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On Mann-type iteration method for a family of hemicontractive mappings in Hilbert spaces

Nawab Hussain[1], Ljubomir B Ćirić[2], Yeol Je Cho[3,4] and Arif Rafiq[5*]

*Correspondence:
aarafiq@gmail.com
[5]Hajvery University, 43-52 Industrial Area Gulberg-III, Lahore, Pakistan
Full list of author information is available at the end of the article

Abstract

Let K be a compact convex subset of a real Hilbert space H and $T_i : K \to K$, $i = 1, 2, \dots, k$, be a family of continuous hemicontractive mappings. Let $\alpha_n, \beta_n^i \in [0, 1]$ be such that $\alpha_n + \sum_{i=1}^{k} \beta_n^i = 1$ and satisfying $\{\alpha_n\}, \beta_n^i \subset [\delta, 1 - \delta]$ for some $\delta \in (0, 1)$, $i = 1, 2, \dots, k$. For arbitrary $x_0 \in K$, define the sequence $\{x_n\}$ by (1.9) see below, then $\{x_n\}$ converges strongly to a common fixed point in $\bigcap_{i=1}^{k} F(T_i) \neq \emptyset$.
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1 Introduction

Let H be a Hilbert space. A mapping $T : H \to H$ is said to be pseudocontractive (see [1, 2]) if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \left\|(I - T)x - (I - T)y\right\|^2 \tag{1.1}$$

for all $x, y \in H$ and T is said to be strongly pseudocontractive if there exists $k \in (0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\left\|(I - T)x - (I - T)y\right\|^2 \tag{1.2}$$

for all $x, y \in H$.

Let $F(T) := \{x \in H : Tx = x\}$ and K be a nonempty subset of H. A mapping $T : K \to K$ is said to be hemicontractive if $F(T) \neq \emptyset$ and

$$\left\|Tx - x^*\right\|^2 \leq \left\|x - x^*\right\|^2 + \|x - Tx\|^2$$

for all $x \in H$ and $x^* \in F(T)$. It is easy to see that the class of pseudocontractive mappings with fixed points is a subclass of the class of hemicontractive mappings.

The following example, due to Rhoades [3], shows that the inclusion is proper. For any $x \in [0, 1]$, define a mapping $T : [0, 1] \to [0, 1]$ by $Tx = (1 - x^{\frac{2}{3}})^{\frac{3}{2}}$. It is shown in [4] that T is not Lipschitz and so T cannot be nonexpansive. A straightforward computation (see [5]) shows that T is pseudocontractive. For the importance of fixed points of pseudocontractive mappings, the reader may refer to [1].

In the last ten years or so, numerous papers have been published on the iterative approximation of fixed points of Lipschitz strongly pseudocontractive (and, correspondingly, Lipschitz strongly accretive) mappings using the Mann iteration process (see, for example, [6]). The results which were known only in Hilbert spaces and only for Lipschitz mappings have been extended to more general Banach spaces (see [3–5, 7–33]) and the references cited therein).

In 1974, Ishikawa [34] introduced an iteration process which, in some sense, is more general than Mann iteration and which converges, under this setting, to a fixed point of T. He proved the following theorem.

Theorem 1.1 *If K is a compact convex subset of a Hilbert space H, $T : K \mapsto K$ is a Lipschitzian pseudocontractive mapping and x_0 is any point in K, then the sequence $\{x_n\}$ converges strongly to a fixed point of T, where x_n is defined iteratively by*

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n, \\ y_n = (1-\beta_n)x_n + \beta_n T x_n \end{cases} \tag{1.3}$$

for each $n \geq 0$, where $\{\alpha_n\}$, $\{\beta_n\}$ are the sequences of positive numbers satisfying the following conditions:

(a) $0 \leq \alpha_n \leq \beta_n < 1$;
(b) $\lim_{n\to\infty} \beta_n = 0$;
(c) $\sum_{n\geq 0} \alpha_n \beta_n = \infty$.

