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Einstein half lightlike submanifolds of a Lorentzian space form with a semi-symmetric non-metric connection

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Abstract

We study screen conformal Einstein half lightlike submanifolds M of a Lorentzian space form $\widetilde{M}(c)$ of constant curvature c admitting a semi-symmetric non-metric connection subject to the conditions; (1) the structure vector field of $\widetilde{M}$ is tangent to M, and (2) the canonical normal vector field of M is conformal Killing. The main result is a characterization theorem for such a half lightlike submanifold.

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Keywords: semi-symmetric non-metric connection; screen conformal; conformal Killing distribution; half lightlike submanifold

1 Introduction

The theory of lightlike submanifolds is used in mathematical physics, in particular, in general relativity as lightlike submanifolds produce models of different types of horizons [1, 2]. Lightlike submanifolds are also studied in the theory of electromagnetism [3]. Thus, large number of applications but limited information available, motivated us to do the research on this subject matter. As for any semi-Riemannian manifold, there is a natural existence of lightlike subspaces, Duggal and Bejancu published their work [3] on the general theory of lightlike submanifolds to fill a gap in the study of submanifolds. Since then, there has been very active study on lightlike geometry of submanifolds (see up-to date results in two books [4, 5]). The class of lightlike submanifolds of codimension 2 is composed of two classes by virtue of the rank of its radical distribution, named by half lightlike and coisotropic submanifolds [6, 7]. Half lightlike submanifold is a special case of general r-lightlike submanifold such that $r = 1$, and its geometry is more general form than that of coisotropic submanifold or lightlike hypersurface. Much of the works on half lightlike submanifolds will be immediately generalized in a formal way to general r-lightlike submanifolds of arbitrary codimension n and arbitrary rank r. For this reason, we study half lightlike submanifold M of a semi-Riemannian manifold $\widetilde{M}$.

Ageshe and Chafle [8] introduced the notion of a semi-symmetric non-metric connection on a Riemannian manifold. Although now, we have lightlike version of a large variety of Riemannian submanifolds, the theory of lightlike submanifolds of semi-Riemannian manifolds, equipped with semi-symmetric metric connections, has not been introduced until quite recently. Yasar *et al.* [9] studied lightlike hypersurfaces in a semi-Riemannian manifold admitting a semi-symmetric non-metric connection. Recently, Jin and Lee [10] and Jin [11–13] studied half lightlike and r-lightlike submanifolds of a semi-Riemannian manifold with a semi-symmetric non-metric connection.

In this paper, we study the geometry of screen conformal Einstein half lightlike submanifolds M of a Lorentzian space form $\widetilde{M}(c)$ of constant curvature c admitting a semi-symmetric non-metric connection subject to the conditions; (1) the structure vector field of $\widetilde{M}$ is tangent to M, and (2) the canonical normal vector field of M is conformal Killing. The reason for this geometric restriction on M is due to the fact that such a class admits an integrable screen distribution and a symmetric Ricci tensor of M. We prove a characterization theorem for such a half lightlike submanifold.

2 Semi-symmetric non-metric connection

Let $(\widetilde{M},\widetilde{g})$ be a semi-Riemannian manifold. A connection $\widetilde{\nabla}$ on $\widetilde{M}$ is called a *semi-symmetric non-metric connection* [8] if $\widetilde{\nabla}$ and its torsion tensor $\widetilde{T}$ satisfy

$$(\widetilde{\nabla}_X\widetilde{g})(Y,Z) = -\pi(Y)\widetilde{g}(X,Z) - \pi(Z)\widetilde{g}(X,Y), \tag{2.1}$$

$$\widetilde{T}(X,Y) = \pi(Y)X - \pi(X)Y, \tag{2.2}$$

for any vector fields X, Y and Z on $\widetilde{M}$, where π is a 1-form associated with a non-vanishing vector field ζ, which is called the *structure vector field* of $\widetilde{M}$, by

$$\pi(X) = \widetilde{g}(X,\zeta). \tag{2.3}$$

In the entire discussion of this article, we shall assume the structure vector field ζ to be unit spacelike, unless otherwise specified.

A submanifold (M,g) of codimension 2 is called *half lightlike submanifold* if the radical distribution $\mathrm{Rad}(TM) = TM \cap TM^\perp$ is a subbundle of the tangent bundle TM and the normal bundle $TM^\perp$ of rank 1. Therefore, there exist complementary non-degenerate distributions $S(TM)$ and $S(TM^\perp)$ of $\mathrm{Rad}(TM)$ in TM and $TM^\perp$ respectively, which are called the *screen* and *co-screen distributions* of M, respectively, such that

$$TM = \mathrm{Rad}(TM) \oplus_{\mathrm{orth}} S(TM), \qquad TM^\perp = \mathrm{Rad}(TM) \oplus_{\mathrm{orth}} S\big(TM^\perp\big), \tag{2.4}$$

