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On the improvement of Mocanu's conditions

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Abstract

We estimate $|\operatorname{Arg}\{p(z)\}|$ for functions of the form $p(z) = 1 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots$ in the unit disc $\mathbb{D} = \{z : |z| < 1\}$ under several assumptions. By using Nunokawa's lemma, we improve a few of Mocanu's results obtained by differential subordinations. Some applications for strongly starlikeness and convexity are formulated.

MSC: Primary 30C45; secondary 30C80

Keywords: Nunokawa's lemma; strongly starlike functions of order alpha; strongly convex functions of order alpha; subordination

1 Introduction

Let $\mathcal{H}$ be the class of functions analytic in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, and denote by $\mathcal{A}$ the class of analytic functions in $\mathbb{D}$ and usually normalized, *i.e.*, $\mathcal{A} = \{f \in \mathcal{H} : f(0) = 0, f'(0) = 1\}$.

Let $\mathcal{SS}^*(\beta)$ denote the class of strongly starlike functions of order β, $0 < \beta \le 1$,

$$\mathcal{SS}^*(\beta) := \left\{ f \in \mathcal{A} : \left| \operatorname{Arg} \frac{zf'(z)}{f(z)} \right| < \frac{\beta\pi}{2}, z \in \mathbb{D} \right\},$$

which was introduced in [1] and [2]. We say that $f \in \mathcal{A}$ is in the class $\mathcal{SC}^*(\beta)$ of strongly convex functions of order β when $zf'(z) \in \mathcal{SS}^*(\beta)$. We say that $f \in \mathcal{H}$ is subordinate to $g \in \mathcal{H}$ in the unit disc $\mathbb{D}$, written $f \prec g$ if and only if there exists an analytic function $w \in \mathcal{H}$ such that $w(0) = 0$, $|w(z)| < 1$ and $f(z) = g[w(z)]$ for $z \in \mathbb{D} \subseteq g(\mathbb{D})$. In particular, if g is univalent in $\mathbb{D}$ then the subordination principle says that $f \prec g$ if and only if $f(0) = g(0)$ and $f(|z| < r) \subseteq g(|z| < r)$ for all $r \in (0,1)$.

2 Main result

In this section, we investigate conditions, under which a function $f \in \mathcal{A}$ is strongly starlike or strongly convex. We also estimate $|\operatorname{Arg}\{p(z)\}|$ for functions of the form $p(z) = 1 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots$ in the unit disc $\mathbb{D}$, under several assumptions, and then we use this estimation for the case $p(z) = zf'(z)/f(z)$. By using Nunokawa's lemma [3], we improve a few Mocanu's [4, 5] results obtained by differential subordinations. Some sufficient conditions for functions to be in several subclasses of strongly starlike functions can also be found in the recent papers [6] and [7–11].

Theorem 2.1 *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in the unit disc $\mathbb{D}$. If*

$$\left|\operatorname{Arg}\{f'(z)\}\right| < \frac{\alpha\pi}{2} \approx 1.0076658, \quad z \in \mathbb{D}, \tag{2.1}$$

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where $\alpha = 1/(1+\beta) = 1/(2-(\log 4)/\pi) \approx 0.641548$, $\beta = 1-(\log 4)/\pi \approx 0.5587$, *then*

$$\left|\operatorname{Arg}\left\{\frac{zf'(z)}{f(z)}\right\}\right| < \frac{\pi}{2}, \quad z \in \mathbb{D}, \tag{2.2}$$

or f *is starlike in* $\mathbb{D}$.

