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A Liouville-type theorem for an integral system on a half-space R_+^n

Linfen Cao[1*] and Zhaohui Dai[2]

*Correspondence: caolf2010@yahoo.com
[1]College of Mathematics and Information Science, Henan Normal University, Xinxiang, Henan 453007, P.R. China
Full list of author information is available at the end of the article

Abstract

Let R_+^n be an n-dimensional upper half Euclidean space, and let α be any real number satisfying $0 < \alpha < n$. In our previous paper (Cao and Dai in J. Math. Anal. Appl. 389:1365-1373, 2012), we considered the single equation

$$u(x) = \int_{R_+^n} \left(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x^*-y|^{n-\alpha}} \right) u^r(y)\,dy, \tag{0.1}$$

where $x^* = (x_1, \dots, x_{n-1}, -x_n)$ is the reflection of the point x about the ∂R_+^n. We obtained the monotonicity and nonexistence of positive solutions to equation (0.1) under some integrability conditions when $r > \frac{n}{n-\alpha}$. In (Zhuo and Li in J. Math. Anal. Appl. 381:392-401, 2011), the authors discussed the following system of integral equations in R_+^n:

$$\begin{cases} u(x) = \int_{R_+^n} \left(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x^*-y|^{n-\alpha}}\right) v^q(y)\,dy, \\ v(x) = \int_{R_+^n} \left(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x^*-y|^{n-\alpha}}\right) u^p(y)\,dy \end{cases} \tag{0.2}$$

with $\frac{1}{q+1} + \frac{1}{p+1} = \frac{n-\alpha}{n}$. They obtained rotational symmetry of positive solutions of (0.2) about some line parallel to x_n-axis under the assumption $u \in L^{p+1}(R_+^n)$ and $v \in L^{q+1}(R_+^n)$. In this paper, we derive nonexistence results of such positive solutions for (0.2). In particular, we present a simple and more general method for the study of symmetry and monotonicity which has been extensively used in various forms on a half-space.

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1 Introduction

By a Liouville-type theorem, we here mean the statement of nonexistence of nontrivial (bounded or not) solutions on the whole space or on a half-space. In the last two decades, Liouville-type theorems have been widely used, in conjunction with rescaling arguments, to derive *a priori* estimates for solutions of boundary value problems.

Let R_+^n be the n-dimensional upper half Euclidean space

$$R_+^n = \left\{ x = (x_1, x_2, \dots, x_n) \in R^n \mid x_n > 0 \right\}.$$

In our previous paper [1], we studied the integral equation in R^n_+:

$$u(x) = \int_{R^n_+} \left(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x^* - y|^{n-\alpha}} \right) u^r(y)\,dy, \quad u(x) > 0, x \in R^n_+. \tag{1.1}$$

For $r > \frac{n}{n-\alpha}$ $(0 < \alpha < n)$, we obtained the following Liouville-type theorem.

Theorem 1.1 [1] *Suppose $r > \frac{n}{n-\alpha}$. If the solution u of* (1.1) *satisfies $u \in L^{\frac{n(r-1)}{\alpha}}(R^n_+)$ and is nonnegative, then $u \equiv 0$.*

The result above motivates us to further study positive solutions of the systems of integral equations in R^n_+,

$$\begin{cases} u(x) = \int_{R^n_+} (\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x^*-y|^{n-\alpha}}) v^q(y)\,dy, \\ v(x) = \int_{R^n_+} (\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x^*-y|^{n-\alpha}}) u^p(y)\,dy, \end{cases} \tag{1.2}$$

where p and q satisfy

$$\frac{1}{q+1} + \frac{1}{p+1} = \frac{n-\alpha}{n}.$$

This is the so-called critical case.

In [2] Zhuo and Li discussed regularity and rotational symmetry of solutions for integral system (1.2).

Theorem 1.2 [2] *Let (u, v) be a pair of positive solutions of* (1.2) *with $p, q \geq 1$. Assume that $u \in L^{p+1}(R^n_+)$ and $v \in L^{q+1}(R^n_+)$, then every positive solution (u, v) of* (1.2) *is rotationally symmetric about some line parallel to x_n-axis.*

They also showed close relationships between integral equation (1.2) and the following PDEs system:

$$\begin{cases} (-\triangle)^{\frac{\alpha}{2}} u = v^q, \quad u > 0, \text{ in } R^n_+; \\ (-\triangle)^{\frac{\alpha}{2}} v = u^p, \quad v > 0, \text{ in } R^n_+; \\ u = (-\triangle)u = \cdots = (-\triangle)^{\frac{\alpha}{2}-1} u = 0, \quad \text{on } \partial R^n_+; \\ v = (-\triangle)v = \cdots = (-\triangle)^{\frac{\alpha}{2}-1} v = 0, \quad \text{on } \partial R^n_+, \end{cases} \tag{1.3}$$

where α is an even number.

