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Existence conditions for symmetric generalized quasi-variational inclusion problems

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Abstract

In this paper, we establish an existence theorem by using the Kakutani-Fan-Glicksberg fixed-point theorem for a symmetric generalized quasi-variational inclusion problem in real locally convex Hausdorff topological vector spaces. Moreover, the closedness of the solution set for this problem is obtained. As special cases, we also derive the existence results for symmetric weak and strong quasi-equilibrium problems. The results presented in the paper improve and extend the main results in the literature.

MSC: 90B20; 49J40

Keywords: symmetric generalized quasi-variational inclusion problem; symmetric weak quasi-equilibrium problem; symmetric strong quasi-equilibrium problem; Kakutani-Fan-Glicksberg fixed-point theorem; existence; closedness

1 Introduction

Let X and Z be real locally convex Hausdorff spaces, $A \subset X$ be a nonempty subset and $C \subset Z$ be a closed convex pointed cone. Let $F : A \times A \to 2^Z$ be a given set-valued mapping. Ansari *et al.* [1] introduced the following two problems (in short, (VEP) and (SVEP)), respectively:

Find $\bar{x} \in A$ such that

$$F(x, y) \not\subset -\operatorname{int} C, \quad \forall y \in A,$$

and find $\bar{x} \in A$ such that

$$F(x, y) \subset C, \quad \forall y \in A.$$

The problem (VEP) is called the weak vector equilibrium problem and the problem (SVEP) is called the strong vector equilibrium problem. Later, these two problems have been studied by many authors; see, for example, [2, 3] and references.

In 2008, Long *et al.* [4] introduced a generalized strong vector quasi-equilibrium problem (for short, (GSVQEP)). Let X, Y and Z be real locally convex Hausdorff topological vector spaces, $A \subset X$ and $B \subset Y$ be nonempty compact convex subsets and $C \subset Z$ be a nonempty closed convex cone, and let $S : A \to 2^A$, $T : A \to 2^B$, $F : A \times B \times A \to 2^Z$ be set-valued mappings.

(GSVQEP): Find $\bar{x} \in A$ and $\bar{y} \in T(\bar{x})$ such that $\bar{x} \in S(\bar{x})$ and

$$F(\bar{x}, \bar{y}, x) \subset C, \quad \forall x \in S(\bar{x}),$$

where $\bar{x}$ is a strong solution of (GSVQEP).

Recently, Plubtieng and Sitthithakerngkietet [5] considered the system of generalized strong vector quasi-equilibrium problems (in short, (SGSVQEPs)). This model is a general problem which contains (GVEP) and (QSVQEP). Let X, Y, Z be real locally convex Hausdorff topological vector spaces, $A \subset X$ and $B \subset Y$ be nonempty compact convex subsets and $C \subset Z$ be a nonempty closed convex cone. Let $S_1, S_2 : A \to 2^A$, $T_1, T_2 : A \to 2^B$ and $F_1, F_2 : A \times B \times A \to 2^Z$ be set-valued mappings. They considered (SGSVQEPs) as follows.

Find $(\bar{x}, \bar{u}) \in A \times A$ and $\bar{z} \in T_1(\bar{x})$, $\bar{v} \in T_2(\bar{u})$ such that $\bar{x} \in S_1(\bar{x})$, $\bar{u} \in S_2(\bar{u})$ and

$$F_1(\bar{x}, \bar{z}, y) \subset C, \quad \forall y \in S_1(\bar{x})$$

and

$$F_2(\bar{u}, \bar{v}, y) \subset C, \quad \forall y \in S_2(\bar{u}),$$

where $(\bar{x}, \bar{u})$ is a strong solution of (SGSVQEPs).

Very recently, new symmetric strong vector quasi-equilibrium problems (in short, (SSVQEP)) in Hausdorff locally convex spaces were introduced by Chen *et al.* [6]. Let X, Y, Z be real locally convex Hausdorff topological vector spaces, $A \subset X$ and $B \subset Y$ be nonempty compact convex subsets and $C \subset Z$ be a nonempty closed convex cone. Let $S_1, S_2 : A \times A \to 2^A$, $T_1, T_2 : A \times A \to 2^B$ and $F_1, F_2 : A \times B \times A \to 2^Z$ be set-valued mappings. They considered the following (SSVQEP):

Find $(\bar{x}, \bar{u}) \in A \times A$ and $\bar{z} \in T_1(\bar{x}, \bar{u})$, $\bar{v} \in T_2(\bar{x}, \bar{u})$ such that $\bar{x} \in S_1(\bar{x}, \bar{u})$, $\bar{u} \in S_2(\bar{x}, \bar{u})$ and

$$F_1(\bar{x}, \bar{z}, y) \subset C, \quad \forall y \in S_1(\bar{x}, \bar{u})$$

and

$$F_2(\bar{u}, \bar{v}, y) \subset C, \quad \forall y \in S_2(\bar{x}, \bar{u}),$$

where $(\bar{x}, \bar{u})$ is a strong solution for the (SSVQEP).

