1-\gamma^2(1+\delta_1^2)}{2(1-\gamma^2\delta_1^2)} & (\frac{1}{1+\delta_1} \leqq \gamma \leqq \frac{1}{\sqrt{1+\delta_1^2}}) \end{cases} \tag{2.4}$$

and $\delta_1 = \delta'/(1 + \delta')$. *Further, for* $0 < \gamma \leqq 1/(1 + 2\delta_1)$, *we have*

$$\bigl|\varphi(z) - 1\bigr| \leqq \frac{\gamma(1 + \delta_1)}{1 - \gamma\delta_1} \quad (z \in \mathbb{U}). \tag{2.5}$$

The value of β *in* (2.4) *and the bound in* (2.5) *are best possible.*

3 Inclusion relationships for the class $\mathcal{S}^n_{p,\lambda,\mu}(\alpha; A, B)$

Unless otherwise mentioned, we assume *throughout* the sequel that

$$n \in \mathbb{Z}, \qquad \lambda, \mu > -p, \qquad \alpha > 0 \quad \text{and} \quad -1 \leqq B < A \leqq 1.$$

Theorem 1 *We have*

$$\mathcal{S}^n_{p,\lambda,\mu}(\alpha; A, B) \subset \mathcal{S}^n_{p,\lambda,\mu}(A, B).$$

Further, for $f \in \mathcal{S}^n_{p,\lambda,\mu}(\alpha; A, B)$, *we also have*

$$\frac{I^n_p(\lambda, \mu)f(z)}{I^{n+1}_p(\lambda, \mu)f(z)} \prec q(z) = \frac{\alpha}{(p + \lambda)Q(z)} \quad (z \in \mathbb{U}), \tag{3.1}$$

where

$$Q(z) = \begin{cases} \int_0^1 t^{\frac{p+\lambda}{\alpha}-1}(\frac{1+tBz}{1+Bz})^{\frac{(p+\lambda)(B-A)}{\alpha B}}\,dt & (B \neq 0), \\ \int_0^1 t^{\frac{p+\lambda}{\alpha}-1}\exp(\frac{(p+\lambda)(t-1)A}{\alpha}z)\,dt & (B = 0) \end{cases} \tag{3.2}$$

and q *is the best dominant of* (3.1). *Moreover, if* $A \leqq -\frac{\alpha B}{p+\lambda}$ $(-1 \leqq B < 0)$, *then*

$$\mathcal{S}^n_{p,\lambda,\mu}(\alpha; A, B) \subset \mathcal{S}^n_{p,\lambda,\mu}(1 - 2\rho, -1), \tag{3.3}$$

where $\rho = [{}_2F_1\,(1, \frac{(p+\lambda)(B-A)}{\alpha B}, \frac{p+\lambda}{\alpha} + 1; \frac{B}{B-1})]^{-1}$. *The bound* ρ *is best possible.*

Proof Let $f \in \mathcal{S}^n_{p,\lambda,\mu}(\alpha; A, B)$. Suppose that

$$g(z) = z\biggl(\frac{I^{n+1}_p(\lambda, \mu)f(z)}{z^p}\biggr)^{1/(p+\lambda)} \tag{3.4}$$

and $r_1 = \sup\{r : g(z) \neq 0, 0 < |z| < r < 1\}$. Choosing the principal branch of g, we note that g is single-valued and analytic in $\mathbb{U}_{r_1} = \{z : |z| < r_1\}$. Taking the logarithmic differentiation in (3.4) and using identity (1.6) in the resulting equation, we get that

$$\varphi(z) = \frac{zg'(z)}{g(z)} = \frac{I_p^n(\lambda,\mu)f(z)}{I_p^{n+1}(\lambda,\mu)f(z)} \tag{3.5}$$

is of the form (1.2) and is analytic in $\mathbb{U}_{r_1}$. Again, carrying out logarithmic differentiation in (3.5) and using (1.6), we deduce that

$$(1-\alpha)\frac{I_p^n(\lambda,\mu)f(z)}{I_p^{n+1}(\lambda,\mu)f(z)} + \alpha\frac{I_p^{n-1}(\lambda,\mu)f(z)}{I_p^n(\lambda,\mu)f(z)} = \varphi(z) + \frac{\alpha}{p+\lambda}\frac{z\varphi'(z)}{\varphi(z)} \prec \frac{1+Az}{1+Bz} \quad (z \in \mathbb{U}_{r_1}). \tag{3.6}$$

Hence, by applying the result [12, Corollary 3.2], we obtain

$$\varphi(z) \prec q(z) = \frac{\alpha}{(p+\lambda)Q(z)} \prec \frac{1+Az}{1+Bz} \quad (z \in \mathbb{U}_{r_1}),$$

where q is the best dominant of (3.1) and Q is given by (3.2).

The proof of the remaining part can now be deduced along the same lines as in [13, Theorem 1]. The bound ρ in (3.3) is best possible as q is the best dominant of (3.1). This evidently completes the proof of the theorem. □

Setting $n = -1$, $\lambda = 0$ and $\mu = 1 - p$ in Theorem 1, we get the following corollary.

Corollary 1 *If* $A \leqq -\alpha B/p$ $(-1 \leqq B < 0)$ *and* $f \in \mathcal{A}_p$ *satisfies*

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \frac{p(1+Az)}{1+Bz} \quad (z \in \mathbb{U}),$$

then

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > p\left[{}_2F_1\left(1, \frac{p(B-A)}{\alpha B}, \frac{p}{\alpha}+1; \frac{B}{B-1}\right)\right]^{-1} \quad (z \in \mathbb{U}).$$

The result is best possible.

In the special case when $n = 0$, $\mu = 1-p$, $A = 1 - (2(\eta + \lambda\alpha)/(p+\lambda))$ $(0 \leqq \eta < 1)$ and $B = -1$, Theorem 1 gives the following.

Corollary 2 *If* $\max\{-\lambda\alpha, (p+\lambda-2\lambda\alpha-\alpha)/2\} < \eta < p+\lambda-\lambda\alpha$ *and* $f \in \mathcal{A}_p$ *satisfies*

$$\operatorname{Re}\left((1-\alpha)\frac{z^\lambda f(z)}{\int_0^z t^{\lambda-1}f(t)\,dt} + \alpha\frac{zf'(z)}{f(z)}\right) > \eta \quad (z \in \mathbb{U}),$$

then

$$\operatorname{Re}\left(\frac{z^\lambda f(z)}{\int_0^z t^{\lambda-1}f(t)\,dt}\right) > (p+\lambda)\left[{}_2F_1\left(1, \frac{2(p+\lambda-\lambda\alpha-\eta)}{\alpha}, \frac{p+\lambda}{\alpha}+1; \frac{1}{2}\right)\right]^{-1} \quad (z \in \mathbb{U}).$$

The result is best possible.