In [35], Qihou extended Theorem 1.1 to a slightly more general class of Lipschitz hemicontractive mappings and, in [25], Reich proved, under the setting of Theorem 1.1, the convergence of the recursion formula (1.3) to a fixed point of T, when T is a continuous hemicontractive mapping, under an additional hypothesis that the number of fixed points of T is finite. The iteration process (1.3) is generally referred to as the Ishikawa iteration process in light of Ishikawa [34]. Another iteration process which has been studied extensively in connection with fixed points of pseudocontractive mappings is the following.

Let K be a nonempty convex subset of E and $T : K \to K$ be a mapping.

The sequence $\{x_n\}$ is defined iteratively by

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1-c_n)x_n + c_n T x_n \end{cases} \tag{1.4}$$

for each $n \geq 1$, where $\{c_n\}$ is a real sequence satisfying the following conditions:

(d) $0 \leq c_n < 1$;
(e) $\lim c_n = 0$;
(f) $\sum_{n=1}^{\infty} c_n = \infty$.

The iteration process (1.4) is generally referred to as the Mann iteration process in light of [36].

In 1995, Liu [37] introduced the iteration process with errors as follows.

(I-a) The sequence $\{x_n\}$ defined by

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n Ty_n + u_n, \\ y_n = (1-\beta_n)x_n + \beta_n Tx_n + v_n \end{cases} \tag{1.5}$$

for each $n \geq 1$, where $\{\alpha_n\}$, $\{\beta_n\}$ are the sequences in $[0,1]$ satisfying appropriate conditions and $\sum \|u_n\| < \infty$, $\sum \|v_n\| < \infty$, is called the Ishikawa iteration process with errors.

(I-b) The sequence $\{x_n\}$ defined by

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n Tx_n + u_n \end{cases} \tag{1.6}$$

for each $n \geq 1$, where $\{\alpha_n\}$ is a sequence in $[0,1]$ satisfying appropriate conditions and $\sum \|u_n\| < \infty$, is called the Mann iteration process with errors.

While it is known that the consideration of error terms in the iterative processes (1.5), (1.6) is an important part of the theory, it is also clear that the iterative processes with errors introduced by Liu in (I-a) and (I-b) are unsatisfactory. The occurrence of errors is random so the conditions imposed on the error terms in (I-a) and (I-b), which imply, in particular, that they tend to zero as n tends to infinity, are unreasonable. In 1997, Xu [32] introduced the following more satisfactory definitions.

(I-c) The sequence $\{x_n\}$ defined iteratively by

$$\begin{cases} x_1 \in K, \\ x_{n+1} = a_n x_n + b_n Ty_n + c_n u_n, \\ y_n = a'_n x_n + b'_n Tx_n + c'_n v_n \end{cases} \tag{1.7}$$

for each $n \geq 1$, where $\{u_n\}$, $\{v_n\}$ are the bounded sequences in K and $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{a'_n\}$, $\{b'_n\}$ and $\{c'_n\}$ are the sequences in $[0,1]$ such that $a_n + b_n + c_n = a'_n + b'_n + c'_n = 1$ for each $n \geq 1$, is called the Ishikawa iteration sequence with errors in the sense of Xu.

(I-d) If, with the same notations and definitions as in (I-c), $b'_n = c'_n = 0$ for each $n \geq 1$, then the sequence $\{x_n\}$ now defined by

$$\begin{cases} x_1 \in K, \\ x_{n+1} = a_n x_n + b_n Tx_n + c_n u_n \end{cases} \tag{1.8}$$

for each $n \geq 1$ is called the Mann iteration sequence with errors in the sense of Xu.

We remark that if K is bounded (as is generally the case), then the error terms u_n, v_n are arbitrary in K.

In [11], Chidume and Chika Moore proved the following theorem.