Proof By (2.1), we have

$$\{f'(z)\}^{1/\alpha} \prec \frac{1+z}{1-z}, \quad z \in \mathbb{D}.$$

Let $z = \rho e^{i\theta}$, $\rho \in [0,1)$, $\theta \in (-\pi,\pi]$. The function $w(z) = (1+z)/(1-z)$ is univalent in $\mathbb{D}$ and maps $|z| < \rho < 1$ onto the open disc $D(C,R)$ with the center $C = (1+\rho^2)/(1-\rho^2)$ and the radius $R = (2\rho)/(1-\rho^2)$. Then by the subordination principle under univalent function,

$$\{f'(xe^{i\theta})\}^{1/\alpha} \in D(C,R) \quad \text{for all } x \in [0,\rho), \theta \in (-\pi,\pi]. \tag{2.3}$$

A simple geometric observation yields to

$$\left|\operatorname{Arg}\left\{\left(f'(\rho e^{i\theta})\right)^{1/\alpha}\right\}\right| \le \sin^{-1}\frac{R}{C} = \sin^{-1}\frac{2\rho}{1+\rho^2} \quad \text{for all } \rho \in [0,1), \theta \in (-\pi,\pi]. \tag{2.4}$$

Therefore, applying the same idea as [3, pp.1292-1293] for $z = re^{i\theta}$, $r \in [0,1)$, $\theta \in (-\pi,\pi]$, we have

$$\begin{aligned}
\left|\operatorname{Arg}\left\{\frac{f(z)}{z}\right\}\right| &= \left|\operatorname{Arg}\left\{\int_0^r f'(\rho e^{i\theta})\,\mathrm{d}\rho\right\}\right| \\
&\le \int_0^r \left|\operatorname{Arg}\{f'(\rho e^{i\theta})\}\right| \mathrm{d}\rho \\
&= \alpha \int_0^r \left|\operatorname{Arg}\left\{\left(f'(\rho e^{i\theta})\right)^{1/\alpha}\right\}\right| \mathrm{d}\rho \\
&\le \alpha \int_0^r \sin^{-1}\frac{2\rho}{1+\rho^2}\,\mathrm{d}\rho \\
&= \alpha\left\{\rho\sin^{-1}\frac{2\rho}{1+\rho^2} - \log(1+\rho^2)\right\}\bigg|_{\rho=0}^{\rho=r} \\
&= \alpha\left\{r\sin^{-1}\frac{2r}{1+r^2} - \log(1+r^2)\right\}.
\end{aligned}$$

The function

$$h(r) = r\sin^{-1}\frac{2r}{1+r^2} - \log(1+r^2), \quad r \in [0,1)$$

is increasing because $h'(r) = \sin^{-1}\{2r/(1+r^2)\} > 0$. Now, letting $r \to 1^-$, we obtain

$$\begin{aligned}
\left|\operatorname{Arg}\left\{\frac{f(z)}{z}\right\}\right| &\le \alpha(\pi/2 - \log 2) = \frac{\alpha\pi}{2}\left\{1 - \frac{\log 4}{\pi}\right\} \\
&= \frac{\pi}{2}\alpha\beta, \quad z \in \mathbb{D}.
\end{aligned}$$

Using this and (2.1), we obtain

$$\begin{aligned} \left|\operatorname{Arg}\left\{\frac{zf'(z)}{f(z)}\right\}\right| &\le \left|\operatorname{Arg}\left\{f'(z)\right\}\right| + \left|\operatorname{Arg}\left\{\frac{f(z)}{z}\right\}\right| \\ &< \frac{\alpha\pi}{2} + \frac{\pi}{2}\alpha\beta \\ &= \frac{\alpha(1+\beta)\pi}{2} \\ &= \frac{\pi}{2}, \quad z \in \mathbb{D}. \end{aligned}$$

It completes the proof. □

Remark 2.2 Theorem 2.1 is an improvement of Mocanu's result in [4].