Remarks 1. Putting $A = 1$ and $B = -1$ in Corollary 1, we find that for $\alpha \geqq p$ and $z \in \mathbb{U}$,

$$\operatorname{Re}\left((1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right) > 0 \quad \Longrightarrow \quad f \in \mathcal{S}_p^*\left(\frac{p\Gamma(1 + (2p/\alpha))}{\sqrt{\pi}\,\Gamma(1 + (p/\alpha))}\right),$$

which in turn implies that

$$f \in \mathcal{K}_p\left(\frac{p(\alpha - 1)\Gamma(1 + (2p/\alpha))}{\sqrt{\pi}\,\alpha\Gamma(1 + (p/\alpha))}\right).$$

For $p = 1$, this result is contained in [14].

2. Setting $\alpha = 1$, $A = 1 - (2\eta/p)$ $(0 \leqq \eta < p)$ and $B = -1$ in Corollary 1 and $\alpha = 1$ in Corollary 2, we get the corresponding result obtained in [15].

Theorem 2 *For* $0 \leqq \eta < p$, *we have*

$$f \in \mathcal{S}^n_{p,\lambda,\mu}\left(0; 1 - \frac{2\eta}{p}, -1\right) \quad \Longrightarrow \quad f \in \mathcal{S}^n_{p,\lambda,\mu}\left(\alpha; 1 - \frac{2\eta}{p}, -1\right) \quad \bigl(|z| < R\bigr),$$

where

$$R = \begin{cases} \frac{p\alpha + (p+\lambda)(p-\eta) - \sqrt{(p\alpha + (p+\lambda)(p-\eta))^2 - p(p+\lambda)^2(p-2\eta)}}{(p+\lambda)(p-2\eta)} & (\eta \neq \frac{p}{2}), \\ \frac{p+\lambda}{p+\lambda+2\alpha} & (\eta = \frac{p}{2}). \end{cases} \tag{3.7}$$

The bound R is best possible.

Proof Setting

$$\frac{I_p^n(\lambda,\mu)f(z)}{I_p^{n+1}(\lambda,\mu)f(z)} = \frac{\eta}{p} + \left(1 - \frac{\eta}{p}\right)u(z) \quad (z \in \mathbb{U}), \tag{3.8}$$

we see that u is of the form (1.2), analytic and has a positive real part in $\mathbb{U}$. Taking the logarithmic differentiation in (3.8) and using identity (1.6), we deduce that

$$\begin{aligned} &\operatorname{Re}\left((1-\alpha)\frac{I_p^n(\lambda,\mu)f(z)}{I_p^{n+1}(\lambda,\mu)f(z)} + \alpha\frac{I_p^{n-1}(\lambda,\mu)f(z)}{I_p^n(\lambda,\mu)f(z)}\right) - \frac{\eta}{p} \\ &\quad \geqq \left(1 - \frac{\eta}{p}\right)\left[\operatorname{Re}\bigl(u(z)\bigr) - \frac{p\alpha|zu'(z)|}{(p+\lambda)(|\eta + (p-\eta)u(z)|)}\right]. \end{aligned} \tag{3.9}$$

Now, by using the well-known [16] estimates

$$\bigl|zu'(z)\bigr| \leqq \frac{2r}{1-r^2}\operatorname{Re}\bigl(u(z)\bigr) \quad \text{and} \quad \operatorname{Re}\bigl(u(z)\bigr) \geqq \frac{1-r}{1+r} \quad \bigl(|z| = r < 1\bigr)$$

in (3.9), we obtain

$$\begin{aligned} &\operatorname{Re}\left((1-\alpha)\frac{I_p^n(\lambda,\mu)f(z)}{I_p^{n+1}(\lambda,\mu)f(z)} + \alpha\frac{I_p^{n-1}(\lambda,\mu)f(z)}{I_p^n(\lambda,\mu)f(z)}\right) - \frac{\eta}{p} \\ &\quad \geqq \left(1 - \frac{\eta}{p}\right)\operatorname{Re}\bigl(u(z)\bigr)\left[1 - \frac{2p\alpha r}{(p+\lambda)(\eta(1-r^2) + (p-\eta)(1-r)^2)}\right], \end{aligned}$$

which is certainly positive if $r < R$, where R is given by (3.7).

To show that the bound R is best possible, we consider the function $f \in \mathcal{A}_p$ defined by

$$\frac{I_p^n(\lambda,\mu)f(z)}{I_p^{n+1}(\lambda,\mu)f(z)} = \frac{\eta}{p} + \left(1 - \frac{\eta}{p}\right)\frac{1+z}{1-z} \quad (z \in \mathbb{U}).$$

Noting that

$$\begin{aligned}(1-\alpha)&\frac{I_p^n(\lambda,\mu)f(z)}{I_p^{n+1}(\lambda,\mu)f(z)} + \alpha\frac{I_p^{n-1}(\lambda,\mu)f(z)}{I_p^n(\lambda,\mu)f(z)} - \frac{\eta}{p} \\ &= \left(1-\frac{\eta}{p}\right)\frac{1+z}{1-z}\left[1 + \frac{2p\alpha z}{(p+\lambda)(\eta(1-z^2)+(p-\eta)(1-z)^2)}\right] = 0\end{aligned}$$

for $z = -R$, we complete the proof of Theorem 2. □

Remark For $n = -1$, $\lambda = 0$, $\mu = 1-p$ and $\alpha = 1$, Theorem 2 yields the corresponding result contained in [15].

For a function $f \in \mathcal{A}_p$, the generalized Bernardi-Libera-Livingston integral operator $\mathcal{F}_{\delta,p} : \mathcal{A}_p \longrightarrow \mathcal{A}_p$ is defined by (*cf.*, *e.g.*, [17])

$$\begin{aligned}\mathcal{F}_{\delta,p}(f)(z) &= \frac{\delta+p}{z^\delta}\int_0^z t^{\delta-1}f(t)\,dt = \left(z^p + \sum_{k=1}^{\infty}\frac{\delta+p}{\delta+p+k}z^{p+k}\right) * f(z) \\ &= z^p{}_2F_1(1,\delta+p;\delta+p+1;z) * f(z) \quad (\delta > -p; z \in \mathbb{U}).\end{aligned} \tag{3.10}$$

For convenience, we write $\mathcal{F}_{\delta,p}(f)(z) = \mathcal{F}_{\delta,p}(z)$, $z \in \mathbb{U}$. It readily follows from (3.10) that $f \in \mathcal{A}_p \Rightarrow \mathcal{F}_{\delta,p} \in \mathcal{A}_p$.