Motivated by the research works mentioned above, in this paper, we introduce symmetric generalized quasi-variational inclusion problems. Let X, Y, Z be real locally convex Hausdorff topological vector spaces and $A \subset X$, $B \subset Y$ be nonempty compact convex subsets. Let $K_i, P_i : A \times A \to 2^A$, $T_i : A \times A \to 2^B$ and $F_i : A \times B \times A \to 2^Z$, $i = 1, 2$, be set-valued mappings.

Now, we adopt the following notations (see [7–9]). Letters w, m and s are used for weak, middle and strong problems, respectively. For subsets U and V under consideration, we adopt the following notations:

$$(u,v)\ \mathrm{w}\ U \times V \quad \text{means } \forall u \in U, \exists v \in V,$$

$$(u,v)\ \mathrm{m}\ U \times V \quad \text{means } \exists v \in V, \forall u \in U,$$

$$(u,v)\ \mathrm{s}\ U \times V \quad \text{means } \forall u \in U, \forall v \in V.$$

Let $\alpha \in \{\mathrm{w}, \mathrm{m}, \mathrm{s}\}$. We consider the following for symmetric generalized quasi-variational inclusion problem (in short, (SQVIP_α)).

$(\mathrm{SQVIP}_{\boldsymbol{\alpha}})$: Find $(\bar{x},\bar{u}) \in A \times A$ such that $\bar{x} \in K_1(\bar{x},\bar{u})$, $\bar{u} \in K_2(\bar{x},\bar{u})$ and

$$(y,\bar{z})\alpha P_1(\bar{x},\bar{u}) \times T_1(\bar{x},\bar{u}) \quad \text{satisfying } 0 \in F_1(\bar{x},\bar{z},y),$$

$$(y,\bar{v})\alpha P_2(\bar{x},\bar{u}) \times T_2(\bar{x},\bar{u}) \quad \text{satisfying } 0 \in F_2(\bar{u},\bar{v},y).$$

We denote that $\Xi_\alpha(F)$ is the solution set of (SQVIP_α).

The symmetric generalized quasi-variational inclusion problems include as special cases symmetric generalized vector quasi-equilibrium problems, vector quasi-equilibrium problems, symmetric vector quasi-variational inequality problems, variational relation problems, *etc.* In recent years, a lot of results for the existence of solutions for symmetric vector quasi-equilibrium problems, vector quasi-equilibrium problems, vector quasi-variational inequality problems, variational relation problems and optimization problems have been established by many authors in different ways. For example, equilibrium problems [1–6, 10–21], variational inequality problems [22–24], variational relation problems [7, 25], optimization problems [22, 26] and the references therein.

The structure of our paper is as follows. In the first part of this article, we introduce the model symmetric generalized quasi-variational inclusion problem. In Section 2, we recall definitions for later use. In Section 3, we establish an existence and closedness theorem by using the Kakutani-Fan-Glicksberg fixed-point theorem for a symmetric generalized quasi-variational inclusion problem. Applications to symmetric weak and strong vector quasi-equilibrium problems are presented in Section 4.

2 Preliminaries

In this section, we recall some basic definitions and some of their properties.

Definition 1 [27, 28] Let X, Y be two topological vector spaces, A be a nonempty subset of X and $F : A \to 2^Z$ be a set-valued mapping.

(i) F is said to be lower semicontinuous (lsc) at $x_0 \in A$ if $F(x_0) \cap U \neq \emptyset$ for some open set $U \subseteq Y$ implies the existence of a neighborhood N of x_0 such that $F(x) \cap U \neq \emptyset$, $\forall x \in N$. F is said to be lower semicontinuous in A if it is lower semicontinuous at all $x_0 \in A$.

(ii) F is said to be upper semicontinuous (usc) at $x_0 \in A$ if for each open set $U \supseteq G(x_0)$, there is a neighborhood N of x_0 such that $U \supseteq F(x)$, $\forall x \in N$. F is said to be upper semicontinuous in A if it is upper semicontinuous at all $x_0 \in A$.

(iii) F is said to be continuous in A if it is both lsc and usc in A.

(iv) F is said to be closed if $\mathrm{Graph}(F) = \{(x,y) : x \in A, y \in F(x)\}$ is a closed subset in $A \times Y$.

Definition 2 [27] Let X, Y be two topological vector spaces, A be a nonempty subset of X, $F: A \to 2^Y$ be a multifunction and $C \subset Y$ be a nonempty closed convex cone.

