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A sharp inequality for multilinear singular integral operators with non-smooth kernels

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Abstract
In this paper, we establish a sharp inequality for some multilinear singular integral operators with non-smooth kernels. As an application, we obtain the weighted L^p-norm inequality and $L\log L$-type inequality for the multilinear operators.
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1 Definitions and results

As the development of singular integral operators and their commutators, multilinear singular integral operators have been well studied (see [1–6]). In this paper, we study some multilinear operator associated to the singular integral operators with non-smooth kernels as follows.

Definition 1 A family of operators D_t, $t > 0$, is said to be 'approximations to the identity' if, for every $t > 0$, D_t can be represented by the kernel $a_t(x,y)$ in the following sense:

$$D_t(f)(x) = \int_{R^n} a_t(x,y)f(y)\,dy$$

for every $f \in L^p(R^n)$ with $p \geq 1$, and $a_t(x,y)$ satisfies

$$\left|a_t(x,y)\right| \leq h_t(x,y) = Ct^{-n/2}s\left(|x-y|^2/t\right),$$

where s is a positive, bounded and decreasing function satisfying

$$\lim_{r\to\infty} r^{n+\epsilon}s\left(r^2\right) = 0$$

for some $\epsilon > 0$.

Definition 2 A linear operator T is called a singular integral operator with non-smooth kernel if T is bounded on $L^2(R^n)$ and associated with a kernel $K(x,y)$ such that

$$T(f)(x) = \int_{R^n} K(x,y)f(y)\,dy$$

for every continuous function f with compact support, and for almost all x not in the support of f.

(1) There exists an 'approximation to the identity' $\{B_t, t > 0\}$ such that TB_t has an associated kernel $k_t(x,y)$ and there exist $c_1, c_2 > 0$ so that

$$\int_{|x-y|>c_1 t^{1/2}} \left|K(x,y) - k_t(x,y)\right| dx \le c_2 \quad \text{for all } y \in R^n.$$

(2) There exists an 'approximation to the identity' $\{A_t, t > 0\}$ such that $A_t T$ has an associated kernel $K_t(x,y)$ which satisfies

$$\left|K_t(x,y)\right| \le c_4 t^{-n/2} \quad \text{if } |x-y| \le c_3 t^{1/2},$$

and

$$\left|K(x,y) - K_t(x,y)\right| \le c_4 t^{\delta/2} |x-y|^{-n-\delta} \quad \text{if } |x-y| \ge c_3 t^{1/2}$$

for some $c_3, c_4 > 0$, $\delta > 0$.

Let m_j be positive integers ($j = 1, \dots, l$), $m_1 + \cdots + m_l = m$, and let b_j be functions on R^n ($j = 1, \dots, l$). Set, for $1 \le j \le m$,

$$R_{m_j+1}(b_j; x, y) = b_j(x) - \sum_{|\alpha| \le m_j} \frac{1}{\alpha!} D^\alpha b_j(y)(x-y)^\alpha.$$

The multilinear operator associated to T is defined by

$$T^b(f)(x) = \int_{R^n} \frac{\prod_{j=1}^l R_{m_j+1}(b_j; x, y)}{|x-y|^m} K(x,y) f(y)\, dy.$$

Note that when $m = 0$, T^b is just the multilinear commutator of T and b_j (see [7]). However, when $m > 0$, T_b is a non-trivial generalization of the commutator. It is well known that multilinear operators are of great interest in harmonic analysis and have been widely studied by many authors (see [1–4]). Hu and Yang (see [8]) proved a variant sharp estimate for the multilinear singular integral operators. In [7], Pérez and Trujillo-Gonzalez proved a sharp estimate for the multilinear commutator when $b_j \in Osc_{\exp L^{r_j}}(R^n)$ and noted that $Osc_{\exp L^{r_j}} \subset BMO$. The main purpose of this paper is to prove a sharp function inequality for the multilinear singular integral operator with non-smooth kernel when $D^\alpha b_j \in BMO(R^n)$ for all α with $|\alpha| = m_j$. As an application, we obtain an L^p ($p > 1$) norm inequality and an $L \log L$-type inequality for the multilinear operators. In [9–12], the boundedness of a singular integral operator with non-smooth kernel is obtained. In [13], the boundedness of the commutator associated to the singular integral operator with non-smooth kernel is obtained. Our works are motivated by these papers.

First, let us introduce some notations. Throughout this paper, Q denotes a cube of R^n with sides parallel to the axes. For any locally integrable function f, the sharp function of f is defined by

$$f^\#(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q \left|f(y) - f_Q\right| dy,$$

where, and in what follows, $f_Q = |Q|^{-1}\int_Q f(x)\,dx$. It is well known that (see [14, 15])

$$f^{\#}(x) \approx \sup_{Q\ni x}\inf_{c\in\mathbf{C}} \frac{1}{|Q|}\int_Q \bigl|f(y)-c\bigr|\,dy.$$

We say that f belongs to $BMO(R^n)$ if $f^{\#}$ belongs to $L^\infty(R^n)$ and $\|f\|_{BMO} = \|f^{\#}\|_{L^\infty}$. Let M be a Hardy-Littlewood maximal operator defined by

$$M(f)(x) = \sup_{Q\ni x}\frac{1}{|Q|}\int_Q \bigl|f(y)\bigr|\,dy.$$

For $k\in N$, we denote by M^k the operator M iterated k times, *i.e.*, $M^1(f)(x) = M(f)(x)$ and

$$M^k(f)(x) = M\bigl(M^{k-1}(f)\bigr)(x) \quad \text{when } k\geq 2.$$

The sharp maximal function $M_A(f)$ associated with the 'approximations to the identity' $\{A_t, t>0\}$ is defined by