Theorem 3 *Let δ be a real number satisfying*

$$(\delta-\lambda)(1-B) + (p+\lambda)(1-A) \geqq 0. \tag{3.11}$$

(i) *If $f \in \mathcal{S}_{p,\lambda,\mu}^n(A,B)$, then*

$$\frac{I_p^n(\lambda,\mu)\mathcal{F}_{\delta,p}(z)}{I_p^{n+1}(\lambda,\mu)\mathcal{F}_{\delta,p}(z)} \prec q(z) = \frac{1}{p+\lambda}\left(\frac{1}{Q(z)} - \delta + \lambda\right) \prec \frac{1+Az}{1+Bz} \quad (z \in \mathbb{U}), \tag{3.12}$$

where

$$Q(z) = \begin{cases}\int_0^1 t^{\delta+p-1}\left(\frac{1+Btz}{1+Bz}\right)^{(p+\lambda)(B-A)/B}dt & (B \neq 0), \\ \int_0^1 t^{\delta+p-1}\exp\{(p+\lambda)A(t-1)z\}\,dt & (B = 0)\end{cases}$$

and q is the best dominant of (3.12). *Consequently, the operator $\mathcal{F}_{\delta,p}$ maps the class $\mathcal{S}_{p,\lambda,\mu}^n(A,B)$ into itself.*

(ii) *If $-1 \leqq B < 0$ and $\delta \geqq \max\{-\frac{(p+\lambda)(1-A)}{1-B} + \lambda, \frac{(p+\lambda)(B-A)}{B} - p - 1\}$, then*

$$f \in \mathcal{S}_{p,\lambda,\mu}^n(A,B) \quad \Longrightarrow \quad \mathcal{F}_{\delta,p} \in \mathcal{S}_{p,\lambda,\mu}^n(1-2\tau,-1), \tag{3.13}$$

where

$$\tau = \frac{1}{p+\lambda}\left[(\delta+p)\left({}_2F_1\left(1, \frac{(p+\lambda)(B-A)}{B}; \delta+p+1; \frac{B}{B-1}\right)\right)^{-1} - \delta + \lambda\right].$$

The bound τ is best possible.

Proof From (1.5) and (3.10), it follows that

$$z\left(I_p^{n+1}(\lambda,\mu)\mathcal{F}_{\delta,p}\right)'(z) = (\delta+p)I_p^{n+1}(\lambda,\mu)f(z) - \delta I_p^{n+1}(\lambda,\mu)\mathcal{F}_{\delta,p}(z) \quad (z \in \mathbb{U}). \tag{3.14}$$

We put

$$g(z) = z\left(\frac{I_p^{n+1}(\lambda,\mu)\mathcal{F}_{\delta,p}(z)}{z^p}\right)^{1/(p+\lambda)} \tag{3.15}$$

and $r_1 = \sup\{r : g(z) \neq 0, 0 < |z| < r < 1\}$. Choosing the principal branch of g, it follows that g is a single-valued and is analytic in $\mathbb{U}_{r_1}$. Taking the logarithmic differentiation in (3.15) and using identity (1.6) for $\mathcal{F}_{\delta,p}$, we deduce that the function

$$\varphi(z) = \frac{zg'(z)}{g(z)} = \frac{I_p^{n}(\lambda,\mu)\mathcal{F}_{\delta,p}(z)}{I_p^{n+1}(\lambda,\mu)\mathcal{F}_{\delta,p}(z)} \tag{3.16}$$

is analytic in $\mathbb{U}_{r_1}$ and $\varphi(0) = 1$. Using identity (3.14) in (3.16), we obtain

$$(\delta+p)\frac{I_p^{n+1}(\lambda,\mu)f(z)}{I_p^{n+1}(\lambda,\mu)\mathcal{F}_{\delta,p}(z)} = (p+\lambda)\frac{I_p^{n}(\lambda,\mu)\mathcal{F}_{\delta,p}(z)}{I_p^{n+1}(\lambda,\mu)\mathcal{F}_{\delta,p}(z)} + (\delta-p) \quad (z \in \mathbb{U}_{r_1}). \tag{3.17}$$

Since $f \in \mathcal{S}_{p,\lambda,\mu}^n(A,B)$, it is clear that $I_p^{n+1}(\lambda,\mu)f(z) \neq 0$ in $0 < |z| < 1$. So, by (3.17), we get

$$\frac{I_p^{n+1}(\lambda,\mu)\mathcal{F}_{\delta,p}(z)}{I_p^{n+1}(\lambda,\mu)f(z)} = \frac{\delta+p}{(p+\lambda)\varphi(z) + (\delta-\lambda)} \quad (z \in \mathbb{U}_{r_1}). \tag{3.18}$$

Again, by taking the logarithmic differentiation in (3.18) followed by the use of identity (1.6) in the resulting equation, we get

$$\frac{I_p^{n}(\lambda,\mu)f(z)}{I_p^{n+1}(\lambda,\mu)f(z)} = \varphi(z) + \frac{z\varphi'(z)}{(p+\lambda)\varphi(z) + (\delta-\lambda)} \prec \frac{1+Az}{1+Bz} \quad (z \in \mathbb{U}_{r_1}).$$

The proof of the remaining part is the same as that of [13, Theorem 1], and we choose to omit the details. The result is best possible as q is the best dominant of (3.12). □

Remark Letting $n = -1$, $\lambda = 0$, $\mu = 1 - p$, $A = 1 - (2\eta/p)$ $(0 \leqq \eta < p)$ and $B = -1$ in Theorem 3, we have the following implications [15, Corollary 3.4 and Remark 3.2]:

$$\mathcal{F}_{\delta,p}\left(\mathcal{S}_p^*(\eta)\right) \subset \mathcal{S}_p^*(\sigma) \quad \text{and} \quad \mathcal{F}_{\delta,p}\left(\mathcal{K}_p(\eta)\right) \subset \mathcal{K}_p(\sigma),$$

where $\delta \geqq \max\{-\eta, p - 2\eta - 1\}$ and $\sigma = (\delta+p)({}_2F_1\,(1, 2(p-\eta); \delta+p+1; 1/2))^{-1} - \delta$. The containment relations are best possible, and they improve the corresponding work in [18] for suitable values of the parameters p, η and δ.

