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A new explicit iteration method for variational inequalities on the set of common fixed points for a finite family of nonexpansive mappings

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Abstract
In this paper, we introduce a new explicit iteration method based on the steepest descent method and Krasnoselskii-Mann type method for finding a solution of a variational inequality involving a Lipschitz continuous and strongly monotone mapping on the set of common fixed points for a finite family of nonexpansive mappings in a real Hilbert space.

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1 Introduction and preliminaries

Let C be a nonempty closed and convex subset of a real Hilbert space H with the inner product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$, and let $F : H \to H$ be a nonlinear mapping. The variational inequality problem is to find a point $p^* \in C$ such that

$$\langle F(p^*), p - p^*\rangle \geq 0, \quad \forall p \in C. \tag{1.1}$$

Variational inequalities were initially studied by Kinderlehrer and Stampacchia in [1], and since then have been widely investigated. They cover partial differential equations, optimal control, optimization, mathematical programming, mechanics, and finance (see [1–3]).

It is well known that if F is an L-Lipschitz continuous and η-strongly monotone, *i.e.*, F satisfies the following conditions:

$$\begin{aligned} &\|F(x) - F(y)\| \leq L\|x - y\|, \\ &\langle F(x) - F(y), x - y\rangle \geq \eta\|x - y\|^2, \end{aligned}$$

where L and η are fixed positive numbers, then (1.1) has a unique solution. It is also known that (1.1) is equivalent to the fixed point equation

$$p = P_C(p - \mu F(p)), \tag{1.2}$$

where P_C denotes the metric projection from $x \in H$ onto C and μ is an arbitrarily positive constant.

The fixed point formulation (1.2) involves the metric projection P_C. To overcome the complexity caused by P_C, Yamada [4] introduced a hybrid steepest descent method for solving (1.1). His idea is stated as follows. Assume that $C = \bigcap_{i=1}^N \mathrm{Fix}(T_i)$, the set of common fixed points of a finite family of nonexpansive mappings T_i on H with an integer $N \geq 1$.

Recall that $T : H \to H$ is nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in H$$

and

$$\mathrm{Fix}(T) = \{x \in H : x = Tx\}$$

denotes the fixed point set of T. Yamada proposed the following algorithm in [4]

$$u_{k+1} = T_{[k+1]}u_k - \lambda_{k+1}\mu F(T_{[k+1]}u_k), \tag{1.3}$$

where $T_{[n]} = T_{n \bmod N}$, for integer $n \geq 1$, with the mod-function taking values in the set $\{1, 2, \ldots, N\}$, $\mu \in (0, 2\eta/L^2)$ and $\{\lambda_k\} \subset (0, 1)$, and proved that the sequence $\{u_k\}$ in (1.3) converges strongly to p^* under the following conditions:

(L1) $\lim \lambda_k = 0$;
(L2) $\sum \lambda_k = \infty$;
(L3) $\sum |\lambda_k - \lambda_{k+N}| < \infty$.

Further, Zeng and Yao [5] proved the same result with (L3) replaced by

(L4) $\lim(\lambda_k - \lambda_{k+N})/\lambda_{k+N} = 0$.

Theorem 1.1 [5] *Let H be a real Hilbert space, and let $F : H \to H$ be an L-Lipschitz continuous and η-strongly monotone mapping for some constants $L, \eta > 0$. Let $\{T_i\}_{i=1}^N$ be N nonexpansive self-maps of H such that $C = \bigcap_{i=1}^N \mathrm{Fix}(T_i) \neq \emptyset$, $\mu \in (0, 2\eta/L^2)$, and let conditions* (L1), (L2), (L4) *be satisfied. Assume in addition that*

$$\begin{aligned} C = \bigcap_{i=1}^N \mathrm{Fix}(T_i) &= \mathrm{Fix}(T_1 T_2 \cdots T_N) \\ &= \mathrm{Fix}(T_N T_1 T_2 \cdots T_{N-1}) \\ &= \cdots \\ &= \mathrm{Fix}(T_2 T_3 \cdots T_N T_1). \end{aligned} \tag{1.4}$$

Then the sequence $\{u_k\}$ defined by (1.3) *converges strongly to the unique element p^* in* (1.1).

It is not difficult to show that (L3) implies (L4) if $\lim \lambda_k/\lambda_{k+N}$ exists. However, in general, conditions (L3) and (L4) are not comparable, *i.e.*, neither one of them implies the other (see [6] for details).