(i) F is called upper C-continuous at $x_0 \in A$ if for any neighborhood U of the origin in Y, there is a neighborhood V of x_0 such that

$$F(x) \subset F(x_0) + U + C, \quad \forall x \in V.$$

(ii) F is called lower C-continuous at $x_0 \in A$ if for any neighborhood U of the origin in Y, there is a neighborhood V of x_0 such that

$$F(x_0) \subset F(x) + U - C, \quad \forall x \in V.$$

Definition 3 [6] Let X, Y be two topological vector spaces and A be a nonempty subset of X and $C \subset Y$ be a nonempty closed convex cone. A set-valued mapping $F: A \to 2^Y$ is said to be type II C-lower semicontinuous at $x_0 \in A$ if for each $y \in F(x_0)$ and any neighborhood U of the origin in Y, there exists a neighborhood $U(x_0)$ of x_0 such that

$$F(x) \cap (y + U - C) \neq \emptyset, \quad \forall x \in U(x_0) \cap A.$$

Definition 4 [6, 28] Let X and Y be two topological vector spaces and A be a nonempty convex subset of X. A set-valued mapping $F: A \to 2^Y$ is said to be C-convex if for any $x, y \in A$ and $t \in [0,1]$, one has

$$F\bigl(tx + (1-t)y\bigr) \subset tF(x) + (1-t)F(y) - C.$$

F is said to be C-concave if -F is C-convex.

Definition 5 [28] Let X and Y be two topological vector spaces and A be a nonempty convex subset of X. A set-valued mapping $F: A \to 2^Y$ is said to be properly C-quasiconvex if for any $x, y \in A$ and $t \in [0,1]$, we have

$$\begin{aligned} \text{either} \quad & F(x) \subset F\bigl(tx + (1-t)y\bigr) + C, \\ \text{or} \quad & F(y) \subset F\bigl(tx + (1-t)y\bigr) + C. \end{aligned}$$

Lemma 6 [28] *Let X, Y be two topological vector spaces, A be a nonempty convex subset of X and $F: A \to 2^Y$ be a multifunction.*

(i) *If F is upper semicontinuous at $x_0 \in A$ with closed values, then F is closed at $x_0 \in A$;*
(ii) *If F is closed at $x_0 \in A$ and Y is compact, then F is upper semicontinuous at $x_0 \in A$.*
(iii) *If F has compact values, then F is usc at $x_0 \in A$ if and only if, for each net $\{x_\alpha\} \subseteq A$ which converges to $x_0 \in A$ and for each net $\{y_\alpha\} \subseteq F(x_\alpha)$, there are $y_0 \in F(x_0)$ and a subnet $\{y_\beta\}$ of $\{y_\alpha\}$ such that $y_\beta \to y_0$.*

Lemma 7 (Kakutani-Fan-Glickcberg (see [29])) *Let A be a nonempty compact convex subset of a locally convex Hausdorff vector topological space X. If $F: A \longrightarrow 2^A$ is upper semicontinuous and for any $x \in A, F(x)$ is nonempty, convex and closed, then there exists an $x^* \in A$ such that $x^* \in F(x^*)$.*

3 Main results

In this section, we discuss the existence and closedness of the solution sets of symmetric generalized quasi-variational inclusion problems by using the Kakutani-Fan-Glicksberg fixed point theorem.

Theorem 8 *For each* $\{i = 1, 2\}$, *assume for the problem* (SQVIP_α) *that*

(i) K_i *is usc in* $A \times A$ *with nonempty convex closed values and* P_i *is lsc in* $A \times A$ *with nonempty closed values*;

(ii) T_i *is usc in* $A \times A$ *with nonempty convex compact values if* $\alpha = w$ (*or* $\alpha = m$) *and* T_i *is lsc in* $A \times A$ *with nonempty convex values if* $\alpha = s$;

(iii) *for all* $(x, z, u) \in A \times B \times A$, $0 \in F_i(x, z, P_i(x, u))$;

(iv) *for all* $(x, z, u) \in A \times B \times A$, *the set* $\{a \in K_i(x, u) : 0 \in F_i(a, z, y), \forall y \in P_i(x, u)\}$ *is convex*;

(v) *the set* $\{(x, z, y) \in A \times B \times A : 0 \in F_i(x, z, y)\}$ *is closed*.

Then the (SQVIP_α) *has a solution, i.e., there exist* $(\bar{x}, \bar{u}) \in A \times A$ *such that* $\bar{x} \in K_1(\bar{x}, \bar{u})$, $\bar{u} \in K_2(\bar{x}, \bar{u})$ *and*

$$(y, \bar{z})\alpha P_1(\bar{x}, \bar{u}) \times T_1(\bar{x}, \bar{u}) \quad \textit{satisfying } 0 \in F_1(\bar{x}, \bar{z}, y),$$

$$(y, \bar{v})\alpha P_2(\bar{x}, \bar{u}) \times T_2(\bar{x}, \bar{u}) \quad \textit{satisfying } 0 \in F_2(\bar{u}, \bar{v}, y).$$

Moreover, the solution set of the (SQVIP_α) *is closed*.

Proof Similar arguments can be applied to three cases. We present only the proof for the case where $\alpha = m$.