$$M_A^{\#}(f)(x) = \sup_{Q\ni x}\frac{1}{|Q|}\int_Q \bigl|f(y)-A_{t_Q}(f)(y)\bigr|\,dy,$$

where $t_Q = l(Q)^2$ and $l(Q)$ denotes the side length of Q. For $0<r<\infty$, we denote $M_A^{\#}(f)_r$ by

$$M_A^{\#}(f)_r = \bigl[M_A^{\#}\bigl(|f|^r\bigr)\bigr]^{1/r}.$$

Let Φ be a Young function and $\tilde{\Phi}$ be the complementary associated to Φ. For a function f, we denote the Φ-average by

$$\|f\|_{\Phi,Q} = \inf\left\{\lambda>0 : \frac{1}{|Q|}\int_Q \Phi\left(\frac{|f(y)|}{\lambda}\right)dy \leq 1\right\}$$

and the maximal function associated to Φ by

$$M_\Phi(f)(x) = \sup_{Q\ni x}\|f\|_{\Phi,Q}.$$

The Young functions used in this paper are $\Phi(t) = t(1+\log t)^r$ and $\tilde{\Phi}(t) = \exp(t^{1/r})$, the corresponding average and maximal functions are denoted by $\|\cdot\|_{L(\log L)^r,Q}$, $M_{L(\log L)^r}$ and $\|\cdot\|_{\exp L^{1/r},Q}$, $M_{\exp L^{1/r}}$. Following [11, 12, 16], we know the generalized Hölder inequality

$$\frac{1}{|Q|}\int_Q \bigl|f(y)g(y)\bigr|\,dy \leq \|f\|_{\Phi,Q}\|g\|_{\tilde{\Phi},Q}$$

and the following inequality, for $r, r_j\geq 1$, $j=1,\ldots,l$ with $1/r = 1/r_1+\cdots+1/r_l$, and any $x\in R^n$, $b\in BMO(R^n)$,

$$\|f\|_{L(\log L)^{1/r},Q} \leq M_{L(\log L)^{1/r}}(f) \leq CM_{L(\log L)^l}(f) \leq CM^{l+1}(f),$$

$$\|b-b_Q\|_{\exp L^r,Q} \leq C\|b\|_{BMO},$$

$$|b_{2^{k+1}Q} - b_{2Q}| \leq Ck\|b\|_{BMO}.$$

We denote the Muckenhoupt weights by A_p for $1\leq p<\infty$ (see [14]).

We shall prove the following theorems.

Theorem 1 *If T is a singular integral operator with non-smooth kernel as given in Definition 2, let $D^\alpha b_j \in BMO(R^n)$ for all α with $|\alpha| = m_j$ and $j = 1, \dots, l$. Then there exists a constant $C > 0$ such that for any $f \in C_0^\infty(R^n)$, $0 < r < 1$ and $\tilde{x} \in R^n$,*

$$M_A^\#\big(T^b(f)\big)_r(\tilde{x}) \le C\prod_{j=1}^{l}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{BMO}\right)M^{l+1}(f)(\tilde{x}).$$

Theorem 2 *If T is a singular integral operator with non-smooth kernel as given in Definition 2, let $D^\alpha b_j \in BMO(R^n)$ for all α with $|\alpha| = m_j$ and $j = 1, \dots, l$. Then T^b is bounded on $L^p(w)$ for any $1 < p < \infty$ and $w \in A_p$, that is,*

$$\|T^b(f)\|_{L^p(w)} \le C\prod_{j=1}^{l}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{BMO}\right)\|f\|_{L^p(w)}.$$

Theorem 3 *If T is a singular integral operator with non-smooth kernel as given in Definition 2, let $w \in A_1$, $D^\alpha b_j \in BMO(R^n)$ for all α with $|\alpha| = m_j$ and $j = 1, \dots, l$. Then there exists a constant $C > 0$ such that for all $\lambda > 0$,*

$$w\big(\{x \in R^n : |T^b(f)(x)| > \lambda\}\big) \le C\int_{R^n}\frac{|f(x)|}{\lambda}\left[1 + \log^+\left(\frac{|f(x)|}{\lambda}\right)\right]^l w(x)\,dx.$$

2 Proof of the theorem

To prove the theorems, we need the following lemma.

Lemma 1 (see [1]) *Let b be a function on R^n and $D^\alpha b \in L^q(R^n)$ for all α with $|\alpha| = m$ and some $q > n$. Then*

$$|R_m(b;x,y)| \le C|x-y|^m\sum_{|\alpha|=m}\left(\frac{1}{|\tilde{Q}(x,y)|}\int_{\tilde{Q}(x,y)}|D^\alpha b(z)|^q\,dz\right)^{1/q},$$

where $\tilde{Q}$ is the cube centered at x and having side length $5\sqrt{n}|x-y|$.

Lemma 2 ([14, p.485]) *Let $0 < p < q < \infty$ and for any function $f \ge 0$, we define that for $1/r = 1/p - 1/q$,*

$$\|f\|_{WL^q} = \sup_{\lambda>0}\lambda\big|\{x \in R^n : f(x) > \lambda\}\big|^{1/q}, \qquad N_{p,q}(f) = \sup_E \|f\chi_E\|_{L^p}/\|\chi_E\|_{L^r},$$

where the sup *is taken for all measurable sets E with $0 < |E| < \infty$. Then*

$$\|f\|_{WL^q} \le N_{p,q}(f) \le \big(q/(q-p)\big)^{1/p}\|f\|_{WL^q}.$$

Lemma 3 (see [17]) *Let $r_j \ge 1$ for $j = 1, \dots, l$, we denote that $1/r = 1/r_1 + \cdots + 1/r_l$. Then*

$$\frac{1}{|Q|}\int_Q|f_1(x)\cdots f_l(x)g(x)|\,dx \le \|f\|_{\exp L^{r_1},Q}\cdots\|f\|_{\exp L^{r_l},Q}\|g\|_{L(\log L)^{1/r},Q}.$$