Recently, Zeng *et al.* [7] proposed the following iterative scheme:

$$u_{k+1} = T_{[k+1]}u_k - \lambda_{k+1}\mu_{k+1} F(T_{[k+1]}u_k), \tag{1.5}$$

and proved the following result.

Theorem 1.2 [7] *Let H be a real Hilbert space, and let $F : H \to H$ be an L-Lipschitz continuous and η-strongly monotone mapping for some constants $L, \eta > 0$. Let $\{T_i\}_{i=1}^N$ be N nonexpansive self-maps of H such that*

$$C = \bigcap_{i=1}^N \operatorname{Fix}(T_i) \neq \emptyset,$$

and let $\mu_k \in (0, 2\eta/L^2)$. Assume that the following conditions hold:

(i) $\sum \lambda_k = \infty$, *where* $\{\lambda_k\} \subset (0,1)$;
(ii) $|\mu_k - \eta/L^2| \leq \sqrt{\eta^2 - cL^2}/L^2$ *for some* $c \in (0, \eta^2/L^2)$;
(iii) $\lim(\mu_{k+N} - (\lambda_k/\lambda_{k+N})\mu_k) = 0$.

Assume in addition that (1.4) *holds. If*

$$\limsup_{k\to\infty} \langle T_{[k+N]} \cdots T_{[k+1]} u_k - u_{k+N}, T_{[k+N]} \cdots T_{[k+1]} u_k - u_k \rangle \leq 0, \tag{1.6}$$

then the sequence $\{u_k\}$ defined by (1.5) *converges strongly to the unique element p^* in* (1.1).

They also showed that conditions (L1), (L2) and (L4) are sufficient for $\{u_k\}$ to be bounded and

$$\lim_{k\to\infty} \|u_k - T_{[k+N]} \cdots T_{[k+1]} u_k\| = 0.$$

So, (1.6) is satisfied. They did not give another sufficient condition different from (L1), (L2) and (L4).

Let $Fx = Ax - u$, where A is a self-adjoint bounded linear mapping such that A is strongly positive, *i.e.*,

$$\langle Ax, x \rangle \geq \eta \|x\|^2, \quad \forall x \in H$$

and u is some fixed element in H. Xu [6] introduced the following iteration process:

$$u_{k+1} = (I - \lambda_{k+1} A) T_{k+1} u_k + \lambda_{k+1} u, \tag{1.7}$$

where I is the identity mapping of H, and proved the following result.

Theorem 1.3 [6] *Let conditions* (L1), (L2) *and* (L3) *or* (L4) *be satisfied. Assume in addition that* (1.4) *holds. Then the sequence $\{u_k\}$ generated by algorithm* (1.7) *converges strongly to the unique solution of* (1.1) *with $Fx = Ax - u$.*

Very recently, Liu and Cui [8] showed that the condition

$$C = \bigcap_{i=1}^N \operatorname{Fix}(T_i) = \operatorname{Fix}(T_1 T_2 \cdots T_N) \tag{1.8}$$

is sufficient for (1.4) if $C \neq \emptyset$.

In this paper, we introduce a new algorithm based on a combination of the steepest descent method for variational inequalities with the Krasnoselskii-Mann method for fixed point problems to solve (1.1) with $C = \bigcap_{i=1}^{N} \operatorname{Fix}(T_i)$, where T_i is a nonexpansive mapping on H for each i.

Given a starting point $x_1 \in H$, the iteration is defined by

$$\begin{cases} x_1 \in H, \\ y_k^0 = (I - \lambda_k \mu F)x_k, \\ y_k^i = (1 - \beta_k^i)x_k + \beta_k^i T_i y_k^{i-1}, \quad i = 1, \dots, N, \\ x_{k+1} = y_k^N, \end{cases} \tag{1.9}$$

and the sequences of parameters $\{\lambda_k\}$ and $\{\beta_k^i\} \subset (0,1)$ satisfy the following conditions:

$$\begin{cases} \lambda_k \to 0 \quad \text{as } k \to \infty \quad \text{and} \quad \sum \lambda_k = \infty; \\ \beta_k^i \to \beta^i, \quad 0 < \beta^i < 1, i = 1, \dots, N-1; \\ 0 < \liminf_{k\to\infty} \beta_k^N \le \limsup_{k\to\infty} \beta_k^N < 1. \end{cases} \tag{1.10}$$

In Section 2, we prove the strong convergence theorem for (1.9)-(1.10) without conditions (L3), (L4) and (1.8). An application to the case that T_i is a γ_i-strictly pseudocontractive mapping is given in Section 3.