Theorem 1.2 *Let K be a compact convex subset of a real Hilbert space H and $T : K \to K$ be a continuous hemicontractive mapping. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{a'_n\}$, $\{b'_n\}$ and $\{c'_n\}$ be the real sequences in* $[0,1]$ *satisfying the following conditions*:

(g) $a_n + b_n + c_n = 1 = a'_n + b'_n + c'_n$;

(h) $\lim b_n = \lim b'_n = 0$;
(i) $\sum c_n < \infty$; $\sum c'_n < \infty$;
(j) $\sum \alpha_n \beta_n = \infty$ *and* $\sum \alpha_n \beta_n \delta_n < \infty$, *where* $\delta_n := \|Tx_n - Ty_n\|^2$;
(k) $0 \le \alpha_n \le \beta_n < 1$ *for each* $n \ge 1$, *where* $\alpha_n := b_n + c_n$ *and* $\beta_n := b'_n + c'_n$.

For arbitrary $x_1 \in K$, *define the sequence* $\{x_n\}$ *iteratively by*

$$\begin{cases} x_{n+1} = a_n x_n + b_n T y_n + c_n u_n, \\ y_n = a'_n x_n + b'_n T x_n + c'_n v_n \end{cases}$$

for each $n \ge 1$, *where* $\{u_n\}$ *and* $\{v_n\}$ *are the arbitrary sequences in* K. *Then* $\{x_n\}$ *converges strongly to a fixed point of* T.

They also gave the following remark in [11].

Remark 1.1 (1) In connection with the iterative approximation of fixed points of pseudocontractive mappings, the following question is still open.

Does the Mann iteration process always converge for continuous pseudocontractive mappings or for even Lipschitz pseudocontractive mappings?

(2) Let E be a Banach space and K be a nonempty compact convex subset of E. Let $T : K \to K$ be a Lipschitz pseudocontractive mapping. Under this setting, even for $E = H$, a Hilbert space, the answer to the above question is not known. There is, however, an example [34] of a discontinuous pseudocontractive mapping T with a unique fixed point for which the Mann iteration process does not always converge to the fixed point of T.

Let H be the complex plane and $K := \{z \in H : |z| \le 1\}$. Define a mapping $T : K \to K$ by

$$T\left(re^{i\theta}\right) = \begin{cases} 2re^{i(\theta + \frac{\pi}{3})} & \text{for } 0 \le r \le \frac{1}{2}, \\ e^{i(\theta + \frac{2\pi}{3})} & \text{for } \frac{1}{2} < r \le 1. \end{cases}$$

Then zero is the only fixed point of T. It is shown in [20] that T is pseudocontractive and, with $c_n = \frac{1}{n+1}$, the sequence $\{z_n\}$ defined by

$$\begin{cases} z_0 \in K, \\ z_{n+1} = (1 - c_n) z_n + c_n T z_n \end{cases}$$

for each $n \ge 1$ does not converge to zero. Since the T in this example is not continuous, the above question remains open.

In [14], Chidume and Mutangadura provide an example of a Lipschitz pseudocontractive mapping with a unique fixed point for which the Mann iteration sequence failed to converge and they stated that 'This resolves a long standing open problem'. However, in [38, 39], Rafiq provided affirmative answers to the above questions (see also [40]) and proved the following result.

Theorem 1.3 *Let* K *be a compact convex subset of a real Hilbert space* H *and* $T : K \to K$ *be a continuous hemicontractive mapping. Let* $\{\alpha_n\}$ *be a real sequence in* $[0,1]$ *satisfying*

$\{\alpha_n\} \subset [\delta, 1-\delta]$ for some $\delta \in (0,1)$. For arbitrary $x_0 \in K$, define the sequence $\{x_n\}$ by

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)Tx_n$$

for each $n \geq 1$. Then $\{x_n\}$ converges strongly to a fixed point of T.

The purpose of this paper is to introduce the following Mann-type implicit iteration process associated with a family of continuous hemicontractive mappings to have a strong convergence in the setting of Hilbert spaces.

Let K be a closed convex subset of a real normed space H and $T_i : K \to K$, $i = 1, 2, \dots, k$ be a family of mappings. Then we define the sequence $\{x_n\}$ in the following way:

$$\begin{cases} x_0 \in K, \\ x_n = \alpha_n x_{n-1} + \sum_{i=1}^k \beta_n^i T_i x_n \end{cases} \tag{1.9}$$

for each $n \geq 1$, where $\alpha_n, \beta_n^i \in [0,1]$, $i = 1, 2, \dots, k$, are such that $\alpha_n + \sum_{i=1}^k \beta_n^i = 1$ and some appropriate conditions hold.