Lemma 4 ([9, 10]) *Let T be a singular integral operator with non-smooth kernel as given in Definition* 2. *Then T is bounded on $L^p(R^n)$ for every $1 < p < \infty$ and bounded from $L^1(R^n)$ to $WL^1(R^n)$.*

Lemma 5 (see [9, 12]) *For any $\gamma > 0$, there exists a constant $C > 0$ independent of γ such that*

$$\left|\left\{x \in R^n : M(f)(x) > D\lambda, M_A^{\#}(f)(x) \le \gamma\lambda\right\}\right| \le C\gamma\left|\left\{x \in R^n : M(f)(x) > \lambda\right\}\right|$$

for $\lambda > 0$, where D is a fixed constant which only depends on n. Thus

$$\left\|M(f)\right\|_{L^p} \le C\left\|M_A^{\#}(f)\right\|_{L^p}$$

for every $f \in L^p(R^n)$, $1 < p < \infty$.

Lemma 6 *Let $\{A_t, t > 0\}$ be an 'approximation to the identity' and $b \in BMO(R^n)$. Then, for every $f \in L^p(R^n)$, $p > 1$ and $\tilde{x} \in R^n$,*

$$\sup_{Q \ni \tilde{x}} \frac{1}{|Q|}\int_Q \left|A_{t_Q}\left((b - b_Q)f\right)(x)\right| dx \le C\|b\|_{BMO} M^2(f)(\tilde{x}),$$

where $t_Q = l(Q)^2$ and $l(Q)$ denotes the side length of Q.

Proof We write, for any cube Q with $\tilde{x} \in Q$,

$$\begin{aligned}
\frac{1}{|Q|}\int_Q \left|A_{t_Q}\left((b - b_Q)f\right)(x)\right| dx &\le \frac{1}{|Q|}\int_Q\int_{R^n} h_{t_Q}(x,y)\left|\left(b(y) - b_Q\right)f(y)\right| dy\,dx \\
&\le \frac{1}{|Q|}\int_Q\int_{2Q} h_{t_Q}(x,y)\left|\left(b(y) - b_Q\right)f(y)\right| dy\,dx \\
&\quad + \sum_{k=1}^{\infty}\frac{1}{|Q|}\int_Q\int_{2^{k+1}Q\setminus 2^kQ} h_{t_Q}(x,y)\left|\left(b(y) - b_Q\right)f(y)\right| dy\,dx \\
&= I_1 + I_2.
\end{aligned}$$

We have, by the generalized Hölder inequality,

$$\begin{aligned}
I_1 &\le C\frac{1}{|Q||2Q|}\int_Q\int_{2Q}\left|\left(b(y) - b_Q\right)f(y)\right| dy\,dx \\
&\le C\|b - b_Q\|_{\exp L, 2Q}\|f\|_{L(\log L), 2Q} \\
&\le C\|b\|_{BMO} M^2(f)(\tilde{x}).
\end{aligned}$$

For I_2, notice for $x \in Q$ and $y \in 2^{k+1}Q\setminus 2^kQ$, then $|x - y| \ge 2^{k-1}t_Q$ and $h_{t_Q}(x,y) \le C\frac{s(2^{2(k-1)})}{|Q|}$, then

$$\begin{aligned}
I_2 &\le C\sum_{k=1}^{\infty} s\left(2^{2(k-1)}\right)\frac{1}{|Q|^2}\int_Q\int_{2^{k+1}Q}\left|\left(b(y) - b_Q\right)f(y)\right| dy\,dx \\
&\le C\sum_{k=1}^{\infty} 2^{kn} s\left(2^{2(k-1)}\right)\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}\left|\left(b(y) - b_Q\right)f(y)\right| dy
\end{aligned}$$

$$\begin{aligned}
&\le C\sum_{k=1}^{\infty} 2^{kn} s\bigl(2^{2(k-1)}\bigr)\|b-b_Q\|_{\exp L,2^{k+1}Q}\|f\|_{L(\log L),2^{k+1}Q} \\
&\le C\sum_{k=1}^{\infty} 2^{(k-1)n} s\bigl(2^{2(k-1)}\bigr)\|b\|_{BMO}M^2(f)(\tilde{x}) \\
&\le C\|b\|_{BMO}M^2(f)(\tilde{x}),
\end{aligned}$$

where the last inequality follows from

$$\sum_{k=1}^{\infty} 2^{(k-1)n} s\bigl(2^{2(k-1)}\bigr) \le C\sum_{k=1}^{\infty} 2^{-(k-1)\varepsilon} < \infty$$

for some $\epsilon > 0$. This completes the proof. □

Proof of Theorem 1 It suffices to prove for $f \in C_0^\infty(R^n)$ and some constant C_0 that the following inequality holds:

$$\left(\frac{1}{|Q|}\int_Q \bigl|\bigl|T^b(f)(x)\bigr|^r - \bigl|A_{t_Q}T^b(f)(x)\bigr|^r\bigr|\,dx\right)^{1/r} \le C\prod_{j=1}^{l}\left(\sum_{|\alpha_j|=m_j}\bigl\|D^{\alpha_j}b_j\bigr\|_{BMO}\right)M^{l+1}(f)(x).$$