4 Properties involving the operator $I_p^n(\lambda, \mu)$

Theorem 4 *If $f \in \mathcal{A}_p$ satisfies*

$$(1-\alpha)\frac{I_p^{n+1}(\lambda,\mu)f(z)}{z^p} + \alpha\frac{I_p^n(\lambda,\mu)f(z)}{z^p} \prec \frac{1+Az}{1+Bz} \quad (z \in \mathbb{U}), \tag{4.1}$$

then

$$\operatorname{Re}\left(\frac{I_p^{n+1}(\lambda,\mu)f(z)}{z^p}\right) > \varrho \quad (z \in \mathbb{U}), \tag{4.2}$$

where

$$\varrho = \begin{cases} \frac{A}{B} + (1-\frac{A}{B})(1-B)^{-1}{}_2F_1(1,1;\frac{p+\lambda}{\alpha}+1;\frac{B}{B-1}) & (B \neq 0), \\ 1 - \frac{(p+\lambda)A}{p+\lambda+\alpha} & (B = 0). \end{cases}$$

The result is best possible.

Proof Setting

$$\varphi(z) = \frac{I_p^{n+1}(\lambda,\mu)f(z)}{z^p} \quad (z \in \mathbb{U}), \tag{4.3}$$

we note that φ is of the form (1.2) and is analytic in $\mathbb{U}$. On differentiating (4.3) and using identity (1.6) in the resulting equation, we deduce that

$$(1-\alpha)\frac{I_p^{n+1}(\lambda,\mu)f(z)}{z^p} + \alpha\frac{I_p^n(\lambda,\mu)f(z)}{z^p} = \varphi(z) + \frac{\alpha}{p+\lambda}z\varphi'(z) \prec \frac{1+Az}{1+Bz} \quad (z \in \mathbb{U}). \tag{4.4}$$

The proof of the remaining part of the theorem follows by using Lemma 1 and the techniques that proved Theorem 4 in [13].

With a view to stating a well-known result, we denote by $\mathcal{P}(\gamma)$ the class of functions φ of the form (1.2) which are analytic in $\mathbb{U}$ and satisfy the inequality

$$\operatorname{Re}\big(\varphi(z)\big) > \gamma \quad (0 \leqq \gamma < 1; z \in \mathbb{U}).$$

It is known [19] that if $\varphi_j \in \mathcal{P}(\gamma_j)$ $(0 \leqq \gamma_j < 1; j = 1,2)$, then

$$(\varphi_1 * \varphi_2) \in \mathcal{P}(\gamma_3), \tag{4.5}$$

where $\gamma_3 = 1 - 2(1-\gamma_1)(1-\gamma_2)$. The bound γ_3 is best possible. □

Theorem 5 *If the functions $I_p^n(\lambda,\mu)f_j/z^p \in \mathcal{P}(A_j, B_j)$ $(-1 \leqq B_j < A_j \leqq 1, f_j \in \mathcal{A}_p; j = 1,2)$, then the function $\mathfrak{h}$ defined in $\mathbb{U}$ by*

$$\mathfrak{h}(z) = I_p^{n+1}(\lambda,\mu)(f_1 * f_2)(z) \tag{4.6}$$

satisfies

$$\operatorname{Re}\left(\frac{I_p^n(\lambda,\mu)\mathfrak{h}(z)}{I_p^{n+1}(\lambda,\mu)\mathfrak{h}(z)}\right) > 0 \quad (z \in \mathbb{U}),$$

provided

$$\frac{(A_1-B_1)(A_2-B_2)}{(1-B_1)(1-B_2)} < \frac{2(p+\lambda)+1}{2[({}_2F_1\,(1,1;p+\lambda+1;1/2)-2)^2+2(p+\lambda)]}. \tag{4.7}$$

Proof We have

$$\frac{I_p^n(\lambda,\mu)f_j}{z^p} \in \mathcal{P}(\gamma_j) \quad \left(\gamma_j = \frac{1-A_j}{1-B_j}; j=1,2\right).$$

Hence, by using (4.5), we deduce that

$$\begin{aligned}
&\operatorname{Re}\left(\frac{I_p^n(\lambda,\mu)\mathfrak{h}(z)}{z^p} + \frac{z}{p+\lambda}\left(\frac{I_p^n(\lambda,\mu)\mathfrak{h}(z)}{z^p}\right)'\right) \\
&\quad = \operatorname{Re}\left(\frac{I_p^n(\lambda,\mu)f_1(z)}{z^p} * \frac{I_p^n(\lambda,\mu)f_2(z)}{z^p}\right) > 1 - 2\frac{(A_1-B_1)(A_2-B_2)}{(1-B_1)(1-B_2)} \quad (z\in\mathbb{U}),
\end{aligned} \tag{4.8}$$

which, in view of Lemma 1 for

$$\kappa = p+\lambda, \qquad A = -1 + 4\frac{(A_1-B_1)(A_2-B_2)}{(1-B_1)(1-B_2)} \quad \text{and} \quad B=-1,$$

yields

$$\operatorname{Re}\left(\frac{I_p^n(\lambda,\mu)\mathfrak{h}(z)}{z^p}\right) > 1 + \frac{(A_1-B_1)(A_2-B_2)}{(1-B_1)(1-B_2)}\left[{}_2F_1\left(1,1;p+\lambda+1;\frac{1}{2}\right)-2\right] \quad (z\in\mathbb{U}). \tag{4.9}$$