2 Main results

We need the following lemmas for the proof of our main result.

Lemma 2.1 [9]

(i) $\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y\rangle$, $\forall x, y \in H$.

(ii) $\|(1-t)x + ty\|^2 = (1-t)\|x\|^2 + t\|y\|^2 - (1-t)t\|x - y\|^2$, $\forall x, y \in H$, *and for any fixed* $t \in [0,1]$.

From [4], we have the following lemma.

Lemma 2.2 [4] $\|T^\lambda x - T^\lambda y\| \le (1 - \lambda\tau)\|x - y\|$, $\forall x, y \in H$ *and for a fixed number* $\mu \in (0, 2\eta/L^2)$, $\lambda \in (0,1)$, *where* $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu L^2)} \in (0,1)$ *and* $T^\lambda x = (I - \lambda\mu F)x$ *for* $x \in H$.

Lemma 2.3 [10, 11] *Assume that T is a nonexpansive self-map of a closed convex subset K of a real Hilbert space H. If T has a fixed point, then $I - T$ is demiclosed; that is, whenever $\{x_k\}$ is a sequence in K weakly converging to some $x \in K$ and the sequence $\{(I - T)x_k\}$ strongly converges to some y, it follows that $(I - T)x = y$.*

Lemma 2.4 [12] *Let $\{x_k\}$ and $\{z_k\}$ be bounded sequences in a Banach space E such that*

$$x_{k+1} = (1 - \beta_k)x_k + \beta_k z_k$$

for $k \ge 1$, where $\{\beta_k\}$ is in $[0,1]$ such that

$$0 < \liminf_{k\to\infty} \beta_k \le \limsup_{k\to\infty} \beta_k < 1.$$

Assume that

$$\limsup_{k\to\infty} \|z_{k+1} - z_k\| - \|x_{k+1} - x_k\| \le 0.$$

Then $\lim_{k\to\infty} \|x_k - z_k\| = 0$.

Lemma 2.5 [6] *Let* $\{a_k\}$ *be a sequence of nonnegative real numbers satisfying the condition*

$$a_{k+1} \le (1 - b_k)a_k + b_k c_k,$$

where $\{b_k\}$ *and* $\{c_k\}$ *are sequences of real numbers such that*
(i) $b_k \in [0,1]$ *and* $\sum_{k=1}^{\infty} b_k = \infty$;
(ii) $\limsup_{k\to\infty} c_k \le 0$.
Then $\lim_{k\to\infty} a_k = 0$.

Now, we are in a position to prove the following main result.

Theorem 2.6 *Let* H *be a real Hilbert space, and let* $F : H \to H$ *be an L-Lipschitz continuous and* η*-strongly monotone mapping for some constants* $L, \eta > 0$. *Let* $\{T_i\}_{i=1}^N$ *be* N *nonexpansive self-maps of* H *such that*

$$C = \bigcap_{i=1}^{N} \operatorname{Fix}(T_i) \neq \emptyset.$$

Then the sequence $\{x_k\}$ *defined by* (1.9)-(1.10) *converges strongly to the unique element* p^* *in* (1.1).

Proof First, we prove that $\{x_k\}$ is bounded. By Lemma 2.2, we have, for any $p \in C$, from (1.9) that

$$\begin{aligned} \|y_k^0 - p\| &= \|(I - \lambda_k\mu F)x_k - p\| = \|(I - \lambda_k\mu F)x_k - (I - \lambda_k\mu F)p - \lambda_k\mu F(p)\| \\ &\le (1 - \lambda_k\tau)\|x_k - p\| + \lambda_k\mu\|F(p)\|. \end{aligned}$$

Put $M_p = \max\{\|x_1 - p\|, \frac{\mu}{\tau}\|F(p)\|\}$. Then $\|x_1 - p\| \le M_p$. So, if $\|x_k - p\| \le M_p$, then $\|y_k^0 - p\| \le M_p$. This conclusion has a place for $\{y_k^i\}$ with $i = 1, \ldots, N-1$. Indeed,

$$\begin{aligned} \|y_k^i - p\| &= \|(1 - \beta_k^i)(x_k - p) + \beta_k^i(T_i y_k^{i-1} - T_i p)\| \\ &\le (1 - \beta_k^i)\|x_k - p\| + \beta_k^i\|y_k^{i-1} - p\| \\ &\le (1 - \beta_k^i)M_p + \beta_k^i M_p \\ &= M_p. \end{aligned}$$