Without loss of generality, we may assume $l = 2$. Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. Let $\tilde{Q} = 5\sqrt{n}Q$ and $\tilde{b}_j(x) = b_j(x) - \sum_{|\alpha|=m_j}\frac{1}{\alpha!}(D^\alpha b_j)_{\tilde{Q}}x^\alpha$, then $R_{m_j}(b_j;x,y) = R_{m_j}(\tilde{b}_j;x,y)$ and $D^\alpha\tilde{b}_j = D^\alpha b_j - (D^\alpha b_j)_{\tilde{Q}}$ for $|\alpha| = m_j$. We write, for $f = f\chi_{\tilde{Q}} + f\chi_{R^n\setminus\tilde{Q}} = f_1 + f_2$,

$$\begin{aligned}
T^b(f)(x) &= \int_{R^n}\frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{b}_j;x,y)}{|x-y|^m}K(x,y)f(y)\,dy = \int_{R^n}\frac{\prod_{j=1}^2 R_{m_j}(\tilde{b}_j;x,y)}{|x-y|^m}K(x,y)f_1(y)\,dy \\
&\quad - \sum_{|\alpha_1|=m_1}\frac{1}{\alpha_1!}\int_{R^n}\frac{R_{m_2}(\tilde{b}_2;x,y)(x-y)^{\alpha_1}D^{\alpha_1}\tilde{b}_1(y)}{|x-y|^m}K(x,y)f_1(y)\,dy \\
&\quad - \sum_{|\alpha_2|=m_2}\frac{1}{\alpha_2!}\int_{R^n}\frac{R_{m_1}(\tilde{b}_1;x,y)(x-y)^{\alpha_2}D^{\alpha_2}\tilde{b}_2(y)}{|x-y|^m}K(x,y)f_1(y)\,dy \\
&\quad + \sum_{|\alpha_1|=m_1,\,|\alpha_2|=m_2}\frac{1}{\alpha_1!\alpha_2!}\int_{R^n}\frac{(x-y)^{\alpha_1+\alpha_2}D^{\alpha_1}\tilde{b}_1(y)D^{\alpha_2}\tilde{b}_2(y)}{|x-y|^m}K(x,y)f_1(y)\,dy \\
&\quad + \int_{R^n}\frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{b}_j;x,y)}{|x-y|^m}K(x,y)f_2(y)\,dy \\
&= T\left(\frac{\prod_{j=1}^2 R_{m_j}(\tilde{b}_j;x,\cdot)}{|x-\cdot|^m}f_1\right) - T\left(\sum_{|\alpha_1|=m_1}\frac{1}{\alpha_1!}\frac{R_{m_2}(\tilde{b}_2;x,\cdot)(x-\cdot)^{\alpha_1}D^{\alpha_1}\tilde{b}_1}{|x-\cdot|^m}f_1\right) \\
&\quad - T\left(\sum_{|\alpha_2|=m_2}\frac{1}{\alpha_2!}\frac{R_{m_1}(\tilde{b}_1;x,\cdot)(x-\cdot)^{\alpha_2}D^{\alpha_2}\tilde{b}_2}{|x-\cdot|^m}f_1\right) \\
&\quad + T\left(\sum_{|\alpha_1|=m_1,\,|\alpha_2|=m_2}\frac{1}{\alpha_1!\alpha_2!}\frac{(x-\cdot)^{\alpha_1+\alpha_2}D^{\alpha_1}\tilde{b}_1D^{\alpha_2}\tilde{b}_2}{|x-\cdot|^m}f_1\right) \\
&\quad + T\left(\frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{b}_j;x,\cdot)}{|x-\cdot|^m}f_2\right)
\end{aligned}$$

and

$$
\begin{aligned}
A_{t_Q}T^b(f)(x) &= \int_{R^n}\frac{\prod_{j=1}^2 R_{m_j}(\tilde{b}_j;x,y)}{|x-y|^m}K_t(x,y)f_1(y)\,dy \\
&\quad - \sum_{|\alpha_1|=m_1}\frac{1}{\alpha_1!}\int_{R^n}\frac{R_{m_2}(\tilde{b}_2;x,y)(x-y)^{\alpha_1}D^{\alpha_1}\tilde{b}_1(y)}{|x-y|^m}K_t(x,y)f_1(y)\,dy \\
&\quad - \sum_{|\alpha_2|=m_2}\frac{1}{\alpha_2!}\int_{R^n}\frac{R_{m_1}(\tilde{b}_1;x,y)(x-y)^{\alpha_2}D^{\alpha_2}\tilde{b}_2(y)}{|x-y|^m}K_t(x,y)f_1(y)\,dy \\
&\quad + \sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2}\frac{1}{\alpha_1!\alpha_2!}\int_{R^n}\frac{(x-y)^{\alpha_1+\alpha_2}D^{\alpha_1}\tilde{b}_1(y)D^{\alpha_2}\tilde{b}_2(y)}{|x-y|^m}K_t(x,y)f_1(y)\,dy \\
&\quad + \int_{R^n}\frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{b}_j;x,y)}{|x-y|^m}K_t(x,y)f_2(y)\,dy \\
&= A_{t_Q}T\left(\frac{\prod_{j=1}^2 R_{m_j}(\tilde{b}_j;x,\cdot)}{|x-\cdot|^m}f_1\right) \\
&\quad - A_{t_Q}T\left(\sum_{|\alpha_1|=m_1}\frac{1}{\alpha_1!}\frac{R_{m_2}(\tilde{b}_2;x,\cdot)(x-\cdot)^{\alpha_1}D^{\alpha_1}\tilde{b}_1}{|x-\cdot|^m}f_1\right) \\
&\quad - A_{t_Q}T\left(\sum_{|\alpha_2|=m_2}\frac{1}{\alpha_2!}\frac{R_{m_1}(\tilde{b}_1;x,\cdot)(x-\cdot)^{\alpha_2}D^{\alpha_2}\tilde{b}_2}{|x-\cdot|^m}f_1\right) \\
&\quad + A_{t_Q}T\left(\sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2}\frac{1}{\alpha_1!\alpha_2!}\frac{(x-\cdot)^{\alpha_1+\alpha_2}D^{\alpha_1}\tilde{b}_1D^{\alpha_2}\tilde{b}_2}{|x-\cdot|^m}f_1\right) \\
&\quad + A_{t_Q}T\left(\frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{b}_j;x,\cdot)}{|x-\cdot|^m}f_2\right),
\end{aligned}
$$