Theorem 2.3 *Let* $p(z) = 1 + \sum_{n=1}^{\infty} a_n z^n$ *be analytic in the unit disc* $\mathbb{D}$. *If*

$$\mathfrak{Re}\left\{p(z) + zp'(z)\right\} > 0, \quad z \in \mathbb{D}, \tag{2.5}$$

then

$$\left|\operatorname{Arg}\left\{p(z)\right\}\right| < \frac{\pi}{2} - \log 2 = 0.877649\ldots, \quad z \in \mathbb{D}. \tag{2.6}$$

Proof By (2.5), we have

$$p(z) + zp'(z) \prec \frac{1+z}{1-z}, \quad z \in \mathbb{D}. \tag{2.7}$$

Let $z = \rho e^{i\theta}$, $\rho \in [0,1)$, $\theta \in (-\pi, \pi]$. The subordination principle used for (2.7) gives

$$\left|p\left(xe^{i\theta}\right) + xe^{i\theta}p'\left(xe^{i\theta}\right) - \frac{1+\rho^2}{1-\rho^2}\right| < \frac{2\rho}{1-\rho^2} \quad \text{for all } x \in [0,\rho), \theta \in (-\pi,\pi]. \tag{2.8}$$

A simple geometric observation yields to

$$\left|\operatorname{Arg}\left\{p\left(\rho e^{i\theta}\right) + \rho e^{i\theta}p'\left(\rho e^{i\theta}\right)\right\}\right| \le \sin^{-1}\frac{2\rho}{1+\rho^2} \quad \text{for all } \rho \in [0,1), \theta \in (-\pi,\pi]. \tag{2.9}$$

Therefore, for $z = re^{i\theta}$, $r \in [0,1)$, $\theta \in (-\pi,\pi]$, we have

$$\begin{aligned}
\left|\operatorname{Arg}\{p(z)\}\right| &= \left|\operatorname{Arg}\left\{\frac{zp(z)}{z}\right\}\right| \\
&= \left|\operatorname{Arg}\left\{\frac{\int_0^z (tp(t))'\,\mathrm{d}t}{z}\right\}\right| \\
&= \left|\operatorname{Arg}\left\{\frac{\int_0^z (p(t)+tp'(t))\,\mathrm{d}t}{z}\right\}\right| \\
&= \left|\operatorname{Arg}\left\{\frac{\int_0^r (p(\rho e^{i\theta})+\rho e^{i\theta}p'(\rho e^{i\theta}))e^{i\theta}\,\mathrm{d}\rho}{re^{i\theta}}\right\}\right| \\
&= \left|\operatorname{Arg}\left\{\int_0^r \left(p\left(\rho e^{i\theta}\right)+\rho e^{i\theta}p'\left(\rho e^{i\theta}\right)\right)\mathrm{d}\rho\right\} - \operatorname{Arg}\{r\}\right| \\
&\le \int_0^r \left|\operatorname{Arg}\left\{p\left(\rho e^{i\theta}\right)+\rho e^{i\theta}p'\left(\rho e^{i\theta}\right)\right\}\right|\mathrm{d}\rho.
\end{aligned}$$

Therefore, by using (2.9), we have

$$\begin{aligned}
\left|\operatorname{Arg}\{p(z)\}\right| &\le \alpha\int_0^r \sin^{-1}\frac{2\rho}{1+\rho^2}\,\mathrm{d}\rho \\
&= \alpha\left\{\rho\sin^{-1}\frac{2\rho}{1+\rho^2} - \log\left(1+\rho^2\right)\right\}\Bigg|_{\rho=0}^{\rho=r} \\
&< \alpha\left\{\rho\sin^{-1}\frac{2\rho}{1+\rho^2} - \log\left(1+\rho^2\right)\right\}\Bigg|_{\rho=0}^{\rho=1} \\
&= \frac{\pi}{2} - \log 2 = 0.8776491464\ldots, \quad z \in \mathbb{D}.
\end{aligned}$$

It leads to the desired conclusion. □

Remark 2.4 Theorem 2.3 is an improvement of Mocanu's result in [5], where instead of $\gamma_0 = \frac{\pi}{2} - \log 2 = 0.8776491464\ldots$ is

$$\theta_1 = \max\left\{\theta : \left|\operatorname{Arg}\left\{\frac{2}{e^{i\theta}}\log\left(1+e^{i\theta}\right) - 1\right\}\right|\right\} = 0.91106219\ldots.$$

Substituting $p(z) = f(z)/z, f \in \mathcal{A}$, in Theorem 2.3 leads to the following corollary.