Then

$$\begin{aligned} \|x_{k+1} - p\| &= \|(1 - \beta_k^N)(x_k - p) + \beta_k^N(T_i y_k^{i-1} - T_i p)\| \\ &\le (1 - \beta_k^N)\|x_k - p\| + \beta_k^i N\|y_k^{N-1} - p\| \end{aligned}$$

$$\begin{aligned}
&\le \left(1-\beta_k^i\right)M_p + \beta_k^i M_p \\
&= M_p.
\end{aligned}$$

Therefore, the sequence $\{x_k\}$ is bounded. So, the sequences $\{F(x_k)\}$, $\{y_k^i\}$, and $\{T_i y_k^{i-1}\}$ ($i = 1, 2, \ldots, N$) are also bounded. Without loss of generality, we assume that they are bounded by a positive constant M_1.

Let $z_k = T_N y_k^{N-1}$. Then we have from (1.9) that

$$x_{k+1} = \left(1-\beta_k^N\right)x_k + \beta_k^N z_k$$

and

$$\begin{aligned}
\|z_{k+1} - z_k\| &= \left\| T_N y_{k+1}^{N-1} - T_N y_k^{N-1} \right\| \le \left\| y_{k+1}^{N-1} - y_k^{N-1} \right\| \\
&= \Big\| \left(1-\beta_{k+1}^{N-1}\right)x_{k+1} + \beta_{k+1}^{N-1} T_{N-1} y_{k+1}^{N-2} \\
&\quad - \left[\left(1-\beta_k^{N-1}\right)x_k + \beta_k^{N-1} T_{N-1} y_k^{N-2}\right] \Big\| \\
&\le \left(1-\beta_{k+1}^{N-1}\right)\|x_{k+1} - x_k\| + 2M_1 \left|\beta_{k+1}^{N-1} - \beta_k^{N-1}\right| \\
&\quad + \beta_{k+1}^{N-1} \left\| T_{N-1} y_{k+1}^{N-2} - T_{N-1} y_k^{N-2} \right\| \\
&\le \left(1-\beta_{k+1}^{N-1}\right)\|x_{k+1} - x_k\| + 2M_1 \left|\beta_{k+1}^{N-1} - \beta_k^{N-1}\right| \\
&\quad + \beta_{k+1}^{N-1} \left\| y_{k+1}^{N-2} - y_k^{N-2} \right\| \\
&\le \left(1-\beta_{k+1}^{N-1}\right)\|x_{k+1} - x_k\| + 2M_1 \left|\beta_{k+1}^{N-1} - \beta_k^{N-1}\right| \\
&\quad + \beta_{k+1}^{N-1} \Big[\left(1-\beta_{k+1}^{N-2}\right)\|x_{k+1} - x_k\| + 2M_1 \left|\beta_{k+1}^{N-2} - \beta_k^{N-2}\right| \\
&\quad + \beta_{k+1}^{N-2} \left\| y_{k+1}^{N-3} - y_k^{N-3} \right\| \Big] \\
&\le \left(1-\beta_{k+1}^{N-1}\beta_{k+1}^{N-2}\right)\|x_{k+1} - x_k\| + 2M_1 \Big(\left|\beta_{k+1}^{N-1} - \beta_k^{N-1}\right| \\
&\quad + \left|\beta_{k+1}^{N-2} - \beta_k^{N-2}\right|\Big) + \beta_{k+1}^{N-1}\beta_{k+1}^{N-2} \left\| y_{k+1}^{N-3} - y_k^{N-3} \right\| \\
&\le \cdots \\
&\le \left(1 - \prod_{i=1}^{N-1} \beta_{k+1}^i\right)\|x_{k+1} - x_k\| + \prod_{i=1}^{N-1} \beta_{k+1}^i \left\| y_{k+1}^0 - y_k^0 \right\| \\
&\quad + 2M_1 \sum_{i=1}^{N-1} \left|\beta_{k+1}^i - \beta_k^i\right|.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\left\| y_{k+1}^0 - y_k^0 \right\| &= \left\| (I - \lambda_{k+1}\mu F)x_{k+1} - (I - \lambda_k \mu F)x_k \right\| \\
&\le \|x_{k+1} - x_k\| + M_1(\lambda_{k+1} + \lambda_k).
\end{aligned}$$

So, we obtain that

$$\|z_{k+1} - z_k\| - \|x_{k+1} - x_k\| \le M_1(\lambda_{k+1} + \lambda_k) \prod_{i=1}^{N-1} \beta_{k+1}^i + 2M_1 \sum_{i=1}^{N-1} \left|\beta_{k+1}^i - \beta_k^i\right|.$$

Since $\lambda_k \to 0$ and $\beta_k^i \to \beta^i$, $i = 1, \ldots, N-1$, we have

$$\limsup_{k\to\infty} \|z_{k+1} - z_k\| - \|x_{k+1} - x_k\| \le 0.$$

By Lemma 2.4, $\|x_k - z_k\| \to 0$ as $k \to \infty$, *i.e.*, $\|x_k - T_N y_k^{N-1}\| \to 0$.