then

$$
\begin{aligned}
&\left[\frac{1}{|Q|}\int_Q\left|\left|T^b(f)(x)\right|^r-\left|A_{t_Q}T^b(f)(x)\right|^r\right|dx\right]^{1/r} \\
&\quad\le\left[\frac{1}{|Q|}\int_Q\left|T^b(f)(x)-A_{t_Q}T^b(f)(x)\right|^r dx\right]^{1/r} \\
&\quad\le\left[\frac{C}{|Q|}\int_Q\left|T\left(\frac{\prod_{j=1}^2 R_{m_j}(\tilde{b}_j;x,\cdot)}{|x-\cdot|^m}f_1\right)\right|^r dx\right]^{1/r} \\
&\qquad+\left[\frac{C}{|Q|}\int_Q\left|T\left(\sum_{|\alpha_1|=m_1}\frac{R_{m_2}(\tilde{b}_2;x,\cdot)(x-\cdot)^{\alpha_1}D^{\alpha_1}\tilde{b}_1}{|x-\cdot|^m}f_1\right)\right|^r dx\right]^{1/r} \\
&\qquad+\left[\frac{C}{|Q|}\int_Q\left|T\left(\sum_{|\alpha_2|=m_2}\frac{R_{m_1}(\tilde{b}_1;x,\cdot)(x-\cdot)^{\alpha_2}D^{\alpha_2}\tilde{b}_2}{|x-\cdot|^m}f_1\right)\right|^r dx\right]^{1/r} \\
&\qquad+\left[\frac{C}{|Q|}\int_Q\left|T\left(\sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2}\int_Q\frac{(x-\cdot)^{\alpha_1+\alpha_2}D^{\alpha_1}\tilde{b}_1D^{\alpha_2}\tilde{b}_2}{|x-\cdot|^m}f_1\right)\right|^r dx\right]^{1/r} \\
&\qquad+\left[\frac{C}{|Q|}\int_Q\left|A_{t_Q}T\left(\frac{\prod_{j=1}^2 R_{m_j}(\tilde{b}_j;x,\cdot)}{|x-\cdot|^m}f_1\right)\right|^r dx\right]^{1/r}
\end{aligned}
$$

$$
\begin{aligned}
&+\left[\frac{C}{|Q|}\int_Q\left|A_{t_Q}T\left(\sum_{|\alpha_1|=m_1}\frac{1}{\alpha_1!}\frac{R_{m_2}(\tilde{b}_2;x,\cdot)(x-\cdot)^{\alpha_1}D^{\alpha_1}\tilde{b}_1}{|x-\cdot|^m}f_1\right)\right|^r dx\right]^{1/r}\\
&+\left[\frac{C}{|Q|}\int_Q\left|A_{t_Q}T\left(\sum_{|\alpha_2|=m_2}\frac{1}{\alpha_2!}\frac{R_{m_1}(\tilde{b}_1;x,\cdot)(x-\cdot)^{\alpha_2}D^{\alpha_2}\tilde{b}_2}{|x-\cdot|^m}f_1\right)\right|^r dx\right]^{1/r}\\
&+\left[\frac{C}{|Q|}\int_Q\left|A_{t_Q}T\left(\sum_{|\alpha_1|=m_1,\,|\alpha_2|=m_2}\frac{1}{\alpha_1!\alpha_2!}\frac{(x-\cdot)^{\alpha_1+\alpha_2}D^{\alpha_1}\tilde{b}_1D^{\alpha_2}\tilde{b}_2}{|x-\cdot|^m}f_1\right)\right|^r dx\right]^{1/r}\\
&+\left[\frac{C}{|Q|}\int_Q\left|(T-A_{t_Q}T)\left(\frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{b}_j;x,\cdot)}{|x-\cdot|^m}f_2\right)\right|^r dx\right]^{1/r}\\
&:= I_1+I_2+I_3+I_4+I_5+I_6+I_7+I_8+I_9.
\end{aligned}
$$

Now, let us estimate $I_1, I_2, I_3, I_4, I_5, I_6, I_7, I_8$ and I_9, respectively. First, for $x\in Q$ and $y\in\tilde{Q}$, by Lemma 1, we get

$$R_m(\tilde{b}_j;x,y)\le C|x-y|^m\sum_{|\alpha_j|=m}\left\|D^{\alpha_j}b_j\right\|_{BMO},$$

by Lemma 2 and the weak type $(1,1)$ of T (Lemma 4), we obtain

$$
\begin{aligned}
I_1&\le C\prod_{j=1}^2\left(\sum_{|\alpha_j|=m_j}\left\|D^{\alpha_j}b_j\right\|_{BMO}\right)\left(\frac{1}{|Q|}\int_{R^n}\left|T(f_1)(x)\right|^r dx\right)^{1/r}\\
&\le C\prod_{j=1}^2\left(\sum_{|\alpha_j|=m_j}\left\|D^{\alpha_j}b_j\right\|_{BMO}\right)|Q|^{-1}\frac{\|T(f_1)\chi_Q\|_{L^r}}{|Q|^{1/r-1}}\\
&\le C\prod_{j=1}^2\left(\sum_{|\alpha_j|=m_j}\left\|D^{\alpha_j}b_j\right\|_{BMO}\right)|Q|^{-1}\left\|T(f_1)\right\|_{WL^1}\\
&\le C\prod_{j=1}^2\left(\sum_{|\alpha_j|=m_j}\left\|D^{\alpha_j}b_j\right\|_{BMO}\right)|\tilde{Q}|^{-1}\|f_1\|_{L^1}\\
&\le C\prod_{j=1}^2\left(\sum_{|\alpha_j|=m_j}\left\|D^{\alpha_j}b_j\right\|_{BMO}\right)M(f)(\tilde{x}).
\end{aligned}
$$