Theorem 1.3 [2] *Let (u, v) be a pair of solutions of* (1.2) *up to a constant, then (u, v) satisfies* (1.3).

In this paper, we use a simple and more general method to derive that the solution pair (u, v) of (1.2) is strictly monotonically increasing with respect to the variable x_n and further present the nonexistence of positive solutions of (1.2) under some integrability conditions.

Theorem 1.4 *Let* (u,v) *be a pair of positive solutions of* (1.2) *with* $p,q \geq 1$. *Assume that* $u \in L^{p+1}(R^n_+)$ *and* $v \in L^{q+1}(R^n_+)$, *then both* u *and* v *are strictly monotonically increasing with respect to the variable* x_n.

Theorem 1.5 *Let* (u,v) *be a pair of positive solutions of* (1.2) *with* $p,q \geq 1$. *Assume that* $u \in L^{p+1}(R^n_+)$ *and* $v \in L^{q+1}(R^n_+)$ *are nonnegative, then* $u = v \equiv 0$.

2 Properties of the function $G(x,y)$

In this section, we introduce some properties of the function $G(x,y)$ which is defined on a half-space. By using the properties, one could find a simple and general method for the study of symmetry and monotonicity which has been used in various forms defined in a half-space. More precisely, for $x,y \in R^n_+$, define

$$G(x,y) = \frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x^* - y|^{n-\alpha}},$$

where $x^* = (x_1,\ldots,x_{n-1},-x_n)$ is a reflection of the point x about the ∂R^n_+.

Let λ be a positive real number. Define

$$\Sigma_\lambda = \{x = (x_1,x_2,\ldots,x_n) \in R^n | 0 < x_n < \lambda\},$$
$$T_\lambda = \{x \in R^n_+ | x_n = \lambda\}$$

and

$$\Sigma^C_\lambda = R^n_+ \setminus \Sigma_\lambda,$$

the complement of Σ_λ in R^n_+.

Let

$$x^\lambda = (x_1,x_2,\ldots,x_{n-1},2\lambda - x_n)$$

be a reflection of the point $x = (x_1,x_2,\ldots,x_n)$ about the plane T_λ.

To this end, for $x,y \in R^n_+$, define

$$d(x,y) = |x-y|^2$$

and

$$\theta(x,y) = \begin{cases} 4x_n y_n, & \text{if } x,y \in R^n_+, \\ 0, & x \notin R^n_+ \text{ or } y \notin R^n_+. \end{cases}$$

Then, for $x,y \in R^n_+$, $x \neq y$, we have the following expression:

$$G(x,y) = H\big(d(x,y),\theta(x,y)\big).$$

Here $H : (0,\infty) \times [0,\infty) \to R$,

$$H(s,t) = \frac{1}{s^\gamma} - \frac{1}{(s+t)^\gamma}, \quad \gamma = \frac{n-\alpha}{2}.$$

The following lemma states some properties of the function $G(x,y)$. Here we present a proof.

Lemma 2.1

(i) *For any* $x, y \in \Sigma_\lambda$, $x \neq y$, *we have*

$$G(x^\lambda, y^\lambda) > \max\{G(x^\lambda, y), G(x, y^\lambda)\} \tag{2.1}$$

and

$$G(x^\lambda, y^\lambda) - G(x,y) > \left|G(x^\lambda, y) - G(x, y^\lambda)\right|. \tag{2.2}$$

(ii) *For any* $x \in \Sigma_\lambda$, $y \in \Sigma_\lambda^C$, *it holds*

$$G(x^\lambda, y) > G(x,y). \tag{2.3}$$

Proof Since $x, y \in \Sigma_\lambda$, it is easy to verify that

$$d(x^\lambda, y^\lambda) = d(x,y) < d(x^\lambda, y) \tag{2.4}$$

and

$$\theta(x^\lambda, y^\lambda) > \theta(x^\lambda, y) > \theta(x,y). \tag{2.5}$$

In fact,

$$\begin{aligned}\theta(x^\lambda, y^\lambda) - \theta(x^\lambda, y) &= 4(2\lambda - x_n)(2\lambda - y_n) - 4(2\lambda - x_n)y_n \\ &= 8(2\lambda - x_n)(\lambda - y_n) > 0, \\ \theta(x^\lambda, y) - \theta(x,y) &= 4(2\lambda - x_n)y_n - 4x_n y_n \\ &= 8(\lambda - x_n)y_n > 0,\end{aligned}$$