From (4.9), by using Theorem 4 for

$$\alpha = 1, \quad A = -1-4\frac{(A_1-B_1)(A_2-B_2)}{(1-B_1)(1-B_2)}\left[{}_2F_1\left(1,1;p+\lambda+1;\frac{1}{2}\right)-2\right] \quad \text{and} \quad B=-1,$$

we deduce that

$$\operatorname{Re}\big(\theta(z)\big) > 1 - 2\frac{(A_1-B_1)(A_2-B_2)}{(1-B_1)(1-B_2)}\left[{}_2F_1\left(1,1;p+\lambda+1;\frac{1}{2}\right)-2\right]^2 \quad (z\in\mathbb{U}), \tag{4.10}$$

where $\theta(z) = I_p^n(\lambda,\mu)\mathfrak{h}(z)/z^p$. If we put

$$\varphi(z) = \frac{I_p^n(\lambda,\mu)\mathfrak{h}(z)}{I_p^{n+1}(\lambda,\mu)\mathfrak{h}(z)} \quad (z\in\mathbb{U}),$$

then φ is of the form (1.2) analytic in $\mathbb{U}$, and a simple computation shows that

$$\begin{aligned}
\frac{I_p^n(\lambda,\mu)\mathfrak{h}(z)}{z^p} + \frac{z}{p+\lambda}\left(\frac{I_p^n(\lambda,\mu)\mathfrak{h}(z)}{z^p}\right)' &= \theta(z)\left[\big(\varphi(z)\big)^2 + \frac{1}{p+\lambda}z\varphi'(z)\right] \\
&= \Psi\big(\varphi(z), z\varphi'(z); z\big),
\end{aligned} \tag{4.11}$$

where $\Psi(u,v;z) = \theta(z)(u^2 + (v/(p+\lambda)))$. Thus, by using (4.8) in (4.11), we conclude that

$$\operatorname{Re}\big(\Psi\big(\varphi(z), z\varphi'(z); z\big)\big) > 1 - 2\frac{(A_1-B_1)(A_2-B_2)}{(1-B_1)(1-B_2)} \quad (z\in\mathbb{U}).$$

Now, for all real $x, y \leqq -(1+x^2)/2$, we have

$$\begin{aligned}\operatorname{Re}\bigl(\Psi(ix,y;z)\bigr) &= \left(\frac{y}{p+\lambda} - x^2\right)\operatorname{Re}\bigl(\theta(z)\bigr) \\ &\leqq -\frac{1}{2(p+\lambda)}\bigl[1+\bigl(2(p+\lambda)+1\bigr)x^2\bigr]\operatorname{Re}\bigl(\theta(z)\bigr) \\ &\leqq \frac{1}{2(p+\lambda)}\operatorname{Re}\bigl(\theta(z)\bigr) \leqq 1 - 2\frac{(A_1-B_1)(A_2-B_2)}{(1-B_1)(1-B_2)} \quad (z \in \mathbb{U})\end{aligned}$$

by (4.7) and (4.10). Hence, by making use of Lemma 2, we get $\operatorname{Re}(\varphi(z)) > 0$ in $\mathbb{U}$. This completes the proof of Theorem 5. □

Theorem 6 *If the functions* $I_p^n(\lambda,\mu)f_j/z^p \in \mathcal{P}(A_j,B_j)$ $(-1 \leqq B_j < A_j \leqq 1, f_j \in \mathcal{A}_p; j = 1,2,3)$, *then the function* H *defined in* $\mathbb{U}$ *by*

$$H(z) = I_p^{n+1}(\lambda,\mu)(f_1 * f_2 * f_3)(z)$$

satisfies

$$\operatorname{Re}\left(\frac{I_p^{2n}(\lambda,\mu)H(z)}{I_p^{2n+1}(\lambda,\mu)H(z)}\right) > 0 \quad (z \in \mathbb{U}),$$

provided

$$\frac{(A_1-B_1)(A_2-B_2)(A_3-B_3)}{(1-B_1)(1-B_2)(1-B_3)} < \frac{2(p+\lambda)+1}{4[({}_2F_1\,(1,1;p+\lambda+1;1/2)-2)^2 + 2(p+\lambda)]}.$$

Proof From the definition of the function H, it is easily seen that

$$\begin{aligned}&\operatorname{Re}\left(\frac{I_p^{2n}(\lambda,\mu)H(z)}{z^p} + \frac{z}{p+\lambda}\left(\frac{I_p^{2n}(\lambda,\mu)H(z)}{z^p}\right)'\right) \\ &\quad= \operatorname{Re}\left(\frac{I_p^n(\lambda,\mu)f_1(z)}{z^p} * \frac{I_p^n(\lambda,\mu)f_2(z)}{z^p} * \frac{I_p^n(\lambda,\mu)f_3(z)}{z^p}\right) \\ &\quad> 1 - 4\frac{(A_1-B_1)(A_2-B_2)(A_3-B_3)}{(1-B_1)(1-B_2)(1-B_3)} \quad (z \in \mathbb{U})\end{aligned}$$

and the proof of the theorem is completed similarly to Theorem 5. □

Theorem 7 *Let* $f_j \in \mathcal{A}_p$ $(j = 1,2)$. *If the function* $\mathfrak{h}$ *defined in* $\mathbb{U}$ *by* (4.6) *satisfies*

$$\operatorname{Re}\left(\frac{I_p^n(\lambda,\mu)\mathfrak{h}(z)}{z^p}\right) > 1 - \frac{2(p+\lambda)+1}{2[({}_2F_1\,(1,1;p+\lambda+1;1/2)-2)^2 + 2(p+\lambda)]} \quad (z \in \mathbb{U}),$$

then

$$\operatorname{Re}\left(\frac{I_p^n(\lambda,\mu)\mathcal{G}(z)}{I_p^{n+1}(\lambda,\mu)\mathcal{G}(z)}\right) > 0 \quad (z \in \mathbb{U}),$$

where

$$\mathcal{G}(z) = \frac{p+\lambda}{z^{\lambda}} \int_0^z t^{\lambda-1}\mathfrak{h}(t)\,dt \quad (z \in \mathbb{U}).$$

Proof Using the fact that

$$\begin{aligned}\operatorname{Re}\left(\frac{I_p^n(\lambda,\mu)\mathfrak{h}(z)}{z^p}\right) &= \operatorname{Re}\left(\frac{I_p^n(\lambda,\mu)\mathcal{G}(z)}{z^p} + \frac{z}{p+\lambda}\left(\frac{I_p^n(\lambda,\mu)\mathcal{G}(z)}{z^p}\right)'\right) \\ &> 1 - \frac{2(p+\lambda)+1}{2[({}_2F_1\,(1,1;p+\lambda+1;1/2)-2)^2 + 2(p+\lambda)]} \quad (z \in \mathbb{U})\end{aligned}$$

and by following the same lines of proof as in Theorem 5, we get the required result. □

Remark Putting $n = -1$, $\lambda = 0$ and $\mu = 1$ in Theorems 5, 6 and 7, respectively, we obtain the corresponding results contained in [20].