For I_2, we get, by Lemma 2 and the generalized Hölder inequality,

$$
\begin{aligned}
I_2&\le C\sum_{|\alpha_2|=m_2}\left\|D^{\alpha_2}b_2\right\|_{BMO}\sum_{|\alpha_1|=m_1}\left(\frac{1}{|Q|}\int_{R^n}\left|T\left(D^{\alpha_1}\tilde{b}_1f_1\right)(x)\right|^r dx\right)^{1/r}\\
&\le C\sum_{|\alpha_2|=m_2}\left\|D^{\alpha_2}b_2\right\|_{BMO}\sum_{|\alpha_1|=m_1}|Q|^{-1}\frac{\|T(D^{\alpha_1}\tilde{b}_1f_1)\chi_Q\|_{L^r}}{|Q|^{1/r-1}}\\
&\le C\sum_{|\alpha_2|=m_2}\left\|D^{\alpha_2}b_2\right\|_{BMO}\sum_{|\alpha_1|=m_1}|Q|^{-1}\left\|T\left(D^{\alpha_1}\tilde{b}_1f_1\right)\right\|_{WL^1}\\
&\le C\sum_{|\alpha_2|=m_2}\left\|D^{\alpha_2}b_2\right\|_{BMO}\sum_{|\alpha_1|=m_1}|\tilde{Q}|^{-1}\left\|D^{\alpha_1}\tilde{b}_1f_1\right\|_{L^1}
\end{aligned}
$$

$$\le C \sum_{|\alpha_2|=m_2} \|D^{\alpha_2} b_2\|_{BMO} \sum_{|\alpha_1|=m_1} \|D^{\alpha_1} b_1 - (D^{\alpha} b_1)_{\tilde{Q}}\|_{\exp L, \tilde{Q}} \|f\|_{L(\log L), \tilde{Q}}$$

$$\le C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \right) M^2(f)(\tilde{x}).$$

For I_3, similar to the proof of I_2, we get

$$I_3 \le C \prod_{j=1}^{2} \left(\sum_{|\alpha|=m_j} \|D^{\alpha} b_j\|_{BMO} \right) M^2(f)(\tilde{x}).$$

Similarly, for I_4, taking $r, r_1, r_2 \ge 1$ such that $1/r = 1/r_1 + 1/r_2$, we obtain, by Lemma 3 and the generalized Hölder inequality,

$$\begin{aligned}
I_4 &\le C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \left(\frac{1}{|Q|} \int_{R^n} \left| T\left(D^{\alpha_1} \tilde{b}_1 D^{\alpha_2} \tilde{b}_2 f_1\right)(x)\right|^r dx \right)^{1/r} \\
&\le C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} |Q|^{-1} \frac{\|T(D^{\alpha_1} \tilde{b}_1 D^{\alpha_2} \tilde{b}_2 f_1)\chi_Q\|_{L^r}}{|Q|^{1/r-1}} \\
&\le C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} |Q|^{-1} \left\| T\left(D^{\alpha_1} \tilde{b}_1 D^{\alpha_2} \tilde{b}_2 f_1\right)\right\|_{WL^1} \\
&\le C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} |Q|^{-1} \left\| D^{\alpha_1} \tilde{b}_1 D^{\alpha_2} \tilde{b}_2 f_1 \right\|_{L^1} \\
&\le C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \prod_{j=1}^{2} \left\| D^{\alpha_j} b_j - \left(D^{\alpha_j} b_j\right)_{\tilde{Q}} \right\|_{\exp L^{r_j}, \tilde{Q}} \cdot \|f\|_{L(\log L)^{1/r}, \tilde{Q}} \\
&\le C \prod_{j=1}^{2} \left(\sum_{|\alpha|=m_j} \|D^{\alpha} b_j\|_{BMO} \right) M^3(f)(\tilde{x}).
\end{aligned}$$

For I_5, I_6, I_7 and I_8, by Lemma 6, we get

$$\begin{aligned}
I_5 + I_6 + I_7 + I_8 &\le C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \right) \frac{1}{|Q|} \int_Q \left| A_{t_Q} T(f_1)(x) \right| dx \\
&\quad + C \sum_{|\alpha_2|=m_2} \|D^{\alpha_2} b_2\|_{BMO} \sum_{|\alpha_1|=m_1} \frac{1}{|Q|} \int_Q \left| A_{t_Q} T\left(D^{\alpha_1} \tilde{b}_1 f_1\right)(x) \right| dx \\
&\quad + C \sum_{|\alpha_1|=m_1} \|D^{\alpha_1} b_1\|_{BMO} \sum_{|\alpha_2|=m_2} \frac{1}{|Q|} \int_Q \left| A_{t_Q} T\left(D^{\alpha_2} \tilde{b}_2 f_1\right)(x) \right| dx \\
&\quad + C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \frac{1}{|Q|} \int_Q \left| A_{t_Q} T\left(D^{\alpha_1} \tilde{b}_1 D^{\alpha_2} \tilde{b}_2 f_1\right)(x) \right| dx \\
&\le C \prod_{j=1}^{2} \left(\sum_{|\alpha|=m_j} \|D^{\alpha} b_j\|_{BMO} \right) M^3(f)(\tilde{x}).
\end{aligned}$$