i.e.,

$$\begin{aligned}\theta(x^\lambda, y^\lambda) &\geq \max\{\theta(x^\lambda, y), \theta(x, y^\lambda)\} \\ &\geq \min\{\theta(x^\lambda, y), \theta(x, y^\lambda)\} \geq \theta(x,y).\end{aligned} \tag{2.6}$$

Consider

$$G(x,y) = H(s,t) = \frac{1}{s^\gamma} - \frac{1}{(s+t)^\gamma}$$

with

$$s = d(x,y) \quad \text{and} \quad t = \theta(x,y).$$

Then, for $s, t > 0$, we have

$$\frac{\partial H}{\partial s} = (-\gamma)\left(\frac{1}{s^{\gamma+1}} - \frac{1}{(s+t)^{\gamma+1}}\right) < 0, \tag{2.7}$$

$$\frac{\partial H}{\partial t} = \frac{\gamma}{(s+t)^{\gamma+1}} > 0, \tag{2.8}$$

$$\frac{\partial^2 H}{\partial t\,\partial s} = -\frac{\gamma(\gamma+1)}{(s+t)^{\gamma+2}} < 0. \tag{2.9}$$

(i) From (2.4), (2.5), (2.7) and (2.8), we obtain (2.1).

While by (2.6) and (2.9), we have

$$\begin{aligned} G\big(x^\lambda, y^\lambda\big) - G(x,y) &= \int_{\theta(x,y)}^{\theta(x^\lambda,y^\lambda)} \frac{\partial H(d(x,y),t)}{\partial t}\,dt \\ &> \int_{\theta(x,y)}^{\theta(x^\lambda,y^\lambda)} \frac{\partial H(d(x^\lambda,y),t)}{\partial t}\,dt \\ &\geq \int_{\theta(x,y^\lambda)}^{\theta(x^\lambda,y)} \frac{\partial H(d(x^\lambda,y),t)}{\partial t}\,dt \\ &= \big|H\big(d\big(x^\lambda,y\big),\theta\big(x^\lambda,y\big)\big) - H\big(d\big(x,y^\lambda\big),\theta\big(x,y^\lambda\big)\big)\big| \\ &= \big|G\big(x^\lambda,y\big) - G\big(x,y^\lambda\big)\big|. \end{aligned}$$

Here we have used the fact that $d(x^\lambda, y) = d(x, y^\lambda)$.

(ii) Noticing that for $x \in \Sigma_\lambda$ and $y \in \Sigma_\lambda^C$, we have

$$\big|x^\lambda - y\big| < |x - y| \quad \text{and} \quad \theta\big(x^\lambda, y\big) > \theta(x,y).$$

Then (2.3) follows immediately from (2.7) and (2.8).

This completes the proof of Lemma 2.1. □

Remark The properties of the function $G(x,y)$ defined on a half-space are very similar to the properties of Green's function for a poly-harmonic operator on the ball with Dirichlet boundary conditions. One could find this interesting relation from [3, 4] and [5].

3 The proof of main theorems

In this section, by using the method of moving planes in integral forms, we derive the nonexistence of positive solutions to integral system (1.2) and obtain a new Liouville-type theorem on a half-space. To prove the theorems, we need several lemmas.

Let $\lambda > 0$,

$$\begin{aligned} \Sigma_\lambda &= \big\{x \in R^n \,|\, 0 < x_n < \lambda\big\}, \\ \tilde{\Sigma}_\lambda &= \{x_\lambda \,|\, x \in \Sigma_\lambda\}. \end{aligned}$$

Set

$$u_\lambda(x) = u\big(x^\lambda\big) \quad \text{and} \quad v_\lambda(x) = v\big(x^\lambda\big).$$

Lemma 3.1 *Let (u,v) be any pair of positive solutions of* (1.2). *For any $x \in \Sigma_\lambda$, we have*

$$u(x) - u_\lambda(x) \le \int_{\Sigma_\lambda} \big[G\big(x^\lambda, y^\lambda\big) - G\big(x, y^\lambda\big)\big]\big[v^q(y) - v_\lambda^q(y)\big]\,dy, \tag{3.1}$$

$$v(x) - v_\lambda(x) \le \int_{\Sigma_\lambda} \big[G\big(x^\lambda, y^\lambda\big) - G\big(x, y^\lambda\big)\big]\big[u^p(y) - u_\lambda^p(y)\big]\,dy. \tag{3.2}$$