Corollary 2.5 *If $f \in \mathcal{A}$ and it satisfies*

$$\mathfrak{Re}\left\{f'(z)\right\} > 0, \quad z \in \mathbb{D},$$

then

$$\left|\operatorname{Arg}\left\{\frac{f(z)}{z}\right\}\right| < \frac{\pi}{2} - \log 2 = 0.877649\ldots, \quad z \in \mathbb{D}.$$

Substituting $p(z) = zf'(z)/f(z), f \in \mathcal{A}$, in Theorem 2.3 gives the following corollary.

Corollary 2.6 *If $f \in \mathcal{A}$ and it satisfies*

$$\mathfrak{Re}\left\{\frac{zf'(z)}{f(z)}\left(2+\frac{zf''(z)}{f'(z)}\right)-\left(\frac{zf'(z)}{f(z)}\right)^2\right\}>0,\quad z\in\mathbb{D},$$

then

$$\left|\operatorname{Arg}\left\{\frac{zf'(z)}{f(z)}\right\}\right|<\frac{\pi}{2}-\log 2=0.877649\ldots,\quad z\in\mathbb{D}.$$

This means that f is strongly starlike of order $1-(\log 4)/\pi=0.558728799\ldots$.

Substituting $p(z)=1+zf''(z)/f'(z), f\in\mathcal{A}$, in Theorem 2.3 gives the following corollary.

Corollary 2.7 *If $f \in \mathcal{A}$ and it satisfies*

$$\mathfrak{Re}\left\{1+\frac{zf''(z)}{f'(z)}\left(2+\frac{zf'''(z)}{f''(z)}\right)-\left(\frac{zf''(z)}{f'(z)}\right)^2\right\}>0,\quad z\in\mathbb{D},$$

then

$$\left|\operatorname{Arg}\left\{1+\frac{zf''(z)}{f'(z)}\right\}\right|<\frac{\pi}{2}-\log 2=0.877649\ldots,\quad z\in\mathbb{D}.$$

This means that f is strongly convex of order $1-(\log 4)/\pi=0.558728799\ldots$.

Theorem 2.8 *Let $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ be analytic in the unit disc $\mathbb{D}$, and suppose that*

$$\left|f'(z)-1\right|<1,\quad z\in\mathbb{D}. \tag{2.10}$$

Then we have

$$\left|\operatorname{Arg}\left\{\frac{zf'(z)}{f(z)}\right\}\right|<(1+r)\sin^{-1}r+\sqrt{1-r^2}-1, \tag{2.11}$$

where $r=|z|<1$, and, therefore, we have

$$\mathfrak{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>0\quad \text{for } |z|<r_0, \tag{2.12}$$

where $0.902<r_0<0.903$ is the positive root of the equation

$$\sin^{-1}r=\frac{\pi-2(\sqrt{1-r^2}-1)}{2(1+r)}. \tag{2.13}$$

Proof From (2.10), we have $f'(z)\prec 1+z$, so the subordination principle gives

$$\left|\operatorname{Arg}\left\{f'(z)\right\}\right|\leq\sin^{-1}|z|,\quad z\in\mathbb{D} \tag{2.14}$$

and for $z = re^{i\theta}$,

$$\begin{aligned}\left|\operatorname{Arg}\left\{\frac{f(z)}{z}\right\}\right| &= \left|\operatorname{Arg}\left\{\frac{1}{re^{i\theta}}\int_0^r f'(\rho e^{i\theta})e^{i\theta}\,\mathrm{d}\rho\right\}\right| \\ &= \left|\operatorname{Arg}\left\{\int_0^r f'(\rho e^{i\theta})\,\mathrm{d}\rho\right\}\right| \\ &\le \int_0^r \left|\operatorname{Arg}\{f'(\rho e^{i\theta})\}\right|\mathrm{d}\rho \\ &< \int_0^r \sin^{-1}\rho\,\mathrm{d}\rho.\end{aligned}$$