where $\oplus_{\mathrm{orth}}$ denotes the orthogonal direct sum. We denote such a half lightlike submanifold by $M = (M,g,S(TM))$. Denote by $F(M)$ the algebra of smooth functions on M and by $\Gamma(E)$ the $F(M)$ module of smooth sections of a vector bundle E over M. Choose $L \in \Gamma(S(TM^\perp))$ as a unit vector field with $\widetilde{g}(L,L) = \pm 1$. We may assume that L to be unit spacelike vector field without loss of generality, *i.e.*, $\widetilde{g}(L,L) = 1$. We call L the *canonical normal vector field* of M. Consider the orthogonal complementary distribution $S(TM)^\perp$ to $S(TM)$ in $T\widetilde{M}$. Certainly, $\mathrm{Rad}(TM)$ and $S(TM^\perp)$ are subbundles of $S(TM)^\perp$. As $S(TM^\perp)$ is non-degenerate, we have

$$S(TM)^\perp = S\big(TM^\perp\big) \oplus_{\mathrm{orth}} S\big(TM^\perp\big)^\perp,$$

where $S(TM^\perp)^\perp$ is the orthogonal complementary to $S(TM^\perp)$ in $S(TM)^\perp$. For any null section ξ of $\mathrm{Rad}(TM)$ on a coordinate neighborhood $\mathcal{U} \subset M$, there exists a uniquely defined lightlike vector bundle $\operatorname{ltr}(TM)$ and a null vector field N of $\operatorname{ltr}(TM)_{|\mathcal{U}}$ satisfying

$$\widetilde{g}(\xi, N) = 1, \qquad \widetilde{g}(N,N) = \widetilde{g}(N,X) = \widetilde{g}(N,L) = 0, \quad \forall X \in \Gamma\big(S(TM)\big).$$

We call N, $\operatorname{ltr}(TM)$ and $\operatorname{tr}(TM) = S(TM^\perp) \oplus_{\text{orth}} \operatorname{ltr}(TM)$ the *lightlike transversal vector field, lightlike transversal vector bundle* and *transversal vector bundle* of M with respect to the screen distribution, respectively [6]. Then $T\widetilde{M}$ is decomposed as follows:

$$\begin{aligned} T\widetilde{M} &= TM \oplus \operatorname{tr}(TM) = \big\{\operatorname{Rad}(TM) \oplus \operatorname{tr}(TM)\big\} \oplus_{\text{orth}} S(TM) \\ &= \big\{\operatorname{Rad}(TM) \oplus \operatorname{ltr}(TM)\big\} \oplus_{\text{orth}} S(TM) \oplus_{\text{orth}} S\big(TM^\perp\big). \end{aligned} \tag{2.5}$$

Given a screen distribution $S(TM)$, there exists a unique complementary vector bundle $\operatorname{tr}(TM)$ to TM in $T\widetilde{M}_{|M}$. Using (2.4) and (2.5), there exists a local quasi-orthonormal frame field of $\widetilde{M}$ along M given by

$$F = \{\xi, N, L, W_a\}, \quad a \in \{1, \dots, m\}, \tag{2.6}$$

where $\{W_a\}$ is an orthonormal frame field of $S(TM)_{|\mathcal{U}}$.

In the entire discussion of this article, we shall assume that ζ is tangent to M, and we take $X, Y, Z, W \in \Gamma(TM)$, unless otherwise specified. Let P be the projection morphism of TM on $S(TM)$ with respect to the first decomposition of (2.4). Then the local Gauss and Weingartan formulas of M and $S(TM)$ are given respectively by

$$\widetilde{\nabla}_X Y = \nabla_X Y + B(X,Y)N + D(X,Y)L, \tag{2.7}$$

$$\widetilde{\nabla}_X N = -A_N X + \tau(X)N + \rho(X)L, \tag{2.8}$$

$$\widetilde{\nabla}_X L = -A_L X + \phi(X)N, \tag{2.9}$$

$$\nabla_X PY = \nabla^*_X PY + C(X, PY)\xi, \tag{2.10}$$

$$\nabla_X \xi = -A^*_\xi X - \tau(X)\xi, \tag{2.11}$$

where ∇ and ∇^* are induced linear connections on TM and $S(TM)$, respectively, B and D are called the *local lightlike*, and *screen second fundamental forms* of M, respectively, C is called the *local second fundamental form* on $S(TM)$, A_N, A^*_ξ and A_L are called the *shape operators*, and τ, ρ and ϕ are 1-forms on TM. We say that

$$h(X,Y) = B(X,Y)N + D(X,Y)L$$

is the *second fundamental form tensor* of M. Using (2.1), (2.2) and (2.7), we have

$$(\nabla_X g)(Y,Z) = B(X,Y)\eta(Z) + B(X,Z)\eta(Y) - \pi(Y)g(X,Z) - \pi(Z)g(X,Y), \tag{2.12}$$

$$T(X,Y) = \pi(Y)X - \pi(X)Y, \tag{2.13}$$

and B and D are symmetric on TM, where T is the torsion tensor with respect to the induced connection ∇, and η is a 1-form on TM such that

$$\eta(X) = \widetilde{g}(X,N).$$

From the facts $B(X,Y) = \widetilde{g}(\widetilde{\nabla}_X Y, \xi)$ and $D(X,Y) = \widetilde{g}(\widetilde{\nabla}_X Y, L)$, we know that B and D are independent of the choice of the screen distribution $S(TM)$ and satisfy

$$B(X,\xi) = 0, \qquad D(X,\xi) = -\phi(X). \tag{2.14}$$

In case ζ is tangent to M, the above three local second fundamental forms on M and $S(TM)$ are related to their shape operators by