Theorem 8 *Let* $\delta > 0$ *and* $0 < \gamma \leqq (1+\delta(p+\lambda))/\sqrt{1+2\delta(p+\lambda)+2(\delta(p+\lambda))^2}$. *If* $f \in \mathcal{A}_p$ *satisfies*

$$\frac{I_p^{n-1}(\lambda,\mu)f(z)}{z^p}\left(\frac{I_p^n(\lambda,\mu)f(z)}{z^p}\right)^{\delta-1} \prec 1+\gamma z \quad (z \in \mathbb{U}), \tag{4.12}$$

then $f \in \mathcal{S}^{n-1}_{p,\lambda,\mu}(1-2\varkappa,-1)$, *where*

$$\varkappa = \begin{cases} \frac{1+\delta(p+\lambda)}{1+\delta(p+\lambda)(1+\gamma)} & (0 < \gamma \leqq \frac{1+\delta(p+\lambda)}{1+2\delta(p+\lambda)}), \\ M_p(\lambda,\delta,\gamma) & (\frac{1+\delta(p+\lambda)}{1+2\delta(p+\lambda)} \leqq \gamma \leqq \frac{1+\delta(p+\lambda)}{\sqrt{1+2\delta(p+\lambda)+2(\delta(p+\lambda))^2}}), \end{cases} \tag{4.13}$$

and

$$M_p(\lambda,\delta,\gamma) = \frac{(1+\delta(p+\lambda))^2 - [1+2\delta(p+\lambda)+2(\delta(p+\lambda))^2]\gamma^2}{2[(1+\delta(p+\lambda))^2 - (\delta(p+\lambda))^2\gamma^2]}.$$

Further, for $0 < \gamma \leqq (1+\delta(p+\lambda))/(1+3\delta(p+\lambda))$,

$$\left|\frac{I_p^{n-1}(\lambda,\mu)f(z)}{I_p^n(\lambda,\mu)f(z)} - 1\right| < \frac{(1+2\delta(p+\lambda))\gamma}{1+\delta(p+\lambda)(1-\gamma)} \quad (z \in \mathbb{U}). \tag{4.14}$$

The bound given by (4.13) *and the estimate in* (4.14) *are best possible.*

Proof Setting

$$\Theta(z) = \left(\frac{I_p^n(\lambda,\mu)f(z)}{z^p}\right)^{\delta} \quad (z \in \mathbb{U}) \tag{4.15}$$

and choosing the principal branch in (4.15), we note that Θ is analytic in $\mathbb{U}$ with $\Theta(0) = 1$. A simple computation shows that (4.12) is equivalent to

$$\Theta(z) + \frac{z\Theta'(z)}{\delta(p+\lambda)} \prec 1+\gamma z \quad (z \in \mathbb{U}).$$

Now, by applying Lemma 1 (with $\kappa = \delta(p+\lambda)$, $A = \gamma$ and $B = -1$), we get

$$\Theta(z) \prec 1 + \gamma_1 z \quad \left(\gamma_1 = \frac{\delta(p+\lambda)\gamma}{1+\delta(p+\lambda)}; z \in \mathbb{U}\right).$$

We further observe that

$$1 + \frac{1}{\delta(p+\lambda)} \frac{z\Theta'(z)}{\Theta(z)} = \frac{I_p^{n-1}(\lambda,\mu)f(z)}{I_p^n(\lambda,\mu)f(z)} \quad (z \in \mathbb{U}).$$

Hence, assertion (4.13) follows by using part (i) of Lemma 3. If we put

$$\frac{I_p^{n-1}(\lambda,\mu)f(z)}{I_p^n(\lambda,\mu)f(z)} = 1 + \omega(z) \quad (\omega \in \mathcal{B}; z \in \mathbb{U}),$$

then we obtain (4.14) from part (ii) of Lemma 3.

To show that the estimates are best possible, we consider the function $f \in \mathcal{A}_p$ defined in $\mathbb{U}$ by

$$\left(\frac{I_p^n(\lambda,\mu)f(z)}{z^p}\right)^\delta = 1 + \delta(p+\lambda)\gamma \int_0^1 t^{\delta(p+\lambda)-1}\omega(tz)\,dt \quad (\delta > 0, \omega \in \mathcal{B}; z \in \mathbb{U}).$$

From this, we obtain

$$\frac{I_p^{n-1}(\lambda,\mu)f(z)}{I_p^n(\lambda,\mu)f(z)} = \frac{1+\gamma\omega(z)}{1+\delta(p+\lambda)\gamma\int_0^1 t^{\delta(p+\lambda)-1}\omega(tz)\,dt} \quad (z \in \mathbb{U}),$$

and the sharpness follows from Lemma 4 (for $\delta' = \delta(p+\lambda)$). □

Putting $n = \lambda = 0$, $\mu = 1-p$ and $\delta = 1$ in Theorem 8, we get the following.

Corollary 3 *If* $0 < \gamma \leqq (p+1)/\sqrt{1+2p+2p^2}$ *and* $f \in \mathcal{A}_p$ *satisfies*

$$\left|\frac{f'(z)}{pz^{p-1}} - 1\right| < \gamma \quad (z \in \mathbb{U}),$$

then

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \begin{cases} \frac{p(p+1)(1-\gamma)}{1+p(1+\gamma)} & (0 < \gamma \leqq \frac{p+1}{2p+1}), \\ \frac{p(p+1)^2 - p(1+2p+2p^2)\gamma^2}{2((p+1)^2-(p\gamma)^2)} & (\frac{p+1}{2p+1} \leqq \gamma \leqq \frac{p+1}{\sqrt{1+2p+2p^2}}). \end{cases}$$

Further, for $0 < \gamma \leqq (p+1)/(3p+1)$,

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < \frac{p(2p+1)\gamma}{1+p(1-\gamma)} \quad (z \in \mathbb{U}).$$

The estimates are best possible.