Indeed, for all $(x, z, u, v) \in A \times B \times A \times B$, define mappings: $\Phi_m, \Pi_m : A \times B \times A \to 2^A$ by

$$\Phi_m(x, z, u) = \{a \in K_1(x, u) : 0 \in F_1(a, z, y), \forall y \in P_1(x, u)\},$$

and

$$\Pi_m(x, v, u) = \{b \in K_2(x, u) : 0 \in F_2(b, v, y), \forall y \in P_2(x, u)\}.$$

(a) Show that $\Phi_m(x, z, u)$ and $\Pi_m(x, v, u)$ are nonempty convex sets.

Indeed, for all $(x, z, u) \in A \times B \times A$ and $(x, v, u) \in A \times B \times A$, for each $\{i = 1, 2\}$, $K_i(x, u)$, $P_i(x, u)$ are nonempty. Thus, by assumptions (i), (ii) and (iii), we have $\Phi_m(x, z, u)$ and $\Pi_m(x, v, u)$ are nonempty. On the other hand, by the condition (iv), we also have $\Phi_m(x, z, u)$, $\Pi_m(x, v, u)$ are convex.

(b) We will prove Φ_m and Π_m are upper semicontinuous in $A \times B \times A$ with nonempty closed values.

First, we show that Φ_m is upper semicontinuous in $A \times B \times A$ with nonempty closed values. Since A is a compact set, by Lemma 6(ii), we need only to show that Φ_m is a closed mapping. Indeed, let a net $\{(x_n, z_n, u_n) : n \in I\} \subset A \times B \times A$ such that $(x_n, z_n, u_n) \to (x, z, u) \in A \times B \times A$, and let $a_n \in \Phi_m(x_n, z_n, u_n)$ such that $a_n \to a_0$. Now we need to show that $a_0 \in \Phi_m(x, z, u)$. Since $a_n \in K_1(x_n, u_n)$ and K_1 is upper semicontinuous with nonempty closed values by Lemma 6(i), hence K_1 is closed, thus we have $a_0 \in K_1(x,u)$. Suppose the contrary $a_0 \notin \Phi_m(x,z,u)$. Then $\exists y_0 \in P_1(x,u)$ such that

$$0 \notin F_1(a_0, z, y_0). \tag{1}$$

By the lower semicontinuity of P_1, there is a net $\{y_n\}$ such that $y_n \in P_1(x_n, u_n)$, $y_n \to y_0$. Since $a_n \in \Phi_m(x_n, z_n, u_n)$, we have

$$0 \in F_1(a_n, z_n, y_n). \tag{2}$$

By the condition (v) and (2), we have

$$0 \in F_1(a_0, z, y_0). \tag{3}$$

This is a contradiction between (3) and (1). Thus, $a_0 \in \Phi_m(x,z,u)$. Hence, Φ_m is upper semicontinuous in $A \times B \times A$ with nonempty closed values. Similarly, we also have $\Pi_m(x,v,u)$ is upper semicontinuous in $A \times B \times A$ with nonempty closed values.

(c) Now we need to prove the solution set $\Xi_m(F) \neq \emptyset$.

Define the set-valued mappings $\Theta_m, \Omega_m : A \times B \times A :\to 2^{A\times B}$ by

$$\Theta_m(x,z,u) = \big(\Phi_m(x,z,u), T_1(x,u)\big), \quad \forall (x,z,u) \in A \times B \times A$$

and

$$\Omega_m(x,v,u) = \big(\Pi_m(x,v,u), T_2(x,u)\big), \quad \forall (x,v,u) \in A \times B \times A.$$

Then Θ_m, Ω_m are upper semicontinuous and $\forall (x,z,u) \in A \times B \times A$, $\forall (x,v,u) \in A \times B \times A$, $\Theta_m(x,z,u)$ and $\Theta_m(x,v,u)$ are nonempty closed convex subsets of $A \times B \times A$.

Define the set-valued mapping $H : (A \times B) \times (A \times B) \to 2^{(A\times B)\times(A\times B)}$ by

$$H\big((x,z),(u,v)\big) = \big(\Theta_m(x,z,u), \Omega_m(x,v,u)\big), \quad \forall \big((x,z),(u,v)\big) \in (A \times B) \times (A \times B).$$

Then H is also upper semicontinuous and $\forall ((x,z),(u,v)) \in (A \times B) \times (A \times B)$, $H((x,z),(u,v))$ is a nonempty closed convex subset of $(A \times B) \times (A \times B)$.

By Lemma 7, there exists a point $((x^*,z^*),(v^*,u^*)) \in (A \times B) \times (A \times B)$ such that $((x^*,z^*),(u^*,v^*)) \in H((x^*,z^*),(u^*,v^*))$, that is,

$$\big(x^*,z^*\big) \in \Theta_m\big(x^*,z^*,u^*\big), \qquad \big(u^*,v^*\big) \in \Omega_m\big(x^*,v^*,u^*\big),$$

which implies that $x^* \in \Phi_m(x^*,z^*,u^*)$, $z^* \in T_1(x^*,u^*)$, $u^* \in \Pi_m(x^*,v^*,u^*)$ and $v^* \in T_2(x^*,u^*)$. Hence, there exists $(x^*,u^*) \in A \times A$, $z^* \in T_1(x^*,u^*)$, $v^* \in T_2(x^*,u^*)$ such that $x^* \in K_1(x^*,u^*)$, $u^* \in K_2(x^*,u^*)$, satisfying

$$0 \in F_1\big(x^*,z^*,y\big), \quad \forall y \in P_1\big(x^*,u^*\big),$$

and

$$0 \in F_2(u^*, v^*, y), \quad \forall y \in P_2(x^*, u^*),$$

i.e., (SQVIP$_\alpha$) has a solution.