Now, we prove that $\|x_k - T_i y_k^{i-1}\| \to 0$ for $i = 1, \ldots, N-1$. First, we prove $\|x_k - T_{N-1} y_k^{N-2}\| \to 0$. Let $\{x_{k_n}\}$ be a subsequence of $\{x_k\}$ such that

$$\limsup_{k\to\infty} \|x_k - T_{N-1} y_k^{N-2}\| = \lim_{n\to\infty} \|x_{k_n} - T_{N-1} y_{k_n}^{N-2}\|$$

and let $\{x_{k_j}\}$ be a subsequence of $\{x_{k_n}\}$ such that

$$\limsup_{n\to\infty} \|x_{k_n} - p\| = \lim_{j\to\infty} \|x_{k_j} - p\|.$$

Further,

$$\begin{aligned}
\|x_{k_j} - p\| &\le \|x_{k_j} - T_N y_{k_j}^{N-1}\| + \|T_N y_{k_j}^{N-1} - T_N p\| \\
&\le \|x_{k_j} - T_N y_{k_j}^{N-1}\| + \|y_{k_j}^{N-1} - p\| \\
&\le \|x_{k_j} - T_N y_{k_j}^{N-1}\| + \left(1 - \beta_k^{N-1}\right)\|x_{k_j} - p\| \\
&\quad + \beta_{k_j}^{N-1}\|T_{N-1} y_{k_j}^{N-2} - T_{N-1} p\| \\
&\le \|x_{k_j} - T_N y_{k_j}^{N-1}\| + \left(1 - \beta_k^{N-1}\right)\|x_{k_j} - p\| + \beta_{k_j}^{N-1}\|y_{k_j}^{N-2} - p\| \\
&\le \|x_{k_j} - T_N y_{k_j}^{N-1}\| + \left(1 - \beta_k^{N-1}\right)\|x_{k_j} - p\| \\
&\quad + \beta_{k_j}^{N-1}\left[\left(1 - \beta_k^{N-2}\right)\|x_{k_j} - p\| + \beta_{k_j}^{N-2}\|T_{N-2} y_{k_j}^{N-3} - T_{N-2} p\|\right] \\
&\le \|x_{k_j} - T_N y_{k_j}^{N-1}\| + \left(1 - \beta_k^{N-1}\beta_{k_j}^{N-2}\right)\|x_{k_j} - p\| \\
&\quad + \beta_{k_j}^{N-1}\beta_{k_j}^{N-2}\|y_{k_j}^{N-2} - p\| \\
&\le \cdots \\
&\le \|x_{k_j} - T_N y_{k_j}^{N-1}\| + \left(1 - \prod_{i=1}^{N-1}\beta_{k_j}^i\right)\|x_{k_j} - p\| + \prod_{i=1}^{N-1}\beta_{k_j}^i\|y_{k_j}^0 - p\|.
\end{aligned}$$

Since

$$\|y_{k_j}^0 - p\| \le (1 - \lambda_{k_j}\tau)\|x_{k_j} - p\| + \lambda_{k_j}\mu\|F(p)\|,$$

we have

$$\begin{aligned}
\|x_{k_j} - p\| &\le \|x_{k_j} - T_N y_{k_j}^{N-1}\| + \|y_{k_j}^{N-1} - p\| \\
&\le \|x_{k_j} - T_N y_{k_j}^{N-1}\| + \|x_{k_j} - p\| + \prod_{i=1}^{N-1}\beta_{k_j}^i\lambda_{k_j}\mu\|F(p)\|.
\end{aligned}$$

Therefore,

$$\lim_{j\to\infty} \|x_{k_j} - p\| = \lim_{j\to\infty} \left\| y_{k_j}^{N-1} - p \right\|. \tag{2.1}$$