2 Main results

In the sequel, we will use following results.

Lemma 2.1 [29] *Suppose that $\{\rho_n\}$, $\{\sigma_n\}$ are two sequences of nonnegative numbers such that, for some real number $N_0 \geq 1$,*

$$\rho_{n+1} \leq \rho_n + \sigma_n$$

for all $n \geq N_0$. Then we have the following:

(1) *If $\sum \sigma_n < \infty$, then $\lim \rho_n$ exists.*
(2) *If $\sum \sigma_n < \infty$ and $\{\rho_n\}$ has a subsequence converging to zero, then $\lim \rho_n = 0$.*

Lemma 2.2 [31] *For all $x, y \in H$ and $\lambda \in [0,1]$, the following well-known identity holds*:

$$\|(1-\lambda)x + \lambda y\|^2 = (1-\lambda)\|x\|^2 + \lambda\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2.$$

Now, we prove our main results.

Lemma 2.3 *Let H be a Hilbert space. Then, for all $x, x_i \in H$, $i = 1, 2, \dots, k$,*

$$\begin{aligned} \left\|\alpha x + \sum_{i=1}^k \beta^i x_i\right\|^2 &= \alpha\|x\|^2 + \sum_{i=1}^k \beta^i\|x_i\|^2 - \sum_{i=1}^k \alpha\beta^i\|x_i - x\|^2 \\ &\quad - \sum_{\substack{i,j=1 \\ i \neq j}}^k \beta^i\beta^j\|x_i - x_j\|^2, \end{aligned} \tag{2.1}$$

where $\alpha, \beta^i \in [0,1]$, $i = 1, 2, \dots, k$, and $\alpha + \sum_{i=1}^k \beta^i = 1$.

Proof For any $x_i \in H$, $i = 1, 2, \dots, k$, it can be easily seen that

$$\left\| \sum_{i=1}^{k} x_i \right\|^2 = \sum_{i=1}^{k} \|x_i\|^2 + 2 \sum_{\substack{i,j=1 \\ i \neq j}}^{k} \operatorname{Re}\langle x_i, x_j \rangle. \tag{2.2}$$

Consider the following:

$$\begin{aligned} \left\| \alpha x + \sum_{i=1}^{k} \beta^i x_i \right\|^2 &= \left\| \left(1 - \sum_{i=1}^{k} \beta^i\right) x + \sum_{i=1}^{k} \beta^i x_i \right\|^2 \\ &= \left\| x + \sum_{i=1}^{k} \beta^i (x_i - x) \right\|^2 \\ &= \|x\|^2 + \sum_{i=1}^{k} \beta^{i^2} \|x_i - x\|^2 + 2 \sum_{i=1}^{k} \beta^i \operatorname{Re}\langle x_i - x, x \rangle \\ &\quad + 2 \sum_{\substack{i,j=1 \\ i \neq j}}^{k} \beta^i \beta^j \operatorname{Re}\langle x_i - x, x_j - x \rangle. \end{aligned} \tag{2.3}$$

For all $i, j = 1, 2, \dots, k$, we have

$$2 \operatorname{Re}\langle x_i - x, x \rangle = \|x_i\|^2 - \|x_i - x\|^2 - \|x\|^2, \tag{2.4}$$

and

$$2 \operatorname{Re}\langle x_i - x, x_j - x \rangle = -\|x_i - x_j\|^2 + \|x_i - x\|^2 + \|x_j - x\|^2. \tag{2.5}$$