Proof Obviously, we have

$$\begin{aligned} u(x) &= \int_{\Sigma_\lambda} G(x,y)v^q(y)\,dy + \int_{\Sigma_\lambda} G\big(x,y^\lambda\big)v_\lambda^q(y)\,dy \\ &\quad + \int_{\Sigma_\lambda^C \setminus \tilde{\Sigma}_\lambda} G(x,y)v^q(y)\,dy, \\ u_\lambda(x) &= \int_{\Sigma_\lambda} G\big(x^\lambda,y\big)v^q(y)\,dy + \int_{\Sigma_\lambda} G\big(x^\lambda,y^\lambda\big)v_\lambda^q(y)\,dy \\ &\quad + \int_{\Sigma_\lambda^C \setminus \tilde{\Sigma}_\lambda} G\big(x^\lambda,y\big)v^q(y)\,dy. \end{aligned}$$

Now, by properties (2.2) and (2.3) of the function $G(x,y)$ and the pair of positive solutions of (1.2), we have

$$\begin{aligned} u(x) - u_\lambda(x) &\le \int_{\Sigma_\lambda} \big[G\big(x^\lambda,y^\lambda\big) - G\big(x,y^\lambda\big)\big]\big(v^q(y) - v_\lambda^q(y)\big)\,dy \\ &\quad + \int_{\Sigma_\lambda^C \setminus \tilde{\Sigma}_\lambda} \big[G(x,y) - G\big(x^\lambda,y\big)\big]v^q(y)\,dy \\ &\le \int_{\Sigma_\lambda} \big[G\big(x^\lambda,y^\lambda\big) - G\big(x,y^\lambda\big)\big]\big(v^q(y) - v_\lambda^q(y)\big)\,dy. \end{aligned}$$

Similarly, we could derive the second inequality in the lemma. This completes the proof of Lemma 3.1. □

Proof of Theorem 1.4 To prove Theorem 1.4, we compare $(u(x), v(x))$ and $(u_\lambda(x), v_\lambda(x))$ on Σ_λ. The proof consists of two steps.

In the first step, we start from the very low end of our region $\mathbb{R}^n_+$, *i.e.*, $x_n = 0$. We will show that for λ sufficiently small,

$$u_\lambda(x) \ge u(x) \quad \text{and} \quad v_\lambda(x) \ge v(x), \quad \forall x \in \Sigma_\lambda. \tag{3.3}$$

In the second step, we will move our plane T_λ toward the positive direction of x_n-axis as long as inequality (3.3) holds.

Step 1. Define

$$\Sigma_\lambda^u = \big\{x \mid x \in \Sigma_\lambda, u(x) > u_\lambda(x)\big\},$$

and

$$\Sigma_\lambda^v = \big\{x \mid x \in \Sigma_\lambda, v(x) > v_\lambda(x)\big\}.$$

We show that for sufficiently small positive λ, Σ_λ^u and Σ_λ^v must both be measure zero. In fact, by Lemma 3.1, it is easy to verify that

$$\begin{aligned}
u(x) - u_\lambda(x) &\le \int_{\Sigma_\lambda} \bigl[G(x^\lambda, y^\lambda) - G(x, y^\lambda)\bigr]\bigl(v^p(y) - v_\lambda^p(y)\bigr)\,dy \\
&= \int_{\Sigma_\lambda \setminus \Sigma_\lambda^v} \bigl[G(x^\lambda, y^\lambda) - G(x, y^\lambda)\bigr]\bigl(v^p(y) - v_\lambda^p(y)\bigr)\,dy \\
&\quad + \int_{\Sigma_\lambda^v} \bigl[G(x^\lambda, y^\lambda) - G(x, y^\lambda)\bigr]\bigl(v^p(y) - v_\lambda^p(y)\bigr)\,dy \\
&\le \int_{\Sigma_\lambda^v} \bigl[G(x^\lambda, y^\lambda) - G(x, y^\lambda)\bigr]\bigl(v^p(y) - v_\lambda^p(y)\bigr)\,dy \\
&\le \int_{\Sigma_\lambda^v} G(x^\lambda, y^\lambda)\bigl[v^p(y) - v_\lambda^p(y)\bigr]\,dy \\
&\le p\int_{\Sigma_\lambda^v} \frac{1}{|x-y|^{n-\alpha}} \psi_\lambda^{p-1}(y)\bigl[v(y) - v_\lambda(y)\bigr]\,dy \\
&\le p\int_{\Sigma_\lambda^v} \frac{1}{|x-y|^{n-\alpha}} v^{p-1}(y)\bigl[v(y) - v_\lambda(y)\bigr]\,dy,
\end{aligned}$$