Next, by Lemma 2.1 we have

$$\begin{aligned}
\left\| y_{k_j}^{N-1} - p \right\|^2 &= \left(1 - \beta_{k_j}^{N-1}\right) \|x_{k_j} - p\|^2 + \beta_{k_j}^{N-1} \left\| T_{N-1} y_{k_j}^{N-2} - p \right\|^2 \\
&\quad - \left(1 - \beta_{k_j}^{N-1}\right) \beta_{k_j}^{N-1} \left\| x_{k_j} - T_{N-1} y_{k_j}^{N-2} \right\|^2 \\
&\le \left(1 - \beta_{k_j}^{N-1}\right) \|x_{k_j} - p\|^2 + \beta_{k_j}^{N-1} \left\| y_{k_j}^{N-2} - p \right\|^2 \\
&\quad - \left(1 - \beta_{k_j}^{N-1}\right) \beta_{k_j}^{N-1} \left\| x_{k_j} - T_{N-1} y_{k_j}^{N-2} \right\|^2 \\
&\le \left(1 - \beta_{k_j}^{N-1}\right) \|x_{k_j} - p\|^2 \\
&\quad + \beta_{k_j}^{N-1} \left\| \left(1 - \beta_{k_j}^{N-2}\right)(x_{k_j} - p) + \beta_{k_j}^{N-2} \left(T_{N-2} y_{k_j}^{N-3} - p\right) \right\|^2 \\
&\quad - \left(1 - \beta_{k_j}^{N-1}\right) \beta_{k_j}^{N-1} \left\| x_{k_j} - T_{N-1} y_{k_j}^{N-2} \right\|^2 \\
&\le \left(1 - \beta_{k_j}^{N-1} \beta_{k_j}^{N-2}\right) \|x_{k_j} - p\|^2 + \beta_{k_j}^{N-1} \beta_{k_j}^{N-2} \left\| y_{k_j}^{N-3} - p \right\|^2 \\
&\quad - \left(1 - \beta_{k_j}^{N-1}\right) \beta_{k_j}^{N-1} \left\| x_{k_j} - T_{N-1} y_{k_j}^{N-2} \right\|^2 \\
&\le \cdots \\
&\le \left(1 - \prod_{i=1}^{N-1} \beta_{k_j}^i\right) \|x_{k_j} - p\|^2 + \prod_{i=1}^{N-1} \beta_{k_j}^i \left\| y_{k_j}^0 - p \right\|^2 \\
&\quad - \left(1 - \beta_{k_j}^{N-1}\right) \beta_{k_j}^{N-1} \left\| x_{k_j} - T_{N-1} y_{k_j}^{N-2} \right\|^2.
\end{aligned}$$

On the other hand, by Lemma 2.1 we get

$$\begin{aligned}
\left\| y_k^0 - p \right\|^2 &= \left\| (I - \lambda_k \mu F) x_k - p \right\|^2 \\
&= \left\| x_k - p - (I - \lambda_k \mu F) x_k \right\| \\
&\le \|x_k - p\|^2 - 2\lambda_k \mu \left\langle F(x_k), x_k - p \right\rangle \\
&\le \|x_k - p\|^2 + 2\lambda_k \mu M_1 M_p.
\end{aligned}$$

Without loss of generality, assume that $\alpha \le \beta_k^i \le \beta$, $k \ge 1$ for $i = 1, \dots, N-1$ and some $\alpha, \beta \in (0,1)$. Then we obtain that

$$\alpha(1-\beta) \left\| x_{k_j} - T_{N-1} y_{k_j}^{N-2} \right\|^2 \le \|x_{k_j} - p\|^2 - \left\| y_{k_j}^{N-1} - p \right\|^2 + 2\lambda_{k_j} \mu M_1 M_p \prod_{i=1}^{N-1} \beta_{k_j}^i,$$

which with $\lambda_k \to 0$ and (2.1) implies that $\|x_{k_j} - T_{N-1} y_{k_j}^{N-2}\| \to 0$ as $j \to \infty$. So, $\|x_k - T_{N-1} y_k^{N-2}\| \to 0$ as $k \to \infty$.

Similarly, we obtain that $\|x_k - T_{N-2} y_k^{N-3}\| \to 0, \dots, \|x_k - T_1 y_k^0\| \to 0$ as $k \to \infty$.