Then we have

$$\int_0^r \sin^{-1}\rho\,\mathrm{d}\rho = r\sin^{-1} r + \sqrt{1-r^2} - 1.$$

Therefore, and from (2.14), we have

$$\begin{aligned}&\left|\operatorname{Arg}\left\{\frac{zf'(z)}{f(z)}\right\}\right| \\ &\quad\le \left|\operatorname{Arg}\{f'(z)\}\right| + \left|\operatorname{Arg}\left\{\frac{f(z)}{z}\right\}\right| \\ &\quad< \sin^{-1} r + r\sin^{-1} r + \sqrt{1-r^2} - 1, \quad |z| = r < 1.\end{aligned}$$

The function

$$\begin{aligned}G(r) &= (1+r)\sin^{-1} r + \sqrt{1-r^2} - 1 \\ &= \sin^{-1} r + \int_0^r \sin^{-1}\rho\,\mathrm{d}\rho\end{aligned}$$

increases in $[0,1]$ as the sum of two increasing functions. Moreover, $G(0) = 0$, $G(1) = \pi - 1$, and it satisfies

$$G(0.902) = 1.57030\ldots < \frac{\pi}{2} = 1.5707963\ldots < G(0.903) = 1.573753\ldots.$$

Therefore, the equation (2.13) has the solution r_0, $0.902 < r_0 < 0.903$, and

$$\mathfrak{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0 \quad \text{for } |z| < r_0 \approx 0.903.$$

This completes the proof. □

Theorem 2.9 *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *be analytic in the unit disc* $\mathbb{D}$, *and suppose that*

$$\left|f'(z) - 1\right| < \alpha, \quad z \in \mathbb{D}, \tag{2.15}$$

with $\alpha \in (0, 2/\sqrt{5}]$. *Then* f *is strongly starlike of order* β, *where* $\beta \in (0,1]$ *is the positive root of the equation*

$$\sin^{-1}\left\{\alpha\sqrt{1-\alpha^2/4} + \frac{\alpha}{2}\sqrt{1-\alpha^2}\right\} = \frac{\pi\beta}{2}. \tag{2.16}$$

Proof We have $f'(z) \prec 1 + \alpha z$. Applying the result from [4, p.118] we have also that $f(z)/z \prec 1 + \alpha z/2$ in $\mathbb{D}$. This shows that

$$\left|\operatorname{Arg}\{f'(z)\}\right| \le \sin^{-1}\alpha|z|, \quad z \in \mathbb{D}, \tag{2.17}$$

and

$$\left|\operatorname{Arg}\left\{\frac{f(z)}{z}\right\}\right| \le \sin^{-1}\frac{\alpha|z|}{2}, \quad z \in \mathbb{D}. \tag{2.18}$$

Therefore, using (2.17) and (2.18), we have

$$\begin{aligned} &\left|\operatorname{Arg}\left\{\frac{zf'(z)}{f(z)}\right\}\right| \\ &\quad \le \left|\operatorname{Arg}\{f'(z)\}\right| + \left|\operatorname{Arg}\left\{\frac{f(z)}{z}\right\}\right| \\ &\quad < \sin^{-1}\alpha + \sin^{-1}\frac{\alpha}{2}, \quad z \in \mathbb{D}. \end{aligned}$$

For $\alpha \in (0, 2/\sqrt{5}]$, we have $\alpha^2 + (\alpha/2)^2 \le 1$, so we can use the formula

$$\sin^{-1}\alpha + \sin^{-1}\frac{\alpha}{2} = \sin^{-1}\left\{\alpha\sqrt{1-\alpha^2/4} + \frac{\alpha}{2}\sqrt{1-\alpha^2}\right\}.$$

The function

$$\begin{aligned} H(\alpha) &= \sin^{-1}\left\{\alpha\sqrt{1-\alpha^2/4} + \frac{\alpha}{2}\sqrt{1-\alpha^2}\right\} \\ &= \sin^{-1}\alpha + \sin^{-1}\frac{\alpha}{2} \end{aligned}$$

increases in the segment $[0, 2/\sqrt{5}]$ as the sum of two increasing functions. Moreover, $H(0) = 0$, $H(2/\sqrt{5}) = \pi/2$, so the equation (2.16) has in $(0,1]$ the solution β. This completes the proof. □

