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Some identities of Genocchi polynomials arising from Genocchi basis

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Abstract
In this paper, we give some interesting identities which are derived from the basis of Genocchi. From our methods which are treated in this paper, we can derive some new identities associated with Bernoulli and Euler polynomials.
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1 Introduction

As is well known, the Genocchi polynomials are defined by the generating function to be

$$\frac{2t}{e^t+1}e^{xt} = e^{G(x)t} = \sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!} \quad \bigl(\text{see [1–9]}\bigr) \tag{1}$$

with the usual convention about replacing $G^n(x)$ by $G_n(x)$.

In the special case $x = 0$, $G_n(0) = G_n$ are called the nth Genocchi numbers. From (1), we note that

$$G_0 = 0, \qquad G_n(1) + G_n = 2\delta_{n,1} \quad \bigl(\text{see [10–16]}\bigr), \tag{2}$$

where $\delta_{n,k}$ is the Kronecker symbol.

$$G_n(x) = (G+x)^n = \sum_{l=0}^{n} \binom{n}{l} G_l x^{n-l} \quad \bigl(\text{see [6–8, 17]}\bigr). \tag{3}$$

Thus, by (2) and (3), we see that

$$\frac{d}{dx}G_n(x) = nG_{n-1}(x), \qquad \deg G_n(x) = n-1. \tag{4}$$

The nth Bernoulli polynomials are also defined by the generating function to be

$$\frac{t}{e^t-1}e^{xt} = e^{B(x)t} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} \quad \bigl(\text{see [14–16]}\bigr) \tag{5}$$

with the usual convention about replacing $B^n(x)$ by $B_n(x)$.

In the special case $x = 0$, $B_n(0) = B_n$ are called the nth Bernoulli numbers. By (5), we get

$$B_0 = 1, \qquad B_n(1) - B_n = \delta_{1,n} \quad \bigl(\text{see [8, 9, 17]}\bigr) \tag{6}$$

and

$$B_n(x) = \sum_{l=0}^{n} \binom{n}{l} B_l x^{n-l} = \sum_{l=0}^{n} \binom{n}{l} B_{n-l} x^l. \tag{7}$$

The Euler numbers are defined by

$$E_0 = 1, \qquad (E+1)^n + E_n = 2\delta_{0,n}. \tag{8}$$

The Euler polynomials are defined by

$$E_n(x) = (E+x)^n = \sum_{l=0}^{n} \binom{n}{l} E_{n-l} x^l \quad \bigl(\text{see [7–13, 17]}\bigr). \tag{9}$$

Let $\mathbb{P}_n = \{p(x) \in \mathbb{Q}[x] \mid \deg p(x) \leq n\}$ be the $(n+1)$-dimensional vector space over $\mathbb{Q}$. Probably, $\{1, x, \ldots, x^n\}$ is the most natural basis for $\mathbb{P}_n$. But $\{G_1(x), G_2(x), \ldots, G_{n+1}(x)\}$ is also a good basis for the space $\mathbb{P}_n$ for our purpose of arithmetical applications of Genocchi polynomials. Let $p(x) \in \mathbb{P}_n$. Then $p(x)$ can be expressed by $p(x) = \sum_{1 \leq k \leq n+1} b_k G_k(x)$.

In this paper, we develop some new methods to obtain some new identities and properties of Genocchi polynomials which are derived from the Genocchi basis.

2 Genocchi basis and some identities of Genocchi polynomials

Let us take $p(x) \in \mathbb{P}_n$. Then $p(x)$ can be expressed as a $\mathbb{Q}$-linear combination of $G_1(x)$, $G_2(x), \ldots, G_{n+1}(x)$ as follows:

$$p(x) = \sum_{1 \leq k \leq n+1} b_k G_k(x) = b_1 G_1(x) + b_2 G_2(x) + \cdots + b_{n+1} G_{n+1}(x). \tag{10}$$

Now, let us define the operator $\tilde{\triangle}$ by

$$\tilde{\triangle} p(x) = p(x+1) + p(x). \tag{11}$$

Then, by (10) and (11), we set

$$g(x) = \tilde{\triangle} p(x) = \sum_{1 \leq k \leq n+1} b_k \bigl(G_k(x+1) + G_k(x)\bigr). \tag{12}$$