Further, we prove that $\|x_k - T_i x_k\| \to 0$ as $k \to \infty$ for $i = 1, \dots, N$. First, note that $\|y_k^0 - x_k\| = \lambda_k \|F(x_k)\| \to 0$ as $k \to \infty$ because $\lambda_k \to 0$ and $\|F(x_k)\| \le M_1$, and $\|y_k^i - x_k\| = \beta_k^i \|x_k -$

$T_iy_k^{i-1}\| \to 0$ because $\|x_k - T_iy_k^{i-1}\| \to 0$, for $i = 1,\dots,N$. Now, from

$$\begin{aligned}\|x_k - T_ix_k\| &\le \|x_k - T_iy_k^{i-1}\| + \|T_iy_k^{i-1} - T_ix_k\| \\ &\le \|x_k - T_iy_k^{i-1}\| + \|y_k^{i-1} - x_k\|\end{aligned}$$

and $\|x_k - T_iy_k^{i-1}\|, \|y_k^{i-1} - x_k\| \to 0$, it follows that $\|x_k - T_ix_k\| \to 0$ for $i = 1,\dots,N$.

Further, we have

$$\limsup_{k\to\infty}\langle F(p^*), p^* - x_k\rangle \le 0. \tag{2.2}$$

Indeed, let $\{x_{k_j}\}$ be a subsequence of $\{x_k\}$ that converges weakly to $\tilde{p}$ such that

$$\limsup_{k\to\infty}\langle F(p^*), p^* - x_k\rangle = \lim_{j\to\infty}\langle F(p^*), p^* - x_{k_j}\rangle.$$

Then $\|x_{k_j} - T_ix_{k_j}\| \to 0$. So, by Lemma 2.3, $\tilde{p} \in C$. Therefore, from (1.1) it implies (2.2).

Finally, we estimate the value

$$\begin{aligned}\|x_{k+1} - p^*\|^2 &\le (1 - \beta_k^N)\|x_k - p^*\|^2 + \beta_k^N\|T_Ny_k^{N-1} - T_Np^*\|^2 \\ &\le (1 - \beta_k^N)\|x_k - p^*\|^2 + \beta_k^N\|y_k^{N-1} - T_Np^*\|^2 \\ &\le (1 - \beta_k^N)\|x_k - p^*\|^2 \\ &\quad + \beta_k^N\big[(1 - \beta_k^{N-1})\|x_k - p^*\|^2 + \beta_k^{N-1}\|T_{N-1}y_k^{N-2} - T_{N-1}p^*\|^2\big] \\ &\le (1 - \beta_k^N\beta_k^{N-1})\|x_k - p^*\|^2 + \beta_k^N\beta_k^{N-1}\|y_k^{N-1} - T_Np^*\|^2 \\ &\le (1 - \beta_k^N\beta_k^{N-1})\|x_k - p^*\|^2 + \beta_k^N\beta_k^{N-1}\big[(1 - \beta_k^{N-2})\|x_k - p^*\|^2 \\ &\quad + \beta_k^{N-2}\|T_{N-2}y_k^{N-3} - T_{N-2}p^*\|^2\big] \\ &\quad \le \cdots \\ &\le \left(1 - \prod_{i=1}^N \beta_k^i\right)\|x_k - p^*\|^2 + \prod_{i=1}^N \beta_k^i\|y_k^0 - p^*\|^2.\end{aligned}$$

On the other hand, since

$$\begin{aligned}\|y_k^0 - p^*\|^2 &= \|(I - \lambda_k\mu F)x_k - p^*\|^2 \\ &= \|(I - \lambda_k\mu F)(x_k - p^*) - \lambda_k\mu F(p^*)\|^2 \\ &\le (1 - \lambda_k\tau)\|x_k - p^*\|^2 - 2\lambda_k\mu\langle F(x_k), y_k^0 - p^*\rangle,\end{aligned}$$

we have

$$\begin{aligned}\|x_{k+1} - p^*\|^2 &\le \left(1 - \prod_{i=1}^N \beta_k^i\right)\|x_k - p^*\|^2 + \prod_{i=1}^N \beta_k^i(1 - \lambda_k\tau)\|x_k - p^*\|^2 \\ &\quad + \big[2\lambda_k\mu\langle F(p^*), p^* - x_k\rangle + 2\lambda_k\mu\langle F(p^*), x_k - y_k^0\rangle\big]\prod_{i=1}^N \beta_k^i\end{aligned}$$

$$\le \left(1 - \lambda_k \tau \prod_{i=1}^{N} \beta_k^i\right) \|x_k - p^*\|^2 + \lambda_k \tau \prod_{i=1}^{N} \beta_k^i \left[\frac{2\mu}{\tau}\langle F(p^*), p^* - x_k\rangle + \frac{2\mu}{\tau}\langle F(p^*), x_k - y_k^0\rangle\right].$$