(d) Now we prove that $\Xi_m(F)$ is closed. Indeed, let a net $\{(x_n, u_n), n \in I\} \in \Xi_m(F)$: $(x_n, u_n) \to (x_0, u_0)$. We need to prove that $(x_0, u_0) \in \Xi_m(F)$. Indeed, by the lower semicontinuity of $P_i, i = 1, 2$, for any $y_0 \in P_i(x_0, u_0)$, there exists $y_n \in P_i(x_n, u_n)$ such that $y_n \to y_0$. Since $(x_n, u_n) \in \Xi_m(F)$, there exists $z_n \in T_1(x_n, u_n)$, $v_n \in T_2(x_n, u_n)$, $x_n \in K_1(x_n, u_n)$, $u_n \in K_2(x_n, u_n)$ such that

$$0 \in F_1(x_n, z_n, y_n),$$

and

$$0 \in F_2(u_n, v_n, y_n).$$

Since K_1, K_2 are upper semicontinuous in $A \times A$ with nonempty closed values, by Lemma 6(i), we have K_1, K_2 are closed. Thus, $x_0 \in K_1(x_0, u_0)$, $u_0 \in K_2(x_0, u_0)$. Since T_1, T_2 are upper semicontinuous in $A \times A$ with nonempty compact values, there exists $z_0 \in T_1(x_0, u_0)$ and $v_0 \in T_2(x_0, u_0)$ such that $z_n \to z_0$, $v_n \to v_0$ (taking subnets if necessary). By the condition (v) and $(x_n, z_n, u_n, v_n) \to (x_0, z_0, u_0, v_0)$, we have

$$0 \in F_1(x_0, z_0, y_0),$$

and

$$0 \in F_2(u_0, v_0, y_0).$$

This means that $(x_0, u_0) \in \Xi_m(F)$. Thus, $\Xi_m(F)$ is a closed set. □

If $K_1(x,u) = P_1(x,u) = S_1(x,u)$, $K_2(x,u) = P_2(x,u) = S_2(x,u)$, $\alpha = m$, and $F_1(x,z,y) = G_1(x,z,y) - C$, $F_2(u,z,y) = G_2(u,z,y) - C$, with $S_1, S_2 : A \times A \to 2^A$, $G_1, G_2 : A \times B \times A \to 2^Z$ are set-valued mappings, and $C \subset Z$ is a nonempty closed convex cone. Then (SQVIP$_\alpha$) becomes (SSVQEP) studied in [6].

In this special case, we have the following corollary.

Corollary 9 *For each* $\{i = 1, 2\}$, *assume for the problem* (SSVQEP) *that*

(i) S_i *is continuous in* $A \times A$ *with nonempty convex closed values*;
(ii) T_i *is usc in* $A \times A$ *with nonempty convex compact values*;
(iii) *for all* $(x, z, u) \in A \times B \times A$, $G_i(x, z, S_i(x, u)) \subset C$;
(iv) *for all* $(x, z, u) \in A \times B \times A$, *the set* $\{a \in S_i(x, u) : G_i(a, z, y) \subset C, \forall y \in S_i(x, u)\}$ *is convex*;
(v) *the set* $\{(x, z, y) \in A \times B \times A : G_i(x, z, y) \subset C\}$ *is closed*.

Then the (SSVQEP) *has a solution, i.e., there exist* $(\bar{x}, \bar{u}) \in A \times A$ *and* $\bar{z} \in T_1(\bar{x}, \bar{u})$, $\bar{v} \in T_2(\bar{x}, \bar{u})$ *such that* $\bar{x} \in S_1(\bar{x}, \bar{u})$, $\bar{u} \in S_2(\bar{x}, \bar{u})$ *and*

$$G_1(\bar{x}, \bar{z}, y) \subset C, \quad \forall y \in S_1(\bar{x}, \bar{u}),$$

and

$$G_2(\bar{u},\bar{v},y) \subset C, \quad \forall y \in S_2(\bar{x},\bar{u}).$$

Moreover, the solution set of the (*SSVQEP*) *is closed*.

Remark 10 Chen *et al.* [6] obtained an existence result of (SSVQEP). However, the assumptions in Theorem 3.1 in [6] are different from the assumptions in Corollary 9. The following example shows that all assumptions of Corollary 9 are satisfied. But Theorem 3.1 in [6] is not fulfilled.