Substituting (2.4) and (2.5) in (2.3), we get

$$\begin{aligned} &\left\| \alpha x + \sum_{i=1}^{k} \beta^i x_i \right\|^2 \\ &\quad = \alpha \|x\|^2 + \sum_{i=1}^{k} \beta^i \|x_i\|^2 - \sum_{\substack{i,j=1 \\ i \neq j}}^{k} \beta^i \beta^j \|x_i - x_j\|^2 - \sum_{i=1}^{k} \beta^i \left(1 - \beta^i\right) \|x_i - x\|^2 \\ &\qquad + \sum_{\substack{i,j=1 \\ i \neq j}}^{k} \beta^i \beta^j \left(\|x_i - x\|^2 + \|x_j - x\|^2 \right) \\ &\quad = \alpha \|x\|^2 + \sum_{i=1}^{k} \beta^i \|x_i\|^2 - \sum_{\substack{i,j=1 \\ i \neq j}}^{k} \beta^i \beta^j \|x_i - x_j\|^2 - \sum_{i=1}^{k} \left(1 - \sum_{j=1}^{k} \beta^j\right) \beta^i \|x_i - x\|^2 \\ &\quad = \alpha \|x\|^2 + \sum_{i=1}^{k} \beta^i \|x_i\|^2 - \sum_{\substack{i,j=1 \\ i \neq j}}^{k} \beta^i \beta^j \|x_i - x_j\|^2 - \sum_{i=1}^{k} \alpha \beta^i \|x_i - x\|^2. \end{aligned}$$

This completes the proof. □

Remark 2.1 Lemma 2.2 is now the special case of our result.

Theorem 2.1 *Let K be a compact convex subset of a real Hilbert space H and $T_i : K \to K$, $i = 1, 2, \dots, k$, be a family of continuous hemicontractive mappings. Let $\alpha_n, \beta_n^i \in [0,1]$ be such that $\alpha_n + \sum_{i=1}^k \beta_n^i = 1$ and satisfying $\{\alpha_n\}, \beta_n^i \subset [\delta, 1-\delta]$ for some $\delta \in (0,1)$, $i = 1, 2, \dots, k$.*

Then, for arbitrary $x_0 \in K$, the sequence $\{x_n\}$ defined by (1.9) *converges strongly to a common fixed point in $\bigcap_{i=1}^k F(T_i) \neq \emptyset$.*

Proof Let $x^* \in \bigcap_{i=1}^k F(T_i)$. Using the fact that T_i, $i = 1, 2, \dots, k$ are hemicontractive, we obtain

$$\|T_i x_n - x^*\|^2 \le \|x_n - x^*\|^2 + \|x_n - T_i x_n\|^2. \tag{2.6}$$

With the help of (1.9), Lemma 2.3 and (2.6), we obtain the following estimates:

$$\begin{aligned}
\|x_n - x^*\|^2 &= \left\|\alpha_n x_{n-1} + \sum_{i=1}^k \beta_n^i T_i x_n - x^*\right\|^2 \\
&= \left\|\alpha_n (x_{n-1} - x^*) + \sum_{i=1}^k \beta_n^i (T_i x_n - x^*)\right\|^2 \\
&= \alpha_n \|x_{n-1} - x^*\|^2 + \sum_{i=1}^k \beta_n^i \|T_i x_n - x^*\|^2 - \sum_{i=1}^k \alpha_n \beta_n^i \|x_{n-1} - T_i x_n\|^2 \\
&\quad - \sum_{\substack{i,j=1 \\ i \neq j}}^k \beta_n^i \beta_n^j \|T_i x_n - T_j x_n\|^2 \\
&\le \alpha_n \|x_{n-1} - x^*\|^2 + \sum_{i=1}^k \beta_n^i \|T_i x_n - x^*\|^2 - \sum_{i=1}^k \alpha_n \beta_n^i \|x_{n-1} - T_i x_n\|^2.
\end{aligned} \tag{2.7}$$

Substituting (2.6) in (2.7), we get

$$\begin{aligned}
&\|x_n - x^*\|^2 \\
&\quad \le \alpha_n \|x_{n-1} - x^*\|^2 + \sum_{i=1}^k \beta_n^i \|x_n - x^*\|^2 + \sum_{i=1}^k \beta_n^i \|x_n - T_i x_n\|^2 \\
&\qquad - \sum_{i=1}^k \alpha_n \beta_n^i \|x_{n-1} - T_i x_n\|^2.
\end{aligned} \tag{2.8}$$