Using Lemma 2.5 with

$$a_k = \|x_k - p^*\|,$$
$$b_k = \lambda_k \tau \prod_{i=1}^{N} \beta_k^i,$$
$$c_k = \frac{2\mu}{\tau}\langle F(p^*), p^* - x_k\rangle + \frac{2\mu}{\tau}\langle F(p^*), x_k - y_k^0\rangle,$$

$\|x_k - y_k^0\| \to 0$ and (2.2), we have that $\|x_k - p^*\| \to 0$. This completes the proof. □

3 Application

Recall that a mapping $S : H \to H$ is called γ-strictly pseudocontractive if there exists a constant $\gamma \in [0,1)$ such that

$$\|Sx - Sy\|^2 \le \|x - y\|^2 + \gamma \|(I - S)x - (I - S)y\|^2, \quad \forall x, y \in H.$$

It is well known [9] that a mapping $T : H \to H$ by $Tx = \alpha x + (1 - \alpha)Sx$ with a fixed $\alpha \in [\gamma, 1)$ for all $x \in H$ is a nonexpansive mapping and $\mathrm{Fix}(T) = \mathrm{Fix}(S)$.

Using this fact, we can extend our result to the case $C = \bigcap_{i=1}^N \mathrm{Fix}(S_i)$, where S_i is γ_i-strictly pseudocontractive as follows.

Let $\alpha_i \in [\gamma_i, 1)$ be fixed numbers. Then $C = \bigcap_{i=1}^N \mathrm{Fix}(\tilde{T}_i)$ with

$$\tilde{T}_i y = \alpha_i y + (1 - \alpha_i) S_i y, \tag{3.1}$$

a nonexpansive mapping, for each $i = 1, \dots, N$. So, we have the following result.

Theorem 3.1 *Let H be a real Hilbert space, and let $F : H \to H$ be an L-Lipschitzian and η-strongly monotone mapping for some constants $L, \eta > 0$. Let $\{S_i\}_{i=1}^N$ be N γ_i-strictly pseudocontractive self-maps of H such that*

$$C = \bigcap_{i=1}^{N} \mathrm{Fix}(S_i) \ne \emptyset.$$

Let $\alpha_i \in [\gamma_i, 1)$, $\mu \in (0, 2\eta/L^2)$. Assume that $\{\lambda_k\}, \{\beta_k^i\} \subset (0,1)$ satisfy (1.10). *Then the sequence $\{x_k\}$ defined by* (1.9) *with T_i replaced by $\tilde{T}_i$ of* (3.1) *converges strongly to the unique element p^* of* (1.1).

4 Numerical example

Consider the following optimization problem: find an element

$$p^* \in C : \varphi(p^*) = \min_C \varphi(x),$$

Table 1 Iterations of scheme (1.9), where starting point $x^0 = (4;15)$

*k*th	x_k by algorithm (1.9)
0	(4.00000;15.00000)
10	(4.10755;13.434117)
20	(3.89572; 12.58288)
1,000	(1.52757; 3.110295)
1,500	(1.28179; 2.12718)
2,000	(1.13269; 1.53076)
2,500	(1.03367; 1.13468)
3,000	(0.99933; 0.99738)
3,500	(0.99938; 0.99517)
4,000	(0.99942; 0.99768)
4,500	(0.99945; 0.99781)
5,000	(0.99948; 0.99792)

where $\varphi(x) = \|x\|^2/2$, $x = (x_1, x_2) \in \mathbf{E}^2$, Euclid space, and $C = C_1 \cap C_2$, defined by

$$C_1 = \{(x_1, x_2) \in \mathbf{E}^2 : x_1 - 2x_2 + 1 \leq 0\},$$
$$C_2 = \{(x_1, x_2) \in \mathbf{E}^2 : 4x_1 - x_2 - 3 \geq 0\}.$$

Clearly, the above problem possesses a unique solution $p^* = (1;1)$ and F, the Fréchet derivative of φ, is 1-Lipschitz continuous and (1/2)-strongly monotone. Starting with the point $x^0 = (x_1^0; x_2^0) = (4;15)$, $\mu = 1/10 \in (0; 2\eta/L^2)$ and $\lambda_k = (k+1)^{1/2}$, we obtained the result in Table 1.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The main idea of this paper was proposed by JKK. JKK and NB prepared the manuscript initially and performed all the steps of proof in this research. All authors read and approved the final manuscript.

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