Example 11 Let $X = Y = Z = \mathbb{R}$, $A = B = [0,1]$, $C = [0,+\infty)$ and let $S_1(x) = S_2(x) = [0,1]$, $G_1, G_2, F : [0,1] \times [0,1] \times [0,1] \to 2^{\mathbb{R}}$ and

$$T_1(x,u) = T_2(x,u) = \begin{cases} [0,2] & \text{if } x_0 = u_0 = \frac{1}{2}, \\ [0,1] & \text{otherwise.} \end{cases}$$

and

$$G_1(x,z,y) = G_2(u,z,y) = F(x,z,y) = \begin{cases} [\frac{1}{2},1] & \text{if } x_0 = z_0 = y_0 = \frac{1}{2}, \\ [1,2] & \text{otherwise.} \end{cases}$$

We show that assumptions of Corollary 9 are easily seen to be fulfilled. Hence, by Corollary 9, (SSVQEP) has a solution. But F is neither type II C-lower semicontinuous nor C-concave at $x_0 = \frac{1}{2}$. Thus, Theorem 3.1 in [6] does not work.

If $K_1(x,u) = P_1(x,u) = S_1(x)$, $K_2(x,u) = P_2(x,u) = S_2(u)$, $T_1(x,u) = T_1(x)$, $T_2(x,u) = T_2(u)$, $\alpha = m$ and $F_1(x,z,y) = G_1(x,z,y) - C$, $F_2(u,z,y) = G_2(u,z,y) - C$, with $S_1, S_2 : A \to 2^A$, $G_1, G_2 : A \times B \times A \to 2^Z$ are set-valued mappings, and $C \subset Z$ is a nonempty closed convex cone. Then (SQVIP$_\alpha$) becomes (SGSVQEP) studied in [5].

In this special case, we have the following corollary.

Corollary 12 *For each* $\{i = 1,2\}$, *assume for the problem* (SGSVQEP) *that*

(i) S_i *is continuous in* A *with nonempty convex closed values*;
(ii) T_i *is usc in* A *with nonempty convex compact values*;
(iii) *for all* $(x,z) \in A \times B$, $G_i(x,z,S_i(x)) \subset C$;
(iv) *for all* $(x,z) \in A \times B$, *the set* $\{a \in S_i(x) : G_i(a,z,y) \subset C, \forall y \in S_i(x)\}$ *is convex*;
(v) *the set* $\{(x,z,y) \in A \times B \times A : G_i(x,z,y) \subset C\}$ *is closed*.

Then the (SGSVQEP) *has a solution, i.e., there exist* $(\bar{x},\bar{u}) \in A \times A$ *and* $\bar{z} \in T_1(\bar{x})$, $\bar{v} \in T_2(\bar{u})$ *such that* $\bar{x} \in S_1(\bar{x})$, $\bar{u} \in S_2(\bar{u})$ *and*

$$G_1(\bar{x},\bar{z},y) \subset C, \quad \forall y \in S_1(\bar{x})$$

and

$$G_2(\bar{u},\bar{v},y) \subset C, \quad \forall y \in S_2(\bar{u}).$$

Moreover, the solution set of the (SGSVQEP) *is closed*.

Remark 13 In [5], Plubtieng-Sitthithakerngkiet also obtained an existence result of (SGSVQEP). However, the assumptions in Theorem 3.1 in [5] are different from the assumptions in Corollary 12. The following example shows that in this special case, all assumptions of Corollary 12 are satisfied. But Theorem 3.1 in [5] is not fulfilled.

Example 14 Let $X = Y = Z = \mathbb{R}$, $A = B = [0,1]$, $C = [0,+\infty)$ and let $S_1(x) = S_2(x) = [0,1]$, $F : [0,1] \times [0,1] \times [0,1] \to 2^{\mathbb{R}}$ and

$$T_1(x) = T_2(x) = \begin{cases} [0,2] & \text{if } x_0 = \frac{1}{2}, \\ [0,1] & \text{otherwise.} \end{cases}$$

and

$$G_1(x,z,y) = G_2(u,z,y) = F(x,z,y) = \begin{cases} [\frac{1}{2},1] & \text{if } x_0 = z_0 = y_0 = \frac{1}{2}, \\ [1,2] & \text{otherwise.} \end{cases}$$

We show that all assumptions of Corollary 12 are satisfied. So, by this corollary, the considered problem has solutions. However, F is not lower $(-C)$-continuous at $x_0 = \frac{1}{2}$. Also, Theorem 3.1 in [5] does not work.

If $K_1(\bar{x},\bar{u}) = P_1(\bar{x},\bar{u}) = K_2(\bar{x},\bar{u}) = P_2(\bar{x},\bar{u}) = S(\bar{x})$, $T_1(\bar{x},\bar{u}) = T_2(\bar{x},\bar{u}) = \{z\}$ and $F_1(x,z,y) = F_2(u,z,y) = G(x,y) - C$, for each $\bar{x},\bar{u} \in A$ and $S : A \to 2^A$, $G : A \times A \to 2^Z$ are set-valued mappings, and $C \subset Z$ is a nonempty closed convex cone. Then (SQVIP$_\alpha$) becomes (SVQEP) studied in [21].