For I_9, we write

$$
\begin{aligned}
&(T - A_{t_Q}T)\left(\frac{\prod_{j=1}^{2} R_{m_j+1}(\tilde{b}_j; x, \cdot)}{|x-\cdot|^m} f_2\right) \\
&\quad = \int_{R^n} \frac{\prod_{j=1}^{2} R_{m_j+1}(\tilde{b}_j; x, y)}{|x-y|^m}\big(K(x,y) - K_t(x,y)\big) f_2(y)\,dy \\
&\quad = \int_{R^n} \frac{\prod_{j=1}^{2} R_{m_j}(\tilde{b}_j; x, y)}{|x-y|^m}\big(K(x,y) - K_t(x,y)\big) f_2(y)\,dy \\
&\qquad - \sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1!} \int_{R^n} \frac{D^{\alpha_1}\tilde{b}_1(y)(x-y)^{\alpha_1} R_{m_2}(\tilde{b}_2; x, y)}{|x-y|^m}\big(K(x,y) - K_t(x,y)\big) f_2(y)\,dy \\
&\qquad - \sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2!} \int_{R^n} \frac{D^{\alpha_2}\tilde{b}_2(y)(x-y)^{\alpha_2} R_{m_1}(\tilde{b}_1; x, y)}{|x-y|^m}\big(K(x,y) - K_t(x,y)\big) f_2(y)\,dy \\
&\qquad + \sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2} \frac{1}{\alpha_1!\alpha_2!} \int_{R^n} \frac{D^{\alpha_1}\tilde{b}_1(y) D^{\alpha_2}\tilde{b}_2(y)(x-y)^{\alpha_1+\alpha_2}}{|x-y|^m}\big(K(x,y) - K_t(x,y)\big) f_2(y)\,dy \\
&\quad = I_9^{(1)} + I_9^{(2)} + I_9^{(3)} + I_9^{(4)}.
\end{aligned}
$$

By Lemma 1 and the following inequality (see [15])

$$
|b_{Q_1} - b_{Q_2}| \le C \log\big(|Q_2|/|Q_1|\big)\|b\|_{BMO} \quad \text{for } Q_1 \subset Q_2,
$$

we know that for $x \in Q$ and $y \in 2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q}$,

$$
\begin{aligned}
\big|R_m(\tilde{b}; x, y)\big| &\le C|x-y|^m \sum_{|\alpha|=m} \big(\|D^\alpha b\|_{BMO} + \big|(D^\alpha b)_{\tilde{Q}(x,y)} - (D^\alpha b)_{\tilde{Q}}\big|\big) \\
&\le Ck|x-y|^m \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO}.
\end{aligned}
$$

Note that $|x-y| \ge d = t^{1/2}$ and $|x-y| \sim |x_0 - y|$ for $x \in Q$ and $y \in R^n \setminus \tilde{Q}$. By the conditions on K and K_t, we obtain

$$
\begin{aligned}
\big|I_9^{(1)}\big| &= \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}} \frac{\prod_{j=1}^{2} |R_{m_j}(\tilde{b}_j; x, y)|}{|x-y|^m} \big|K(x,y) - K_t(x,y)\big|\big|f(y)\big|\,dy \\
&\le C \prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j} \|D^\alpha b_j\|_{BMO}\right) \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}} k^2 \frac{d^\delta}{|x_0 - y|^{n+\delta}} \big|f(y)\big|\,dy \\
&\le C \prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j} \|D^\alpha b_j\|_{BMO}\right) \sum_{k=1}^{\infty} k^2 2^{-\delta k} \frac{1}{|2^k\tilde{Q}|} \int_{2^k\tilde{Q}} \big|f(y)\big|\,dy \\
&\le C \prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j} \|D^\alpha b_j\|_{BMO}\right) M(f)(\tilde{x}).
\end{aligned}
$$

For $I_9^{(2)}$, we get, by the generalized Hölder inequality,

$$\begin{aligned}
\left|I_9^{(2)}\right| &\le C\Bigl(\sum_{|\alpha_2|=m_2}\|D^{\alpha_2}b_2\|_{BMO}\Bigr)\sum_{|\alpha_1|=m_1}\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}\frac{kd^{\delta}}{|x_0-y|^{n+\delta}}\left|D^{\alpha_1}\tilde{b}_1(y)\right|\left|f(y)\right|dy \\
&\le C\Bigl(\sum_{|\alpha_2|=m_2}\|D^{\alpha_2}b_2\|_{BMO}\Bigr) \\
&\quad\times\sum_{|\alpha_1|=m_1}\sum_{k=1}^{\infty}k2^{-\delta k}\left\|D^{\alpha_1}b_1-\left(D^{\alpha_1}b_1\right)_{\tilde{Q}}\right\|_{\exp L,2^k\tilde{Q}}\|f\|_{L(\log L),2^k\tilde{Q}} \\
&\le C\prod_{j=1}^{2}\Bigl(\sum_{|\alpha|=m_j}\|D^{\alpha}b_j\|_{BMO}\Bigr)M^2(f)(\tilde{x}).
\end{aligned}$$

Similarly,

$$\left|I_9^{(3)}\right| \le C\prod_{j=1}^{2}\Bigl(\sum_{|\alpha|=m_j}\|D^{\alpha}b_j\|_{BMO}\Bigr)M^2(f)(\tilde{x}).$$

For $I_9^{(4)}$, taking $r, r_1, r_2 \ge 1$ such that $1/r = 1/r_1 + 1/r_2$, by Lemma 3 and the generalized Hölder inequality, we get

$$\begin{aligned}
\left|I_9^{(4)}\right| &\le C\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}\frac{d^{\delta}}{|x_0-y|^{n+\delta}}\left|D^{\alpha_1}\tilde{b}_1(y)\right|\left|D^{\alpha_2}\tilde{b}_2(y)\right|\left|f(y)\right|dy \\
&\le C\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}\sum_{k=1}^{\infty}\prod_{j=1}^{2}\left\|D^{\alpha_j}b_j-\left(D^{\alpha_j}b_j\right)_{\tilde{Q}}\right\|_{\exp L^{r_j},2^k\tilde{Q}}\|f\|_{L(\log L)^{1/r},2^k\tilde{Q}} \\
&\le C\prod_{j=1}^{2}\Bigl(\sum_{|\alpha|=m_j}\|D^{\alpha}b_j\|_{BMO}\Bigr)M^3(f)(\tilde{x}).
\end{aligned}$$

Thus

$$|I_5| \le C\prod_{j=1}^{2}\Bigl(\sum_{|\alpha|=m_j}\|D^{\alpha}b_j\|_{BMO}\Bigr)M^3(f)(\tilde{x}).$$

This completes the proof of Theorem 1. □

By Theorem 1 and the $L^p(w)$-boundedness of M^{l+1}, we may obtain the conclusions of Theorem 2. By Theorem 1 and [16, 17], we may obtain the conclusions of Theorem 3.