Theorem 9 *If $\gamma > 0$ and $f \in \mathcal{A}_p$ satisfies*

$$(1-\alpha)\frac{I_p^n(\lambda,\mu)f(z)}{z^p} + \alpha\frac{I_p^{n-1}(\lambda,\mu)f(z)}{z^p} \prec 1+\gamma z \quad (\alpha > 0; z \in \mathbb{U}), \tag{4.16}$$

then $f \in \mathcal{S}_{p,\lambda,\mu}^{n-1}(1-2\vartheta,-1)$, where

$$\vartheta = \begin{cases} \frac{\alpha((p+\lambda)(1+\gamma)+\alpha-\gamma)-2(p+\lambda)\gamma}{\alpha((p+\lambda)(1+\gamma)+\alpha)} & (0<\gamma \leqq \frac{p+\lambda+\alpha}{2(p+\lambda)+\alpha}), \\ \frac{N_p(\lambda,\alpha,\gamma)-(1-\alpha)}{\alpha} & (\frac{p+\lambda+\alpha}{2(p+\lambda)+\alpha} \leqq \gamma \leqq \frac{p+\lambda+\alpha}{\sqrt{(p+\lambda)^2+(p+\lambda+\alpha)^2}}) \end{cases} \tag{4.17}$$

and

$$N_p(\lambda,\alpha,\gamma) = \frac{(p+\lambda+\alpha)^2 - ((p+\lambda+\alpha)^2+(p+\lambda)^2)\gamma^2}{2((p+\lambda+\alpha)^2-(p+\lambda)^2\gamma^2)}.$$

For $0<\gamma \leqq (p+\lambda+\alpha)/(3(p+\lambda)+\alpha)$, we have

$$\left|\frac{I_p^{n-1}(\lambda,\mu)f(z)}{I_p^n(\lambda,\mu)f(z)} - 1\right| < \frac{(2(p+\lambda)+\alpha)\gamma}{(p+\lambda)(1-\gamma)+\alpha} \quad (z \in \mathbb{U}). \tag{4.18}$$

Further, $f \in \mathcal{S}_{p,\lambda,\mu}^{n}(1-2\widetilde{\varkappa},-1)$, where $\widetilde{\varkappa}$ is obtained from $\varkappa$ (given in (4.13)) for $\delta = 1$ and upon replacing γ by $(p+\lambda)\gamma/(p+\lambda+\alpha)$. Moreover, for $0<\gamma \leqq ((p+\lambda+\alpha)(p+\lambda+1))/[(p+\lambda)(1+2(p+\lambda))]$,

$$\left|\frac{I_p^{n}(\lambda,\mu)f(z)}{I_p^{n+1}(\lambda,\mu)f(z)} - 1\right| < \frac{(p+\lambda)(1+2(p+\lambda))\gamma}{(p+\lambda+\alpha)+(p+\lambda)(\alpha+(p+\lambda)(1-\gamma))} \quad (z \in \mathbb{U}). \tag{4.19}$$

The estimates are best possible.

Proof Since $f \in \mathcal{A}_p$ satisfies (4.16), by Theorem 4 (for $A = \gamma$ and $B = -1$) we obtain

$$\frac{I_p^n(\lambda,\mu)f(z)}{z^p} \prec 1+\gamma_1 z \quad \left(\gamma_1 = \frac{(p+\lambda)\gamma}{p+\lambda+\alpha}; z \in \mathbb{U}\right). \tag{4.20}$$

Again, on writing (4.16) in the form

$$\frac{I_p^n(\lambda,\mu)f(z)}{z^p}\left(1-\alpha+\alpha\frac{I_p^{n-1}(\lambda,\mu)f(z)}{I_p^n(\lambda,\mu)f(z)}\right) \prec 1+\gamma z \quad (z \in \mathbb{U})$$

and using part (ii) of Lemma 3, we deduce that

$$\begin{aligned} &\operatorname{Re}\left(1-\alpha+\alpha\frac{I_p^{n-1}(\lambda,\mu)f(z)}{I_p^n(\lambda,\mu)f(z)}\right) \\ &\quad \geqq \begin{cases} \frac{(p+\lambda+\alpha)(1-\gamma)}{(p+\lambda)(1+\gamma)+\alpha} & (0<\gamma \leqq \frac{p+\lambda+\alpha}{2(p+\lambda)+\alpha}), \\ N_p(\lambda,\alpha,\mu) & (\frac{p+\lambda+\alpha}{2(p+\lambda)+\alpha} \leqq \gamma \leqq \frac{p+\lambda+\alpha}{\sqrt{(p+\lambda)^2+(p+\lambda+\alpha)^2}}), \end{cases} \end{aligned}$$

which implies assertion (4.17). By using part (ii) of Lemma 3 with

$$\omega(z) = \alpha\left(\frac{I_p^{n-1}(\lambda,\mu)f(z)}{I_p^n(\lambda,\mu)f(z)} - 1\right),$$

we obtain (4.18). That $f \in \mathcal{S}_{p,\lambda,\mu}^{n-1}(1-2\widetilde{\varkappa},-1)$ and (4.19) now follow from Theorem 8 and (4.20).

To show the sharpness of the estimates, we consider the function f defined in $\mathbb{U}$ by

$$\frac{I_p^n(\lambda,\mu)f(z)}{z^p} = 1 + \frac{(p+\lambda)\gamma}{\alpha}\int_0^1 t^{\frac{p+\lambda}{\alpha}-1}\omega(tz)\,dt \quad (\omega \in \mathcal{B}; z \in \mathbb{U}).$$

Hence, by using identity (1.6), we get

$$1-\alpha+\alpha\frac{I_p^{n-1}(\lambda,\mu)f(z)}{I_p^n(\lambda,\mu)f(z)} = \frac{1+\gamma\omega(z)}{1+\frac{(p+\lambda)\gamma}{\alpha}\int_0^1 t^{\frac{p+\lambda}{\alpha}-1}\omega(tz)\,dt} \quad (z \in \mathbb{U}),$$

and the sharpness follows from Lemma 4. The fact that $f \in \mathcal{S}_{p,\lambda,\mu}^{n}(1-2\widetilde{\varkappa},-1)$ is sharp follows from (4.20) and the sharpness of Theorem 8. □

Putting $n = -1$, $\lambda = 0$ and $\mu = 1-p$ in Theorem 9, we have the following.

Corollary 4 *If $f \in \mathcal{A}_p$ satisfies*

$$\left(1-\alpha+\frac{\alpha}{p}\right)\frac{f'(z)}{pz^{p-1}} + \alpha\frac{f''(z)}{p^2z^{p-2}} \prec 1+\gamma z \quad (\gamma > 0, \alpha > 0; z \in \mathbb{U}),$$

then

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) > \begin{cases} \frac{p\alpha(p+\alpha+(p-1)\gamma)-2p^2\gamma}{\alpha(p(1+\gamma)+\alpha)} & (0 < \gamma \leqq \frac{p+\alpha}{2p+\alpha}), \\ \frac{p(p+\alpha)^2-p((p+\alpha)^2+p^2)\gamma^2}{2\alpha((p+\alpha)^2-p^2\gamma^2)} - p(\frac{1-\alpha}{\alpha}) & (\frac{p+\alpha}{2p+\alpha} \leqq \gamma \leqq \frac{p+\alpha}{\sqrt{p^2+(p+\alpha)^2}}) \end{cases}$$

and for $0 < \gamma \leqq (p+\alpha)/(3p+\alpha)$,

$$\left|1+\frac{zf''(z)}{f'(z)} - p\right| < \frac{p(2p+\alpha)\gamma}{\alpha(p(1-\gamma)+\alpha)} \quad (z \in \mathbb{U}).$$

The result is sharp.