$$g\big(A^*_\xi X, Y\big) = B(X,Y), \qquad \widetilde{g}\big(A^*_\xi X, N\big) = 0, \tag{2.15}$$

$$g(A_L X, Y) = D(X,Y) + \phi(X)\eta(Y), \qquad \widetilde{g}(A_L X, N) = \rho(X), \tag{2.16}$$

$$g(A_N X, PY) = C(X,PY) - fg(X,PY) - \eta(X)\pi(PY), \qquad \widetilde{g}(A_N X, N) = -f\eta(X), \tag{2.17}$$

where f is the smooth function given by $f = \pi(N)$. From (2.15) and (2.17), we show that A^*_ξ and A_N are $S(TM)$-valued, and A_ξ is self-adjoint operator and satisfies

$$A^*_\xi \xi = 0, \tag{2.18}$$

that is, ξ is an eigenvector field of A^*_ξ corresponding to the eigenvalue 0.

In general, the screen distribution $S(TM)$ is not necessarily integrable. The following result gives equivalent conditions for the integrability of $S(TM)$.

Theorem 2.1 [10] *Let M be a half lightlike submanifold of a semi-Riemannian manifold $\widetilde{M}$ admitting a semi-symmetric non-metric connection. Then the following assertions are equivalent*:

(1) *The screen distribution $S(TM)$ is an integrable distribution.*
(2) *C is symmetric, i.e., $C(X,Y) = C(Y,X)$ for all $X, Y \in \Gamma(S(TM))$.*
(3) *The shape operator A_N is a self-adjoint with respect to g, i.e.,*

$$g(A_N X, Y) = g(X, A_N Y), \quad \forall X, Y \in \Gamma\big(S(TM)\big).$$

Just as in the well-known case of locally product Riemannian or semi-Riemannian manifolds [2–4, 7], if $S(TM)$ is an integrable distribution, then M is locally a product manifold $M = \mathcal{C}_1 \times M^*$, where $\mathcal{C}_1$ is a null curve tangent to $\mathrm{Rad}(TM)$, and M^* is a leaf of the integrable screen distribution $S(TM)$.

3 Structure equations

Denote by $\widetilde{R}$, R and R^* the curvature tensors of the semi-symmetric non-metric connection $\widetilde{\nabla}$ on $\widetilde{M}$, the induced connection ∇ on M and the induced connection ∇^* on $S(TM)$, respectively. Using the Gauss-Weingarten formulas for M and $S(TM)$, we obtain the Gauss-Codazzi equations for M and $S(TM)$:

$$\begin{aligned}\widetilde{R}(X,Y)Z = {} & R(X,Y)Z + B(X,Z)A_N Y - B(Y,Z)A_N X \\ & + D(X,Z)A_L Y - D(Y,Z)A_L X \\ & + \big\{(\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z)\end{aligned}$$

$$
\begin{aligned}
&+ B(Y,Z)\big[\tau(X) - \pi(X)\big] - B(X,Z)\big[\tau(Y) - \pi(Y)\big] \\
&+ D(Y,Z)\phi(X) - D(X,Z)\phi(Y)\big\}N \\
&+ \big\{(\nabla_X D)(Y,Z) - (\nabla_Y D)(X,Z) + B(Y,Z)\rho(X) \\
&- B(X,Z)\rho(Y) - D(Y,Z)\pi(X) + D(X,Z)\pi(Y)\big\}L, \qquad (3.1)
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{R}(X,Y)N = {}& -\nabla_X(A_N Y) + \nabla_Y(A_N X) + A_N[X,Y] \\
&+ \tau(X)A_N Y - \tau(Y)A_N X + \rho(X)A_L Y - \rho(Y)A_L X \\
&+ \big\{B(Y,A_N X) - B(X,A_N Y) + 2d\tau(X,Y) \\
&+ \phi(X)\rho(Y) - \phi(Y)\rho(X)\big\}N \\
&+ \big\{D(Y,A_N X) - D(X,A_N Y) + 2d\rho(X,Y) \\
&+ \rho(X)\tau(Y) - \rho(Y)\tau(X)\big\}L, \qquad (3.2)
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{R}(X,Y)L = {}& -\nabla_X(A_L Y) + \nabla_Y(A_L X) + A_L[X,Y] \\
&+ \phi(X)A_N Y - \phi(Y)A_N X \\
&+ \big\{B(Y,A_L X) - B(X,A_L Y) + 2d\phi(X,Y) \\
&+ \tau(X)\phi(Y) - \tau(Y)\phi(X)\big\}N \\
&+ \big\{D(Y,A_L X) - D(X,A_L Y) + \rho(X)\phi(Y) - \rho(Y)\phi(X)\big\}L, \qquad (3.3)
\end{aligned}
$$

$$
\begin{aligned}
R(X,Y)PZ = {}& R^*(X,Y)PZ + C(X,PZ)A^*_\xi Y - C(Y,PZ)A_\xi X \\
&+ \big\{(\nabla_X C)(Y,PZ) - (\nabla_Y C)(X,PZ) \\
&+ C(X,PZ)\big[\tau(Y) + \pi(Y)\big] - C(Y,PZ)\big[\tau(X) + \pi(X)\big]\big\}\xi, \qquad (3.4)
\end{aligned}
$$

$$
\begin{aligned}
R(X,Y)\xi = {}& -\nabla^*_X\big(A^*_\xi Y\big) + \nabla^*_Y\big(A^*_\xi X\big) + A^*_\xi[X,Y] + \tau(Y)A^*_\xi X \\
&- \tau(X)A^*_\xi Y + \big\{C\big(Y,A^*_\xi X\big) - C\big(X,A^*_\xi Y\big) - 2d\tau(X,Y)\big\}\xi. \qquad (3.5)
\end{aligned}
$$