From (1), we note that

$$\sum_{n=0}^{\infty} \bigl\{G_n(x+1) + G_n(x)\bigr\} \frac{t^n}{n!} = \frac{2t}{e^t+1} e^{(x+1)t} + \frac{2t}{e^t+1} e^{xt}. \tag{13}$$

By (2), (3) and (13), we get

$$\frac{G_{n+1}(x+1) + G_{n+1}(x)}{n+1} = 2x^n. \tag{14}$$

From (12) and (14), we get

$$g(x) = \tilde{\triangle} p(x) = 2 \sum_{1 \leq k \leq n+1} k b_k x^{k-1}. \tag{15}$$

For $r \in \mathbb{N}$, let us take the rth derivative of $g(x)$ in (15) as follows:

$$g^{(r)}(x) = \frac{d^r g(x)}{dx^r} = 2 \sum_{1 \le k \le n+1} k(k-1) \cdots (k-1-r+1) b_k x^{k-1-r}. \tag{16}$$

Thus, by (16), we get

$$g^{(r)}(0) = \left. \frac{d^r g(x)}{dx^r} \right|_{x=0} = 2(r+1)! b_{r+1}. \tag{17}$$

From (11) and (17), we have

$$b_{r+1} = \frac{1}{2(r+1)!} \left\{ p^{(r)}(1) + p^{(r)}(0) \right\}, \tag{18}$$

where $p^{(r)}(a) = \frac{d^r g(x)}{dx^r}|_{x=a}$.

Therefore, by (10) and (18), we obtain the following theorem.

Theorem 1 *For $n \in \mathbb{N}$, let $p(x) \in \mathbb{P}_n$ with $p(x) = \sum_{1 \le k \le n+1} b_k G_k(x)$.*
Then we have

$$b_k = \frac{1}{2k!} \left(p^{(k-1)}(1) + p^{(k-1)}(0) \right).$$

Let us assume that $p(x) = B_n(x)$. Then by Theorem 1, we get

$$B_n(x) = \sum_{1 \le k \le n+1} b_k G_k(x), \tag{19}$$

where

$$b_k = \frac{1}{2k!} \left\{ p^{(k-1)}(1) + p^{(k-1)}(0) \right\} = \frac{1}{2k!} (n)_{k-1} \left\{ B_{n-k+1}(1) + B_{n-k+1} \right\}. \tag{20}$$

From (6) and (20), we have

$$b_k = \frac{1}{2(n+1)} \binom{n+1}{k} \{ \delta_{n,k} + 2B_{n-k+1} \}. \tag{21}$$

By (19) and (21), we get

$$\begin{aligned} B_n(x) &= \frac{1}{n+1} \sum_{1 \le k \le n-1} \binom{n+1}{k} B_{n-k+1} G_k(x) + \frac{1}{2}(1 + 2B_1) G_n(x) + \frac{1}{2(n+1)} 2G_{n+1}(x) \\ &= \frac{1}{n+1} \sum_{1 \le k \le n-1} \binom{n+1}{k} B_{n-k+1} G_k(x) + \frac{1}{n+1} G_{n+1}(x). \end{aligned} \tag{22}$$

Therefore, by (22), we obtain the following theorem.

Theorem 2 *For $n \in \mathbb{N}$, we have*

$$B_n(x) = \frac{1}{n+1} \sum_{1 \le k \le n-1} \binom{n+1}{k} B_{n-k+1} G_k(x) + \frac{1}{n+1} G_{n+1}(x).$$

In particular, if we take $p(x) = E_n(x) \in \mathbb{P}_n$, then we have

$$E_n(x) = \sum_{1 \le k \le n+1} b_k G_k(x), \tag{23}$$

where

$$b_k = \frac{1}{2k!}\left\{p^{(k-1)}(1) + p^{(k-1)}(0)\right\} = \frac{1}{2k!}(n)_{k-1}\left\{E_{n-k+1}(1) + E_{n-k+1}\right\}. \tag{24}$$