In this special case, we also have the following corollary.

Corollary 15 *Assume for the problem* (SVQEP) *that*

(i) *S is continuous in A with nonempty convex closed values*;
(ii) *for all $x \in A$, $G(x,S(x)) \subset C$*;
(iii) *for all $x \in A$, the set $\{a \in S(x) : G(a,y) \subset C, \forall y \in S(x)\}$ is convex*;
(iv) *the set $\{(x,y) \in A \times A : G(x,y) \subset C\}$ is closed.*

Then the (SVQEP) *has a solution, i.e., there exists $\bar{x} \in S(\bar{x})$ such that*

$$G(\bar{x},y) \subset C, \quad \forall y \in S(\bar{x}).$$

Moreover, the solution set of the (SVQEP) *is closed.*

The following example shows that in this special case, all assumptions of Corollary 15 are satisfied. But Theorem 3.3 in [21] is not fulfilled.

Example 16 Let X, Y, Z, A, B, C as in Example 14, and let $S(x) = [0,1]$, $G : [0,1] \times [0,1] \to 2^{\mathbb{R}}$ and

$$G(x,y) = \begin{cases} [1,\frac{5}{2}] & \text{if } x_0 = z_0 = y_0 = \frac{1}{2}, \\ [\frac{1}{5},\frac{3}{4}] & \text{otherwise.} \end{cases}$$

We show that all assumptions of Corollary 15 are satisfied. So, (SVQEP) has a solution. However, G is not upper C-continuous at $x_0 = \frac{1}{2}$. Also, Theorem 3.3 in [21] does not work.

The following example shows that all assumptions of Corollary 9, Corollary 12 and Corollary 15 are satisfied. However, Theorem 3.1 in [6], Theorem 3.1 in [5] and Theorem 3.3 in [21] are not fulfilled. The reason is that G is not properly C-quasiconvex.

Example 17 Let A, B, X, Y, Z, C as in Example 14, and let $S:[0,1]\to 2^{\mathbb{R}}$, $G:[0,1]\times[0,1]\to 2^{\mathbb{R}}$, $S_1(x,u)=S_2(x,u)=S(x)=[0,1]$, $T_1(x,u)=T_2(x,u)=T(x,u)=\{z\}$ and

$$G_1(x,z,y)=G_2(u,z,y)=G(x,y)=\begin{cases}[1,2] & \text{if } x_0=y_0=\frac{1}{2},\\ [\frac{1}{2},1] & \text{otherwise.}\end{cases}$$

We show that all assumptions of Corollary 9, Corollary 12 and Corollary 15 are satisfied. However, G is not properly C-quasiconvex at $x_0=\frac{1}{2}$. Thus, it gives the case where Corollary 9, Corollary 12 and Corollary 15 can be applied but Theorem 3.1 in [6], Theorem 3.1 in [5] and Theorem 3.3 in [21] do not work.

4 Applications

Since our symmetric vector quasi-equilibrium problems include many rather general problems as particular cases mentioned in Section 1, from the results of Section 2 we can derive consequences for such special cases. In this section, we discuss only some corollaries for symmetric weak and strong quasi-equilibrium problems as examples.

Let X, Y, Z, A, B be as in Section 1, and $C\subset Z$ be a nonempty closed convex cone. Let $S_i,P_i:A\times A\to 2^A$, $T_i:A\times A\to 2^B$ be set-valued mappings and $f_i:A\times B\times A\to Z$, $i=1,2$ be vector-valued functions. We consider the two following symmetric weak and strong vector quasi-equilibrium problems (in short, (SWQVEP) and (SSQVEP)), respectively.

(SWQVEP): Find $(\bar{x},\bar{u})\in A\times A$ and $\bar{z}\in T_1(\bar{x},\bar{u})$, $\bar{v}\in T_2(\bar{x},\bar{u})$ such that $\bar{x}\in S_1(\bar{x},\bar{u})$, $\bar{u}\in S_2(\bar{x},\bar{u})$ satisfying

$$f_1(\bar{x},\bar{z},y)\notin -\operatorname{int}C,\quad \forall y\in S_1(\bar{x},\bar{u}),$$
$$f_2(\bar{u},\bar{v},y)\notin -\operatorname{int}C,\quad \forall y\in S_2(\bar{x},\bar{u}).$$

(SSQVEP): Find $(\bar{x},\bar{u})\in A\times A$ and $\bar{z}\in T_1(\bar{x},\bar{u})$, $\bar{v}\in T_2(\bar{x},\bar{u})$ such that $\bar{x}\in S_1(\bar{x},\bar{u})$, $\bar{u}\in S_2(\bar{x},\bar{u})$ satisfying

$$f_1(\bar{x},\bar{z},y)\in C,\quad \forall y\in S_1(\bar{x},\bar{u}),$$
$$f_2(\bar{u},\bar{v},y)\in C,\quad \forall y\in S_2(\bar{x},\bar{u}).$$