A semi-Riemannian manifold $\widetilde{M}$ of constant curvature c is called a *semi-Riemannian space form* and denote it by $\widetilde{M}(c)$. The curvature tensor $\widetilde{R}$ of $\widetilde{M}(c)$ is given by

$$
\widetilde{R}(X,Y)Z = c\big\{\widetilde{g}(Y,Z)X - \widetilde{g}(X,Z)Y\big\}, \quad \forall X,Y,Z \in \Gamma(T\widetilde{M}). \qquad (3.6)
$$

Taking the scalar product with ξ and L to (3.6), we obtain $\widetilde{g}(\widetilde{R}(X,Y)Z,\xi) = 0$ and $\widetilde{g}(\widetilde{R}(X,Y)Z,L) = 0$ for any $X,Y,Z \in \Gamma(TM)$. From these equations and (3.1), we get

$$
\begin{aligned}
\widetilde{R}(X,Y)Z = {}& R(X,Y)Z + B(X,Z)A_N Y - B(Y,Z)A_N X \\
&+ D(X,Z)A_L Y - D(Y,Z)A_L X, \quad \forall X,Y,Z \in \Gamma(TM). \qquad (3.7)
\end{aligned}
$$

4 Screen conformal half lightlike submanifolds

Definition 1 A half lightlike submanifold M of a semi-Riemannian manifold $\widetilde{M}$ is said to be *irrotational* [14] if $\widetilde{\nabla}_X\xi \in \Gamma(TM)$ for any $X \in \Gamma(TM)$.

From (2.7) and (2.14), we show that the above definition is equivalent to

$$
D(X,\xi) = 0 = \phi(X), \quad \forall X \in \Gamma(TM).
$$

Theorem 4.1 *Let M be an irrotational half lightlike submanifold of a semi-Riemannian manifold $\widetilde{M}$ admitting a semi-symmetric non-metric connection such that ζ is tangent to M. Then ζ is conjugate to any vector field X on M, i.e., ζ satisfies $h(X,\zeta) = 0$.*

Proof Taking the scalar product with ξ to (3.2) and N to (3.1) such that $Z = \xi$ by turns and using (2.14), (3.5) and the fact that $\phi = 0$, we obtain

$$\begin{aligned}\widetilde{g}\big(\widetilde{R}(X,Y)\xi,N\big) &= B(X,A_N Y) - B(Y,A_N X) - 2d\tau(X,Y)\\ &= C\big(Y,A^*_\xi X\big) - C\big(X,A^*_\xi Y\big) - 2d\tau(X,Y).\end{aligned}$$

From these two representations, we obtain

$$B(X,A_N Y) - B(Y,A_N X) = C\big(Y,A^*_\xi X\big) - C\big(X,A^*_\xi Y\big).$$

Using $(2.15)_1$, $(2.17)_2$ and the fact that A^*_ξ is self-adjoint, we have

$$\pi\big(A^*_\xi X\big)\eta(Y) = \pi\big(A^*_\xi Y\big)\eta(X).$$

Replacing Y by ξ to this equation and using (2.18), we have

$$B(X,\zeta) = \pi\big(A^*_\xi X\big) = 0. \tag{4.1}$$

As D is symmetric and $\phi = 0$, we show that A_L is self-adjoint. Taking the scalar product with L to (3.2) and N to (3.3) with $\phi = 0$ by turns, we obtain

$$\begin{aligned}\widetilde{g}\big(\widetilde{R}(X,Y)N,L\big) &= \widetilde{g}\big(\nabla_X(A_L Y) - \nabla_Y(A_L X) - A_L[X,Y],N\big)\\ &= D(Y,A_N X) - D(X,A_N Y) + 2d\rho(X,Y) + \rho(X)\tau(Y) - \rho(Y)\tau(X).\end{aligned}$$

Using these two representations and $(2.16)_2$, we show that

$$\begin{aligned}&D(Y,A_N X) - D(X,A_N Y) + 2d\rho(X,Y) + \rho(X)\tau(Y) - \rho(Y)\tau(X)\\ &\quad = \widetilde{g}\big(\nabla_X(A_L Y),N\big) - \widetilde{g}\big(\nabla_Y(A_L X),N\big) - \rho\big([X,Y]\big).\end{aligned}$$

Applying $\widetilde{\nabla}_X$ to $\widetilde{g}(A_L Y,N) = \rho(Y)$ and using (2.1), (2.7) and (2.8), we have

$$\begin{aligned}\widetilde{g}\big(\nabla_X(A_L Y),N\big) &= X\big(\rho(Y)\big) + \pi(A_L Y)\eta(X) + fg(X,A_L Y)\\ &\quad + g(A_L Y,A_N X) - \tau(X)\rho(Y).\end{aligned}$$