Also, we have

$$\begin{aligned}
\|x_n - T_i x_n\|^2 &= \left\|\alpha_n x_{n-1} + \sum_{i=1}^k \beta_n^i T_i x_n - T_i x_n\right\|^2 \\
&= \alpha_n^2 \|x_{n-1} - T_i x_n\|^2.
\end{aligned} \tag{2.9}$$

Substituting (2.9) in (2.8), we get

$$\|x_n - x^*\|^2 \leq \alpha_n \|x_{n-1} - x^*\|^2 + \sum_{i=1}^{k} \beta_n^i \|x_n - x^*\|^2 - \sum_{i=1}^{k} \alpha_n (1-\alpha_n)\beta_n^i \|x_{n-1} - T_i x_n\|^2,$$

which implies that

$$\|x_n - x^*\|^2 \leq \|x_{n-1} - x^*\|^2 - \sum_{i=1}^{k} (1-\alpha_n)\beta_n^i \|x_{n-1} - T_i x_n\|^2.$$

Thus, from the condition $\{\alpha_n\}, \beta_n^i \subset [\delta, 1-\delta]$ for some $\delta \in (0,1)$, $i = 1, 2, \ldots, k$, we obtain

$$\|x_n - x^*\|^2 \leq \|x_{n-1} - x^*\|^2 - \delta(1-\delta) \sum_{i=1}^{k} \|x_{n-1} - T_i x_n\|^2 \tag{2.10}$$

for all fixed points $x^* \in \bigcap_{i=1}^k F(T_i)$. Moreover, we have

$$\delta(1-\delta) \sum_{i=1}^{k} \|x_{n-1} - T_i x_n\|^2 \leq \|x_{n-1} - x^*\|^2 - \|x_n - x^*\|^2,$$

and thus, for all $i = 1, 2, \ldots, k$,

$$\begin{aligned} \delta(1-\delta) \sum_{j=1}^{\infty} \|x_{j-1} - T_i x_j\|^2 &\leq \sum_{j=1}^{\infty} \left(\|x_{j-1} - x^*\|^2 - \|x_j - x^*\|^2 \right) \\ &= \|x_0 - x^*\|^2. \end{aligned}$$

Hence, for all $i = 1, 2, \ldots, k$, we obtain

$$\sum_{j=1}^{\infty} \|x_{j-1} - T_i x_j\|^2 < \infty \tag{2.11}$$

for each $i = 1, 2, \ldots, k$, which implies that

$$\lim_{n\to\infty} \|x_{n-1} - T_i x_n\| = 0$$

for each $i = 1, 2, \ldots, k$. From (2.9), it further implies that

$$\lim_{n\to\infty} \|x_n - T_i x_n\| = 0.$$

By the compactness of K, this immediately implies that there is a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges to a common fixed point of $\bigcap_{i=1}^k F(T_i)$, say y^*. Since (2.10) holds for all fixed points of $\bigcap_{i=1}^k F(T_i)$, we have

$$\|x_n - y^*\|^2 \leq \|x_{n-1} - y^*\|^2 - \delta(1-\delta) \sum_{i=1}^{k} \beta_n^i \|x_{n-1} - T_i x_n\|^2$$

and, in view of (2.11) and Lemma 2.1, we conclude that $\|x_n - y^*\| \to 0$ as $n \to \infty$, that is, $x_n \to y^*$ as $n \to \infty$. This completes the proof. □

Theorem 2.2 *Let H, K, T_i, $i = 1, 2, \ldots, k$, be as in Theorem 2.1 and $\alpha_n, \beta_n^i \in [0,1]$ be such that $\alpha_n + \sum_{i=1}^k \beta_n^i = 1$ and satisfying $\{\alpha_n\}, \beta_n^i \subset [\delta, 1-\delta]$ for some $\delta \in (0,1)$, $i = 1, 2, \ldots, k$.*

If $P_K : H \to K$ is the projection operator of H onto K, then the sequence $\{x_n\}$ defined iteratively by

$$x_n = P_K\left(\alpha_n x_{n-1} + \sum_{i=1}^k \beta_n^i T_i x_n\right)$$

for each $n \geq 0$ converges strongly to a common fixed point in $\bigcap_{i=1}^k F(T_i) \neq \emptyset$.