Putting $\alpha = 2/\sqrt{5}$, we get $\beta = 1$ and Theorem 2.15 becomes the result from [4, p.118]:

$$\left[\left|f'(z) - 1\right| < 2/\sqrt{5}, z \in \mathbb{D}\right] \quad \Rightarrow \quad \left[\mathfrak{Re}\{zf'(z)/f(z)\} > 0, z \in \mathbb{D}\right].$$

Theorem 2.10 *Let* $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ *be analytic in the unit disc* $\mathbb{D}$, *and suppose that*

$$\left|\operatorname{Arg}\left\{p(z) + \alpha\left(\frac{zp'(z)}{p(z)}\right)\right\}\right| < \tan^{-1}\frac{|\alpha|\delta(\beta)\sin((1+\beta)\pi/2)}{1 + |\alpha|\delta(\beta)\cos((1+\beta)\pi/2)} - \frac{\pi\beta}{2}, \quad z \in \mathbb{D}, \tag{2.19}$$

where $\alpha < 0$, $0 < \beta < 1$, *and*

$$\delta(\beta) = \frac{\beta}{2}\left(\left(\frac{1-\beta}{1+\beta}\right)^{\beta+1} + \left(\frac{1-\beta}{1+\beta}\right)^{\beta-1}\right) \quad \textit{and} \quad |\alpha| > \frac{\sin(\pi\beta/2)}{\delta(\beta)}.$$

Then $\operatorname{Arg}\{p(z)\} < \frac{\beta\pi}{2}$ *in* $\mathbb{D}$.

Proof Suppose that there exists a point $z_0 \in \mathbb{D}$ such that

$$\left|\operatorname{Arg}\{p(z)\}\right| < \frac{\pi\beta}{2} \quad \text{for } |z| < |z_0| \tag{2.20}$$

and

$$\left|\operatorname{Arg}\{p(z_0)\}\right| = \frac{\pi\beta}{2},$$

then by Nunokawa's lemma [12], we have

$$\{p(z_0)\}^{1/\beta} = \pm ia, \quad a > 0 \quad \text{and} \quad \frac{z_0 p'(z_0)}{p(z_0)} = ik\beta,$$

where

$$k \geq \frac{1}{2}\left(a + \frac{1}{a}\right) \quad \text{when } \operatorname{Arg}\{p(z_0)\} = \frac{\pi\beta}{2}$$

and

$$k \leq -\frac{1}{2}\left(a + \frac{1}{a}\right) \quad \text{when } \operatorname{Arg}\{p(z_0)\} = -\frac{\pi\beta}{2},$$

moreover,

$$\frac{\beta k}{a^\beta} \geq \delta(\beta). \tag{2.21}$$

For the case $\operatorname{Arg}\{p(z_0)\} = \frac{\pi\beta}{2}$, we have from (2.21),

$$\begin{aligned}
\operatorname{Arg}\left\{p(z_0) + \alpha\left(\frac{zp'(z_0)}{p(z_0)}\right)\right\} &= \operatorname{Arg}\{p(z_0)\} + \operatorname{Arg}\left\{1 + \alpha\left(\frac{zp'(z_0)}{p^2(z_0)}\right)\right\} \\
&= \frac{\pi\beta}{2} + \operatorname{Arg}\left\{1 + \frac{|\alpha|\beta k}{a^\beta} e^{-i\pi(1+\beta)/2}\right\} \\
&\leq -\left\{\tan^{-1}\left(\frac{|\alpha|\delta(\beta)\sin\frac{\pi(1+\beta)}{2}}{1 + |\alpha|\delta(\beta)\cos\frac{\pi(1+\beta)}{2}}\right) - \frac{\pi\beta}{2}\right\}.
\end{aligned}$$