Substituting this equation into the last equation and using $(2.16)_1$, we have

$$\pi(A_L X)\eta(Y) = \pi(A_L Y)\eta(X).$$

Replacing Y by ξ to this equation, we have

$$\pi(A_L X) = \pi(A_L\xi)\eta(X).$$

Taking $X = \xi$ and $Y = \zeta$ to $(2.16)_1$, we get $\pi(A_L\xi) = 0$. Therefore, we have

$$D(X,\zeta) = \pi(A_L X) = 0. \tag{4.2}$$

From (4.1) and (4.2), we show that $h(X,\zeta) = 0$ for all $X \in \Gamma(TM)$. □

Definition 2 A half lightlike submanifold M of a semi-Riemannian manifold $\widetilde{M}$ is *screen conformal* [4, 5, 7] if the second fundamental forms B and C satisfy

$$C(X,PY) = \varphi B(X,Y), \quad \forall X,Y \in \Gamma(TM), \tag{4.3}$$

where φ is a non-vanishing function on a coordinate neighborhood $\mathcal{U}$ in M.

Theorem 4.2 *Let M be an irrotational half lightlike submanifold of a semi-Riemannian space form $\widetilde{M}(c)$ admitting a semi-symmetric non-metric connection such that ζ is tangent to M. If M is screen conformal, then $c = 0$.*

Proof Substituting (3.6) into (3.2) and using the fact that $\phi = 0$, we have

$$\begin{aligned}&(\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z)\\ &\quad = B(Y,Z)\{\pi(X) - \tau(X)\} - B(X,Z)\{\pi(Y) - \tau(Y)\}.\end{aligned} \tag{4.4}$$

Taking the scalar product with N to (3.1) and (3.4) by turns and using $(2.16)_2$, $(2.17)_2$ and (3.6), we have the following two forms of $\widetilde{g}(R(X,Y)PZ,N)$:

$$\begin{aligned}&\{cg(Y,PZ) - fB(Y,PZ)\}\eta(X) - \{cg(X,PZ) - fB(X,PZ)\}\eta(Y)\\ &\qquad + \rho(X)D(Y,PZ) - \rho(Y)D(X,PZ)\\ &\quad = (\nabla_X C)(Y,PZ) - (\nabla_Y C)(X,PZ) + C(X,PZ)\{\pi(Y) + \tau(Y)\}\\ &\qquad - C(Y,PZ)\{\pi(X) + \tau(X)\}.\end{aligned} \tag{4.5}$$

Applying ∇_X to $C(Y,PZ) = \varphi B(Y,PZ)$, we have

$$(\nabla_X C)(Y,PZ) = X[\varphi]B(Y,PZ) + \varphi(\nabla_X B)(Y,PZ).$$

Substituting this into (4.5) and using (4.4), we obtain

$$\begin{aligned}&c\{g(Y,PZ)\eta(X) - g(X,PZ)\eta(Y)\}\\ &\quad = \{X[\varphi] - 2\varphi\tau(X) + f\eta(X)\}B(Y,PZ) - \rho(X)D(Y,PZ)\\ &\qquad - \{Y[\varphi] - 2\varphi\tau(Y) + f\eta(Y)\}B(X,PZ) + \rho(Y)D(X,PZ).\end{aligned} \tag{4.6}$$

Replacing Z by ζ to (4.5) and using (4.1) and (4.2), we have $c = 0$. □

Remark 4.3 If M is screen conformal, then, from (4.3), we show that C is symmetric on $S(TM)$. By Theorem 2.1, $S(TM)$ is integrable and M is locally a product manifold $\mathcal{C}_1 \times M^*$, where $\mathcal{C}_1$ is a null curve tangent to $\mathrm{Rad}(TM)$ and M^* is a leaf of $S(TM)$.

5 Main theorem

Let $\widetilde{Ric}$ be the Ricci curvature tensor of $\widetilde{M}$ and $R^{(0,2)}$ the induced Ricci type tensor on M given respectively by

$$\widetilde{Ric}(X,Y) = \operatorname{trace}\{Z \to \widetilde{R}(Z,X)Y\}, \quad \forall X,Y \in \Gamma(T\widetilde{M}),$$
$$R^{(0,2)}(X,Y) = \operatorname{trace}\{Z \to R(Z,X)Y\}, \quad \forall X,Y \in \Gamma(TM).$$

Using the quasi-orthonormal frame field (2.6) on $\widetilde{M}$, we show [10] that

$$\begin{aligned} R^{(0,2)}(X,Y) = {} & \widetilde{Ric}(X,Y) + B(X,Y)\operatorname{tr}A_N + D(X,Y)\operatorname{tr}A_L \\ & - g\big(A_N X, A^*_\xi Y\big) - g(A_L X, A_L Y) + \rho(X)\phi(Y) \\ & - \widetilde{g}\big(\widetilde{R}(\xi,Y)X,N\big) - \widetilde{g}\big(\widetilde{R}(L,X)Y,L\big), \end{aligned}$$

where $\operatorname{tr}A_N$ is the trace of A_N. From this, we show that $R^{(0,2)}$ is not symmetric. The tensor field $R^{(0,2)}$ is called the *induced Ricci curvature tensor* [4, 5] of M, denoted by Ric, if it is symmetric. M is called *Ricci flat* if its induced Ricci tensor vanishes on M. It is known [10] that $R^{(0,2)}$ is symmetric if and only if the 1-form τ is closed, *i.e.*, $d\tau = 0$.