Corollary 18 *For each* $\{i=1,2\}$, *assume for the problem* (SWQVEP) *that*

(i) S_i *is continuous in* $A\times A$ *with nonempty convex closed values*;
(ii) T_i *is usc in* $A\times A$ *with nonempty convex compact values*;
(iii) *for all* $(x,z,u)\in A\times B\times A$, $f_i(x,z,S_i(x,u))\notin -\operatorname{int}C$;
(iv) *for all* $(x,z,u)\in A\times B\times A$, *the set* $\{a\in S_i(x,u):f_i(a,z,y)\notin -\operatorname{int}C,\forall y\in S_i(x,u)\}$ *is convex*;
(v) *the set* $\{(x,z,y)\in A\times B\times A:f_i(x,z,y)\notin -\operatorname{int}C\}$ *is closed.*

Then the (SWQVEP) *has a solution, i.e., there exist* $(\bar{x},\bar{u}) \in A \times A$ *and* $\bar{z} \in T_1(\bar{x},\bar{u})$, $\bar{v} \in T_2(\bar{x},\bar{u})$ *such that* $\bar{x} \in S_1(\bar{x},\bar{u})$, $\bar{u} \in S_2(\bar{x},\bar{u})$ *satisfying*

$$f_1(\bar{x},\bar{z},y) \notin -\operatorname{int} C, \quad \forall y \in S_1(\bar{x},\bar{u}),$$
$$f_2(\bar{u},\bar{v},y) \notin -\operatorname{int} C, \quad \forall y \in S_2(\bar{x},\bar{u}).$$

Moreover, the solution set of the (SWQVEP) *is closed.*

Proof Setting $\alpha = m$, $F_1(x,z,y) = Z \setminus (f_1(x,z,y) + \operatorname{int} C)$ and $F_2(u,z,y) = Z \setminus (f_2(u,z,y) + \operatorname{int} C)$, problem (SWQVEP) becomes a particular case of (SQVIP_α) and Corollary 18 is a direct consequence of Theorem 8. □

Corollary 19 *Assume for the problem* (SWQVEP) *assumptions* (i), (ii), (iii) *and* (iv) *as in Corollary* 18 *and replace* (v) *by* (v′)

(v′) *for each* $i = \{1,2\}$, f_i *is continuous in* $A \times B \times A$.

Then the (SWQVEP) *has a solution. Moreover, the solution set of the* (SWQVEP) *is closed.*

Proof We omit the proof since the technique is similar to that for Corollary 18 with suitable modifications. □

Corollary 20 *For each* $\{i = 1,2\}$, *assume for the problem* (SSQVEP) *that*

(i) S_i *is continuous in* $A \times A$ *with nonempty convex closed values*;
(ii) T_i *is usc in* $A \times A$ *with nonempty convex compact values*;
(iii) *for all* $(x,z,u) \in A \times B \times A$, $f_i(x,z,S_i(x,u)) \in C$;
(iv) *for all* $(x,z,u) \in A \times B \times A$, *the set* $\{a \in S_i(x,u) : f_i(a,z,y) \in C, \forall y \in S_i(x,u)\}$ *is convex*;
(v) *the set* $\{(x,z,y) \in A \times B \times A : f_i(x,z,y) \in C\}$ *is closed.*

Then the (SSQVEP) *has a solution, i.e., there exist* $(\bar{x},\bar{u}) \in A \times A$ *and* $\bar{z} \in T_1(\bar{x},\bar{u})$, $\bar{v} \in T_2(\bar{x},\bar{u})$ *such that* $\bar{x} \in S_1(\bar{x},\bar{u})$, $\bar{u} \in S_2(\bar{x},\bar{u})$ *satisfying*

$$f_1(\bar{x},\bar{z},y) \in C, \quad \forall y \in S_1(\bar{x},\bar{u}),$$
$$f_2(\bar{u},\bar{v},y) \in C, \quad \forall y \in S_2(\bar{x},\bar{u}).$$

Moreover, the solution set of the (SSQVEP) *is closed.*

Proof Setting $\alpha = m$, $F_1(x,z,y) = f_1(x,z,y) - C$ and $F_2(u,z,y) = f_2(u,z,y) - C$, problem (SSQVEP) becomes a particular case of (SQVIP_α) and the Corollary 20 is a direct consequence of Theorem 8. □

Corollary 21 *Assume for the problem* (SSQVEP) *assumptions* (i), (ii), (iii) *and* (iv) *as in Corollary* 20 *and replace* (v) *by* (v′)

(v′) *for each* $i = \{1,2\}$, f_i *is continuous in* $A \times B \times A$.

Then the (SSQVEP) *has a solution. Moreover, the solution set of the* (SSQVEP) *is closed.*

Competing interests

The author declares that he has no competing interests.

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