This contradicts (2.19), and for the case $\operatorname{Arg}\{p(z_0)\} = -\frac{\pi\beta}{2}$, applying the same method as above, we have

$$\operatorname{Arg}\left\{p(z_0) + \alpha\left(\frac{zp'(z_0)}{p(z_0)}\right)\right\} \geq \left\{\tan^{-1}\left(\frac{|\alpha|\delta(\beta)\sin\frac{\pi(1+\beta)}{2}}{1 + |\alpha|\delta(\beta)\cos\frac{\pi(1+\beta)}{2}}\right) - \frac{\pi\beta}{2}\right\}.$$

This contradicts also (2.19), and, therefore, it completes the proof. □

Theorem 2.11 *Let* $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ *be analytic in the unit disc* $\mathbb{D}$, *and suppose that*

$$\left|\operatorname{Arg}\left\{p(z) + \alpha\left(\frac{zp'(z)}{p(z)}\right)\right\}\right| < \frac{\pi}{2}\left\{\beta + \frac{2}{\pi}\tan^{-1}\frac{|\alpha|\delta(\beta)\sin((1-\beta)\pi/2)}{1+|\alpha|\delta(\beta)\cos((1-\beta)\pi/2)}\right\} \quad \text{for } z \in \mathbb{D}, \tag{2.22}$$

where $0 < \alpha$, $0 < \beta < 1$, *and*

$$\delta(\beta) = \frac{\beta}{2}\left(\left(\frac{1-\beta}{1+\beta}\right)^{\beta+1} + \left(\frac{1-\beta}{1+\beta}\right)^{\beta-1}\right) \quad \text{and} \quad \alpha > \frac{\sin(\pi\beta/2)}{\delta(\beta)}.$$

Then $\operatorname{Arg}\{p(z)\} < \frac{\pi\beta}{2}$ *in* $\mathbb{D}$.

Proof The proof runs as the previous proof, take $\alpha > 0$ into account. Suppose that there exists a point $z_0 \in \mathbb{D}$ such that

$$\left|\operatorname{Arg}\left\{p(z)\right\}\right| < \frac{\pi\beta}{2} \quad \text{for } |z| < |z_0| \tag{2.23}$$

and

$$\left|\operatorname{Arg}\left\{p(z_0)\right\}\right| = \frac{\pi\beta}{2},$$

then by Nunokawa's lemma [12], we have for the case $\operatorname{Arg}\{p(z_0)\} = \frac{\pi\beta}{2}$,

$$\begin{aligned}
\operatorname{Arg}\left\{p(z_0) + \alpha\left(\frac{zp'(z_0)}{p(z_0)}\right)\right\} &= \operatorname{Arg}\left\{p(z_0)\right\} + \operatorname{Arg}\left\{1 + \alpha\left(\frac{zp'(z_0)}{p^2(z_0)}\right)\right\} \\
&= \frac{\pi\beta}{2} + \operatorname{Arg}\left\{1 + \frac{\alpha\beta k}{a^{\beta}} e^{i\pi(1+\beta)/2}\right\} \\
&\leq \frac{\pi\beta}{2} + \tan^{-1}\left(\frac{|\alpha|\delta(\beta)\sin\frac{\pi(1+\beta)}{2}}{1+|\alpha|\delta(\beta)\cos\frac{\pi(1+\beta)}{2}}\right)\frac{\pi\beta}{2}.
\end{aligned}$$

This contradicts (2.22), and for the case $\operatorname{Arg}\{p(z_0)\} = -\frac{\pi\beta}{2}$, applying the same method as above, we have

$$\operatorname{Arg}\left\{p(z_0) + \alpha\left(\frac{zp'(z_0)}{p(z_0)}\right)\right\} \leq -\left\{\frac{\pi\beta}{2} + \tan^{-1}\left(\frac{|\alpha|\delta(\beta)\sin\frac{\pi(1+\beta)}{2}}{1+|\alpha|\delta(\beta)\cos\frac{\pi(1+\beta)}{2}}\right)\frac{\pi\beta}{2}\right\}.$$

This contradicts also (2.22), and, therefore, it completes the proof. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors jointly worked on the results, and they read and approved the final manuscript.

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