Remark 5.1 If the induced Ricci type tensor $R^{(0,2)}$ is symmetric, then there exists a null pair $\{\xi, N\}$ such that the corresponding 1-form τ satisfies $\tau = 0$ [3, 4], which is called a *canonical null pair* of M. Although $S(TM)$ is not unique, it is canonically isomorphic to the factor vector bundle $S(TM)^\sharp = TM/\operatorname{Rad}(TM)$ [14]. This implies that all screen distribution are mutually isomorphic. For this reason, in case $d\tau = 0$, we consider only lightlike hypersurfaces M endow with the canonical null pair such that $\tau = 0$.

We say that M is an *Einstein manifold* if the Ricci tensor of M satisfies

$$Ric = \kappa g.$$

It is well known that if $\dim M > 2$, then κ is a constant.

Let $\dim\widetilde{M} = m + 3$. In case $\widetilde{M}$ is a semi-Riemannian space form $\widetilde{M}(c)$, we have

$$\begin{aligned} R^{(0,2)}(X,Y) = {} & mcg(X,Y) + B(X,Y)\operatorname{tr}A_N + D(X,Y)\operatorname{tr}A_L \\ & - g\big(A_N X, A^*_\xi Y\big) - g(A_L X, A_L Y) + \rho(X)\phi(Y). \end{aligned} \tag{5.1}$$

Due to (2.15) and (2.17), we show that M is screen conformal if and only if the shape operators A_N and A^*_ξ are related by

$$A_N X = \varphi A^*_\xi X - fX - \eta(X)\zeta. \tag{5.2}$$

Assume that $\phi = 0$. As D is symmetric, A_L is self-adjoint. Using this, (5.1) and (5.2), we show that $R^{(0,2)}$ is symmetric. Thus, we can take $\tau = 0$. As $\tau = 0$, (4.4) reduce to

$$(\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) = \pi(X)B(Y,Z) - \pi(Y)B(X,Z). \tag{5.3}$$

Definition 3 A vector field X on $\widetilde{M}$ is said to be *conformal Killing* [3, 5, 15] if $\widetilde{\mathcal{L}}_X\widetilde{g} = -2\delta\widetilde{g}$ for a scalar function δ, where $\widetilde{\mathcal{L}}$ denotes the Lie derivative on $\widetilde{M}$, that is,

$$(\widetilde{\mathcal{L}}_X\widetilde{g})(Y,Z) = X(\widetilde{g}(Y,Z)) - \widetilde{g}([X,Y],Z) - \widetilde{g}(Y,[X,Z]), \quad \forall Y,Z \in \Gamma(T\widetilde{M}).$$

In particular, if $\delta = 0$, then X is called a *Killing vector field* on $\widetilde{M}$.

Theorem 5.2 *Let M be a half lightlike submanifold of a semi-Riemannian manifold $\widetilde{M}$ admitting a semi-symmetric non-metric connection. If the canonical normal vector field L is a conformal Killing one, then L is a Killing vector field.*

Proof Using (2.1) and (2.2), for any $X,Y,Z \in \Gamma(T\widetilde{M})$, we have

$$(\widetilde{\mathcal{L}}_X\widetilde{g})(Y,Z) = \widetilde{g}(\widetilde{\nabla}_Y X, Z) + \widetilde{g}(Y, \widetilde{\nabla}_Z X) - 2\pi(X)\widetilde{g}(Y,Z).$$

As L is a conformal Killing vector field, we have $\widetilde{g}(\widetilde{\nabla}_X L, Y) = -D(X,Y)$ by (2.9) and (2.16). This implies $(\widetilde{\mathcal{L}}_L\widetilde{g})(X,Y) = -2D(X,Y)$ for any $X,Y \in \Gamma(TM)$. Thus, we have

$$D(X,Y) = \delta g(X,Y), \quad \forall X,Y \in \Gamma(TM). \tag{5.4}$$

Taking $X = Y = \zeta$ and using (4.2), we get $\delta = 0$. Thus, L is a Killing vector field. □

Remark 5.3 Călin [16] proved the following result. For any lightlike submanifolds M of indefinite almost contact metric manifolds $\widetilde{M}$, *if ζ is tangent to M, then it belongs to $S(TM)$*. Duggal and Sahin also proved this result (see pp.318-319 of [5]). After Călin's work, many earlier works [17–19], which were written on lightlike submanifolds of indefinite almost contact metric manifolds or lightlike submanifolds of semi-Riemannian manifolds, admitting semi-symmetric non-metric connections, obtained their results by using the Călin's result described in above. However, Jin [12, 13] proved that Călin's result is not true for any lightlike submanifolds M of a semi-Riemannian space form $\widetilde{M}(c)$, admitting a semi-symmetric non-metric connection.

For the rest of this section, we may assume that the structure vector field ζ of $\widetilde{M}$ belongs to the screen distribution $S(TM)$. In this case, we show that $f = 0$.