By (8) and (24), we get

$$\begin{aligned} b_k &= \frac{1}{2(n+1)}\binom{n+1}{k}\{2\delta_{n-k+1,0} - E_{n-k+1} + E_{n-k+1}\} \\ &= \frac{1}{n+1}\binom{n+1}{k}\delta_{n+1,k}. \end{aligned} \tag{25}$$

From (23) and (25), we have

$$E_n(x) = \frac{1}{n+1} G_{n+1}(x).$$

Let us take $p(x) \in \mathbb{P}_n$ with

$$p(x) = \sum_{0 \le k \le n} B_k(x) B_{n-k}(x). \tag{26}$$

Then we have

$$\begin{aligned} \frac{dp(x)}{dx} &= p^{(1)}(x) = \sum_{k=1}^{n} kB_{k-1}(x)B_{n-k}(x) + \sum_{k=0}^{n-1}(n-k)B_k(x)B_{n-k-1}(x) \\ &= (n+1)\sum_{k=1}^{n} B_{k-1}(x)B_{n-k}(x), \\ \frac{d^2p(x)}{dx^2} &= p^{(2)}(x) = (n+1)n\sum_{k=2}^{n} B_{k-2}(x)B_{n-k}(x). \end{aligned}$$

Continuing this process, we get

$$\begin{aligned} \frac{d^k p(x)}{dx^k} &= p^{(k)}(x) = (n+1)n\cdots(n+1-k+1)\sum_{l=k}^{n} B_{l-k}(x)B_{n-l}(x) \\ &= \frac{(n+1)!}{(n+1-k)!}\sum_{l=k}^{n} B_{l-k}(x)B_{n-l}(x). \end{aligned} \tag{27}$$

From (27), we have

$$p^{(k-1)}(1) = \frac{(n+1)!}{(n+2-k)!}\sum_{l=k-1}^{n} B_{l+1-k}(1)B_{n-l}(1). \tag{28}$$

By (6), we get

$$B_{l+1-k}(1)B_{n-l}(1) = (\delta_{l+1-k,1} + B_{l+1-k})(\delta_{n-l,1} + B_{n-l}) \\ = \{\delta_{k,n-1} + B_{n-k} + B_{n-k} + B_{l+1-k}B_{n-l}\}. \tag{29}$$

From (28) and (29), we have

$$p^{(k-1)}(1) = \frac{(n+1)!}{(n+2-k)!}\left\{\delta_{k,n-1} + 2B_{n-k} + \sum_{k-1\le l\le n} B_{l+1-k}B_{n-l}\right\}. \tag{30}$$

By Theorem 1, $p(x) = \sum_{0\le k\le n} B_k(x)B_{n-k}(x)$ can be expressed by

$$p(x) = \sum_{1\le k\le n+1} b_k(x)G_k(x), \tag{31}$$

where

$$b_k = \frac{1}{2k!}\left\{p^{(k-1)}(1) + p^{(k-1)}(0)\right\} \\ = \frac{(n+1)!}{2k!(n+2-k)!}\left\{\delta_{k,n-1} + 2B_{n-k} + 2\sum_{l=k-1} B_{l+1-k}B_{n-l}\right\}. \tag{32}$$

Thus, by (31) and (32), we get

$$p(x) = \frac{n(n+1)}{12}G_{n-1}(x) + \sum_{1\le k\le n+1}\frac{1}{k}\binom{n+1}{k-1}B_{n-k}G_k(x) \\ + \sum_{1\le k\le n+1}\frac{1}{k}\binom{n+1}{k-1}\sum_{l=k-1}^{n} B_{l+1-k}B_{n-l}G_k(x). \tag{33}$$

Therefore, by (31) and (33), we obtain the following theorem.

Theorem 3 *For $n \in \mathbb{N}$, we have*

$$\sum_{k=0}^{n} B_k(x)B_{n-k}(x) = \frac{n(n+1)}{12}G_{n-1}(x) + \sum_{1\le k\le n+1}\frac{1}{k}\binom{n+1}{k-1}B_{n-k}G_k(x) \\ + \sum_{1\le k\le n+1}\left(\sum_{k-1\le l\le n}\frac{1}{k}\binom{n+1}{k-1}B_{l+1-k}B_{n-l}\right)G_k(x).$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript and typed, read, and approved the final manuscript.

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