3 Applications

In this section we shall apply Theorems 1, 2 and 3 of the paper to the holomorphic functional calculus of linear elliptic operators. First, we review some definitions regarding the holomorphic functional calculus (see [9]). Given $0 \le \theta < \pi$, define

$$S_\theta = \left\{z \in C : \left|\arg(z)\right| \le \theta\right\} \cup \{0\}$$

and its interior by S_θ^0. Set $\tilde{S}_\theta = S_\theta \setminus \{0\}$. A closed operator L on some Banach space E is said to be of type θ if its spectrum $\sigma(L) \subset S_\theta$ and if for every $\nu \in (\theta, \pi]$, there exists a constant C_ν such that

$$|\eta| \| (\eta I - L)^{-1} \| \leq C_\nu, \quad \eta \notin \tilde{S}_\theta.$$

For $\nu \in (0, \pi]$, let

$$H_\infty(S_\mu^0) = \{ f : S_\theta^0 \to C : f \text{ is holomorphic and } \|f\|_{L^\infty} < \infty \},$$

where $\|f\|_{L^\infty} = \sup\{|f(z)| : z \in S_\mu^0\}$. Set

$$\Psi(S_\mu^0) = \left\{ g \in H_\infty(S_\mu^0) : \exists s > 0, \exists c > 0 \text{ such that } |g(z)| \leq c \frac{|z|^s}{1 + |z|^{2s}} \right\}.$$

If L is of type θ and $g \in H_\infty(S_\mu^0)$, we define $g(L) \in L(E)$ by

$$g(L) = -(2\pi i)^{-1} \int_\Gamma (\eta I - L)^{-1} g(\eta) \, d\eta,$$

where Γ is the contour $\{\xi = re^{\pm i\phi} : r \geq 0\}$ parameterized clockwise around S_θ with $\theta < \phi < \mu$. If, in addition, L is one-to-one and has a dense range, then, for $f \in H_\infty(S_\mu^0)$,

$$f(L) = [h(L)]^{-1} (fh)(L),$$

where $h(z) = z(1+z)^{-2}$. L is said to have a bounded holomorphic functional calculus on the sector S_μ if

$$\|g(L)\| \leq N \|g\|_{L^\infty}$$

for some $N > 0$ and for all $g \in H_\infty(S_\mu^0)$.

Now, let L be a linear operator on $L^2(R^n)$ with $\theta < \pi/2$ so that $(-L)$ generates a holomorphic semigroup e^{-zL}, $0 \leq |\arg(z)| < \pi/2 - \theta$. Applying Theorem 6 of [9], we get the following.

Theorem 4 *Assume the following conditions are satisfied*:

(i) *The holomorphic semigroup* e^{-zL}, $0 \leq |\arg(z)| < \pi/2 - \theta$ *is represented by the kernels* $a_z(x,y)$ *which satisfy, for all* $\nu > \theta$, *an upper bound*

$$|a_z(x,y)| \leq c_\nu h_{|z|}(x,y)$$

for $x, y \in R^n$, *and* $0 \leq |\arg(z)| < \pi/2 - \theta$, *where* $h_t(x,y) = Ct^{-n/2} s(|x-y|^2/t)$ *and* s *is a positive, bounded and decreasing function satisfying*

$$\lim_{r \to \infty} r^{n+\epsilon} s(r^2) = 0.$$

(ii) *The operator L has a bounded holomorphic functional calculus in $L^2(R^n)$, that is, for all $\nu > \theta$ and $g \in H_\infty(S_\mu^0)$, the operator $g(L)$ satisfies*

$$\|g(L)(f)\|_{L^2} \le c_\nu \|g\|_{L^\infty} \|f\|_{L^2}.$$

Then, for $D^\alpha b_j \in BMO(R^n)$ for all α with $|\alpha| = m_j$ and $j = 1, \ldots, l$, the multilinear operator $g(L)^b$ associated to $g(L)$ and b_j satisfies:

(a) *For $0 < r < 1$ and $\tilde{x} \in R^n$*,

$$M_A^\#\big(g(L)^b(f)\big)_r(\tilde{x}) \le C \prod_{j=1}^{l} \bigg(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \bigg) M^{l+1}(f)(\tilde{x});$$

(b) *$g(L)^b$ is bounded on $L^p(w)$ for any $1 < p < \infty$ and $w \in A_p$, that is*,

$$\|g(L)^b(f)\|_{L^p(w)} \le C \prod_{j=1}^{l} \bigg(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \bigg) \|f\|_{L^p(w)};$$

(c) *There exists a constant $C > 0$ such that for all $\lambda > 0$ and $w \in A_1$*,

$$w\big(\big\{x \in R^n : \big|g(L)^b(f)(x)\big| > \lambda\big\}\big) \le C \int_{R^n} \frac{|f(x)|}{\lambda} \left[1 + \log^+\left(\frac{|f(x)|}{\lambda}\right)\right]^l w(x)\,dx.$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors completed the paper together. They also read and approved the final manuscript.

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