Theorem 5.4 *Let M be a screen conformal Einstein half lightlike submanifold of a Lorentzian space form $\widetilde{M}(c)$, admitting a semi-symmetric non-metric connection such that ζ belongs to $S(TM)$. If the canonical normal vector field L is conformal Killing, then M is Ricci flat. Moreover, if the mean curvature of M is constant, then M is locally a product manifold $\mathcal{C}_1 \times \mathcal{C}_2 \times M^{m-1}$, where $\mathcal{C}_1$ and $\mathcal{C}_2$ are null and non-null curves, and M^{m-1} is an $(m-1)$-dimensional Euclidean space.*

Proof As L is conformal Killing vector field, $D = A_L = 0$ by (5.4) and Theorem 5.2. Therefore, from (2.14), we show that $\phi = 0$, *i.e.*, M is irrotational. By Theorem 4.2, we also have $c = 0$. Using (2.15), (4.1) and (5.2) with $f = 0$, from (5.1), we have

$$g(A^*_\xi X, A^*_\xi Y) - \alpha g(A^*_\xi X, Y) + \varphi^{-1}\kappa g(X,Y) = 0 \tag{5.5}$$

due to $c = 0$, where $\alpha = \operatorname{tr} A^*_\xi$. As $g(A^*_\xi \zeta, X) = B(\zeta, X) = 0$ for all $X \in \Gamma(TM)$ and $S(TM)$ is non-degenerate, we show that

$$A^*_\xi \zeta = 0. \tag{5.6}$$

Taking $X = Y = \zeta$ to (5.5) and using (5.6), we have $\varphi^{-1}\kappa = 0$. Thus, (5.5) reduce to

$$g\big(A^*_\xi X, A^*_\xi Y\big) - \alpha g\big(A^*_\xi X, Y\big) = 0, \qquad \kappa = 0. \tag{5.7}$$

From the second equation of (5.7), we show that M is Ricci flat.

As M is screen conformal and $\widetilde{M}$ is Lorentzian, $S(TM)$ is an integrable Riemannian vector bundle. Since ξ is an eigenvector field of A^*_ξ, corresponding to the eigenvalue 0 due to (2.15), and A^*_ξ is $S(TM)$-valued real self-adjoint operator, A^*_ξ has m real orthonormal eigenvector fields in $S(TM)$ and is diagonalizable. Consider a frame field of eigenvectors $\{\xi, E_1, \dots, E_m\}$ of A^*_ξ such that $\{E_1, \dots, E_m\}$ is an orthonormal frame field of $S(TM)$ and $A^*_\xi E_i = \lambda_i E_i$. Put $X = Y = E_i$ in (5.7), each eigenvalue λ_i is a solution of

$$x^2 - \alpha x = 0.$$

As this equation has at most two distinct solutions 0 and α, there exists $p \in \{0, 1, \dots, m\}$ such that $\lambda_1 = \cdots = \lambda_p = 0$ and $\lambda_{p+1} = \cdots = \lambda_m = \alpha$, by renumbering if necessary. As $\operatorname{tr} A^*_\xi = 0p + (m-p)\alpha$, we have

$$\alpha = \operatorname{tr} A^*_\xi = (m-p)\alpha.$$

So $p = m - 1$, *i.e.*,

$$A^*_\xi = \begin{pmatrix} 0 & & & \\ & \ddots & & \\ & & 0 & \\ & & & \alpha \end{pmatrix}.$$

Consider two distributions D_o and D_α on $S(TM)$ given by

$$D_o = \big\{X \in \Gamma\big(S(TM)\big) \mid A^*_\xi X = 0 \text{ and } X \neq 0\big\},$$
$$D_\alpha = \big\{U \in \Gamma\big(S(TM)\big) \mid A^*_\xi U = \alpha U \text{ and } U \neq 0\big\}.$$

Clearly we show that $D_o \cap D_\alpha = \{0\}$ as $\alpha \neq 0$. In the sequel, we take $X, Y \in \Gamma(D_o)$, $U, V \in \Gamma(D_\alpha)$ and $Z, W \in \Gamma(S(TM))$. Since X and U are eigenvector fields of the real self-adjoint operator A^*_ξ, corresponding to the different eigenvalues 0 and α respectively, we have $g(X, U) = 0$. From this and the fact that $B(X, U) = g(A^*_\xi X, U) = 0$, we show that $D_\alpha \perp_g D_o$ and $D_\alpha \perp_B D_o$, respectively. Since $\{E_i\}_{1 \le i \le m-1}$ and $\{E_m\}$ are vector fields of D_o and D_α, respectively, and D_o and D_α are mutually orthogonal, we show that D_o and D_α are non-degenerate distributions of rank $(m-1)$ and rank 1, respectively. Thus, the screen distribution $S(TM)$ is decomposed as $S(TM) = D_\alpha \oplus_{\text{orth}} D_o$.

From (5.7), we get $A^*_\xi(A^*_\alpha - \alpha P) = 0$. Let $W \in \operatorname{Im} A^*_\xi$. Then there exists $Z \in \Gamma(S(TM))$ such that $W = A^*_\xi Z$. Then $(A^*_\xi - \alpha P)W = 0$ and $W \in \Gamma(D_\alpha)$. Thus, $\operatorname{Im} A^*_\xi \subset \Gamma(D_\alpha)$. By duality, we have $\operatorname{Im}(A^*_\xi - \alpha P) \subset \Gamma(D_o)$.