Remark For $p = 1$ in Corollary 3 and Corollary 4, we get the corresponding results obtained in [11].

Theorem 10 *If $f \in \mathcal{A}_p$ satisfies*

$$\left|\frac{I_p^n(\lambda,\mu)f(z)}{I_p^{n+1}(\lambda,\mu)f(z)} - 1\right|^\gamma \left|\frac{I_p^{n-1}(\lambda,\mu)f(z)}{I_p^n(\lambda,\mu)f(z)} - 1\right|^\beta < \left(\frac{A-B}{1+|B|}\right)^{\gamma+\beta}\left(1+\frac{1}{(p+\lambda)(1+|A|)}\right)^\beta \quad (z \in \mathbb{U}), \tag{4.21}$$

for some real numbers β and γ such that $\beta \geqq 0$, $\gamma \geqq 0$, $\beta+\gamma > 0$, then $f \in \mathcal{S}_{p,\lambda,\mu}^n(A,B)$.

Proof If we set

$$\frac{I_p^n(\lambda,\mu)f(z)}{I_p^{n+1}(\lambda,\mu)f(z)} = \frac{1+A\omega(z)}{1+B\omega(z)} \quad (z \in \mathbb{U}), \tag{4.22}$$

then ω is analytic in $\mathbb{U}$. Differentiating (4.22) logarithmically and using identity (1.6) in the resulting equation, we get

$$\frac{I_p^{n-1}(\lambda,\mu)f(z)}{I_p^n(\lambda,\mu)f(z)} = \frac{1+A\omega(z)}{1+B\omega(z)} + \frac{(A-B)z\omega'(z)}{(p+\lambda)(1+A\omega(z))(1+B\omega(z))} \quad (z \in \mathbb{U}).$$

Now, we have

$$\begin{aligned}
&\left|\frac{I_p^n(\lambda,\mu)f(z)}{I_p^{n+1}(\lambda,\mu)f(z)} - 1\right|^{\gamma} \left|\frac{I_p^{n-1}(\lambda,\mu)f(z)}{I_p^n(\lambda,\mu)f(z)} - 1\right|^{\beta} \\
&\quad = (A-B)^{\gamma+\beta}\left|\frac{\omega(z)}{1+B\omega(z)}\right|^{\gamma+\beta} \\
&\qquad \times \left|1 + \frac{1}{p+\lambda}\frac{z\omega'(z)}{\omega(z)}\frac{1}{1+A\omega(z)}\right|^{\beta} \quad (z \in \mathbb{U}).
\end{aligned} \tag{4.23}$$

We claim that $|\omega(z)| < 1$ for $z \in \mathbb{U}$. Otherwise, there exists a point $z_0 \in \mathbb{U}$ such that $\max_{|z| \leqq |z_0|} |\omega(z)| = |\omega(z_0)| = 1$. By using Jack's lemma [21], we write $\omega(z_0) = e^{i\theta}$, $0 < \theta \leqq 2\pi$ and $z\omega'(z_0) = m\omega(z_0)$, $m \geqq 1$. Thus, from (4.23), it follows that

$$\begin{aligned}
\left|\frac{I_p^n(\lambda,\mu)f(z_0)}{I_p^{n+1}(\lambda,\mu)f(z_0)} - 1\right|^{\gamma} \left|\frac{I_p^{n-1}(\lambda,\mu)f(z_0)}{I_p^n(\lambda,\mu)f(z_0)} - 1\right|^{\beta} &\geqq \left(\frac{A-B}{1+|B|}\right)^{\gamma+\beta}\left(1 + \frac{m}{p+\lambda}\Re\left(\frac{1}{1+Ae^{i\theta}}\right)\right)^{\beta} \\
&\geqq \left(\frac{A-B}{1+|B|}\right)^{\gamma+\beta}\left(1 + \frac{1}{(p+\lambda)(1+|A|)}\right)^{\beta}.
\end{aligned}$$

This contradicts the hypothesis (4.21) and hence $|\omega(z)| < 1$ for $z \in \mathbb{U}$. This proves the theorem. □

Taking $A = 1 - \frac{2\rho}{p}$, $B = -1$, $\mu = 1-p$, $\lambda = 0$, $n = -1$ and $\gamma = 1-\beta$ in Theorem 10, we get the following interesting criterion for starlikeness for multivalent functions, which improves the corresponding work in [22] for $p = 1$.

Corollary 5 *Let $\beta \geqq 0$ and $0 \leqq \rho < p$. If $f \in \mathcal{A}_p$ satisfies*

$$\left|\frac{zf'(z)}{f(z)} - p\right|^{1-\beta}\left|1 + \frac{zf''(z)}{f'(z)} - p\right|^{\beta} < \xi(p,\rho,\beta) = \begin{cases} (p-\rho)(1 + \frac{1}{2(p-\rho)})^{\beta} & (0 \leqq \rho \leqq \frac{p}{2}), \\ (p-\rho)(1 + \frac{1}{2\rho})^{\beta} & (\frac{p}{2} \leqq \rho < p) \end{cases}$$

for all $z \in \mathbb{U}$, then $f \in \mathcal{S}_p^*(\rho)$.

Similarly, by setting $A = 1 - \frac{2\rho}{p}$, $B = -1$, $\mu = 1-p$, $\lambda = 0$, $n = -2$ and $\gamma = 1-\beta$ in Theorem 10, we obtain the following sufficient condition for convexity of multivalent functions.

Corollary 6 *Let $\beta \geqq 0$ and $0 \leqq \rho < p$. If $f \in \mathcal{A}_p$ satisfies*

$$\left|\left(1+\frac{zf''(z)}{f'(z)}\right)-p\right|^{1-\beta}\left|\left(1+\frac{z^2f'''(z)+2zf''(z)}{zf''(z)+f'(z)}\right)-p\right|^{\beta}<\xi(p,\rho,\beta) \quad (z\in\mathbb{U}),$$

where $\xi(p,\rho,\beta)$ is defined as in Corollary 5, then $f \in \mathcal{K}_p(\rho)$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors jointly worked on the results and they read and approved the final manuscript.

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