Proof The mapping P_K is nonexpansive (see [2]) and K is a Chebyshev subset of H and so P_K is a single-valued mapping. Hence, we have the following estimate:

$$\begin{aligned}
\|x_n - x^*\|^2 &= \left\| P_K\left(\alpha_n x_{n-1} + \sum_{i=1}^k \beta_n^i T_i x_n\right) - P_K x^* \right\|^2 \\
&\leq \left\| \alpha_n x_{n-1} + \sum_{i=1}^k \beta_n^i T_i x_n - x^* \right\|^2 \\
&= \left\| \alpha_n (x_{n-1} - x^*) + \sum_{i=1}^k \beta_n^i (T_i x_n - x^*) \right\|^2 \\
&\leq \alpha_n \|x_{n-1} - x^*\|^2 + \sum_{i=1}^k \beta_n^i \|x_n - x^*\|^2 \\
&\quad - \sum_{i=1}^k \alpha_n (1-\alpha_n) \beta_n^i \|x_{n-1} - T_i x_n\|^2,
\end{aligned}$$

which implies that

$$\|x_n - x^*\|^2 \leq \|x_{n-1} - x^*\|^2 - \sum_{i=1}^k (1-\alpha_n) \beta_n^i \|x_{n-1} - T_i x_n\|^2.$$

The set $K \cup T(K)$ is compact and so the sequence $\{\|x_n - T_i x_n\|\}$ is bounded. The rest of the argument follows exactly as in the proof of Theorem 2.1. This completes the proof. □

Theorem 2.3 *Let K be a compact convex subset of a real Hilbert space H and $T_i : K \to K$, $i = 1, 2, \ldots, k$, be a family of Lipschitz hemicontractive mappings. Let $\alpha_n, \beta_n^i \in [0,1]$ be such that $\alpha_n + \sum_{i=1}^k \beta_n^i = 1$ and satisfying $\{\alpha_n\}, \beta_n^i \subset [\delta, 1-\delta]$ for some $\delta \in (0,1)$, $i = 1, 2, \ldots, k$.*

Then, for arbitrary $x_0 \in K$, the sequence $\{x_n\}$ defined by (1.9) converges strongly to a common fixed point in $\bigcap_{i=1}^k F(T_i) \neq \emptyset$.

Theorem 2.4 *Let H, K, T_i, $i = 1, 2, \ldots, k$, be as in Theorem 2.3 and $\alpha_n, \beta_n^i \in [0,1]$ be such that $\alpha_n + \sum_{i=1}^k \beta_n^i = 1$ and satisfying $\{\alpha_n\}, \beta_n^i \subset [\delta, 1-\delta]$ for some $\delta \in (0,1)$, $i = 1, 2, \ldots, k$.*

If $P_K : H \to K$ is the projection operator of H onto K, then the sequence $\{x_n\}$ defined iteratively by

$$x_n = P_K\left(\alpha_n x_{n-1} + \sum_{i=1}^{k} \beta_n^i T_i x_n\right)$$

for each $n \geq 1$ converges strongly to a common fixed point in $\bigcap_{i=1}^{k} F(T_i) \neq \emptyset$.

Example For $k = 2$, we can choose the following control parameters: $\alpha_n = \frac{1}{4} - \frac{1}{(n+2)^2}$, $\beta_n^1 = \frac{1}{4}$ and $\beta_n^2 = \frac{1}{2} + \frac{1}{(n+2)^2}$.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

Author details

[1]Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah, 21589, Saudi Arabia. [2]Faculty of Mechanical Engineering, University of Belgrade, Kraljice Marije 16, Belgrade, Serbia. [3]Department of Mathematics Education and RINS, Gyeongsang National University, Jinju, 660-701, Korea. [4]Department of Mathematics Education, Kyungnam University, Masan, Kyungnam 631-701, Korea. [5]Hajvery University, 43-52 Industrial Area Gulberg-III, Lahore, Pakistan.

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