Applying ∇_X to $B(Y, U) = 0$ and using (2.15) and $A^*_\xi Y = 0$, we obtain

$$(\nabla_X B)(Y, U) = -g\big(A^*_\xi \nabla_X Y, U\big).$$

Substituting this into (5.3) and using (2.12) and $A^*_\xi X = A^*_\xi Y = 0$, we get

$$g\big(A^*_\xi [X, Y], U\big) = 0.$$

As $\operatorname{Im} A^*_\xi \subset \Gamma(D_\alpha)$ and D_α is non-degenerate, we get $A^*_\xi [X, Y] = 0$. This implies that $[X, Y] \in \Gamma(D_o)$. Thus, D_o is an integrable distribution.

Applying ∇_U to $B(X, Y) = 0$ and ∇_X to $B(U, Y) = 0$, we have

$$(\nabla_U B)(X, Y) = 0, \qquad (\nabla_X B)(U, Y) = -\alpha g(\nabla_X Y, U).$$

Substituting this two equations into (5.3), we have $\alpha g(\nabla_X Y, U) = 0$. As

$$g\big(A^*_\xi \nabla_X Y, U\big) = B(\nabla_X Y, U) = \alpha g(\nabla_X Y, U) = 0$$

and $\operatorname{Im} A^*_\xi \subset \Gamma(D_\alpha)$ and D_α is non-degenerate, we get $A^*_\xi \nabla_X Y = 0$. This implies that $\nabla_X Y \in \Gamma(D_o)$. Thus, D_o is an auto-parallel distribution on $S(TM)$.

As $A^*_\xi \zeta = 0$, ζ belongs to D_o. Thus, $\pi(U) = 0$ for any $U \in \Gamma(D_\alpha)$. Applying ∇_X to $g(U, Y) = 0$ and using (2.12) and the fact that D_o is auto-parallel, we get $g(\nabla_X U, Y) = 0$. This implies that $\nabla_X U \in \Gamma(D_\alpha)$.

Applying ∇_U to $B(V, X) = 0$ and using $A^*_\xi X = 0$, we obtain

$$(\nabla_U B)(V, X) = -\alpha g(V, \nabla_U X).$$

Substituting this into (5.3) and using the fact that $D_o \perp_B D_\alpha$, we get

$$g(V, \nabla_U X) = g(U, \nabla_V X).$$

Applying ∇_U to $g(V, X) = 0$ and using (2.12), we get

$$g(\nabla_U V, X) = \pi(X) g(U, V) - g(V, \nabla_V X).$$

Taking the skew-symmetric part of this and using (2.13), we obtain

$$g\big([U, V], X\big) = 0.$$

This implies that $[U, V] \in \Gamma(D_\alpha)$ and D_α is an integrable distribution.

Now we assume that the mean curvature $H = \frac{1}{m+2} \operatorname{tr} B = \frac{1}{m+2} \operatorname{tr} A^*_\xi$ of M is a constant. As $\operatorname{tr} A^*_\xi = \alpha$, we see that α is a constant. Applying ∇_X to $B(U, V) = \alpha g(U, V)$ and ∇_U

to $B(X,V) = 0$ by turns and using the facts that $\nabla_X U \in \Gamma(TM)$, $D_o \perp_g D_\alpha$, $D_o \perp_B D_\alpha$ and $B(X, \nabla_U V) = g(A^*_\xi X, \nabla_U V) = 0$, we have

$$(\nabla_X B)(U,V) = 0, \qquad (\nabla_U B)(X,V) = -\alpha g(\nabla_U X, V).$$

Substituting these two equations into (5.3) and using $D_o \perp_B D_\alpha$, we have

$$g(\nabla_U X, V) = \pi(X) g(U,V).$$

Applying ∇_U to $g(X,V) = 0$ and using (2.12), we obtain

$$g(X, \nabla_U V) = \pi(X) g(U,V) - g(\nabla_U X, V) = 0.$$

Thus, D_α is also an integrable and auto-parallel distribution.

Since the leaf M^* of $S(TM)$ is a Riemannian manifold and $S(TM) = D_\alpha \oplus_{\mathrm{orth}} D_o$, where D_α and D_o are auto-parallel distributions of M^*, by the decomposition of the theorem of de Rham [20], we have $M^* = \mathcal{C}_2 \times M^{m-1}$, where $\mathcal{C}_2$ is a leaf of D_α, and M^{m-1} is a totally geodesic leaf of D_o. Consider the frame field of eigenvectors $\{\xi, E_1, \dots, E_m\}$ of A^*_ξ such that $\{E_i\}_i$ is an orthonormal frame field of $S(TM)$, then $B(E_i,E_j) = C(E_i,E_j) = 0$ for $1 \le i < j \le m$ and $B(E_i,E_i) = C(E_i,E_i) = 0$ for $1 \le i \le m-1$. From (3.1) and (3.4), we have $\widetilde{g}(\widetilde{R}(E_i,E_j)E_j,E_i) = g(R^*(E_i,E_j)E_j,E_i) = 0$. Thus, the sectional curvature K of the leaf M^{m-1} of D_o is given by

$$K(E_i,E_j) = \frac{g(R^*(E_i,E_j)E_j,E_i)}{g(E_i,E_i)g(E_j,E_j) - g^2(E_i,E_j)} = 0.$$

Thus, M is a local product $\mathcal{C}_1 \times \mathcal{C}_2 \times M^{m-1}$, where $\mathcal{C}_1$ is a null curve, $\mathcal{C}_2$ is a non-null curve, and M^{m-1} is an $(m-1)$-dimensional Euclidean space. □

Competing interests
The author declares that he has no competing interests.

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