where $\psi_\lambda(y)$ is valued between $v(y)$ and $v_\lambda(y)$. Therefore, on Σ_λ^v we have

$$0 \le v_\lambda(y) \le \psi_\lambda(y) \le v(y).$$

It follows from the Hardy-Littlewood-Sobolev inequality that

$$\|u_\lambda - u\|_{L^{p+1}(\Sigma_\lambda^u)} \le C\bigl\|v^{q-1}(v_\lambda - v)\bigr\|_{L^{(q+1)/q}(\Sigma_\lambda^v)}. \tag{3.4}$$

Then by the Hölder inequality,

$$\|u_\lambda - u\|_{L^{p+1}(\Sigma_\lambda^u)} \le C\|v\|_{L^{q+1}(\Sigma_\lambda^v)}^{q-1}\|v_\lambda - v\|_{L^{q+1}(\Sigma_\lambda^v)}. \tag{3.5}$$

Similarly, one can show that

$$\|v_\lambda - v\|_{L^{q+1}(\Sigma_\lambda^v)} \le C\|u\|_{L^{p+1}(\Sigma_\lambda^u)}^{p-1}\|u_\lambda - u\|_{L^{p+1}(\Sigma_\lambda^u)}. \tag{3.6}$$

Combining (3.5) and (3.6), we arrive at

$$\|u_\lambda - u\|_{L^{p+1}(\Sigma_\lambda^u)} \le C\|v\|_{L^{q+1}(\Sigma_\lambda^v)}^{q-1}\|u\|_{L^{p+1}(\Sigma_\lambda^u)}^{p-1}\|u_\lambda - u\|_{L^{p+1}(\Sigma_\lambda^u)}. \tag{3.7}$$

By the conditions that $u \in L^{p+1}(R_+^n)$ and $v \in L^{q+1}(R_+^n)$, we can choose sufficiently small positive λ such that

$$C\|v\|_{L^{q+1}(\Sigma_\lambda^v)}^{q-1}\|u\|_{L^{p+1}(\Sigma_\lambda^u)}^{p-1} \le \frac{1}{2}.$$

Now, inequality (3.7) implies $\|u_\lambda - u\|_{L^{p+1}(\Sigma_\lambda^u)} = 0$, and therefore Σ_λ^u must be measure zero. Similarly, one can show that Σ_λ^v is measure zero. Therefore, (3.3) holds. This completes Step 1.

Step 2. (Move the plane to the limiting position to derive symmetry and monotonicity.)

Inequality (3.3) provides a starting point to move the plane T_λ. Now, we start from the neighborhood of $x_n = 0$ and move the plane up as long as (3.3) holds to the limiting position. We will show that the solution $u(x)$ must be symmetric about the limiting plane and be strictly monotonically increasing with respect to the variable x_n. More precisely, define

$$\lambda_0 = \sup\bigl\{\lambda | u(x) \le u_\mu(x) \text{ and } v(x) \le v_\mu(x), \forall x \in \Sigma_\mu, \mu \le \lambda\bigr\}.$$

Suppose that for such a λ_0, we will show that both $u(x)$ and $v(x)$ must be symmetric about the plane T_{λ_0} by using a contradiction argument. Assume that on Σ_{λ_0}, we have

$$u(x) \le u_{\lambda_0}(x) \quad \text{and} \quad v(x) \le v_{\lambda_0}(x), \quad \text{but} \quad u(x) \not\equiv u_{\lambda_0}(x) \quad \text{or} \quad v(x) \not\equiv v_{\lambda_0}(x).$$

We show that the plane can be moved further up. More precisely, there exists an $\epsilon > 0$ depending on n, α, and the solution $(u(x), v(x))$ such that

$$u(x) \le u_\lambda(x) \quad \text{and} \quad v(x) \le v_\lambda(x) \quad \text{on } \Sigma_\lambda \text{ for all } \lambda \text{ in } [\lambda_0, \lambda_0 + \epsilon). \tag{3.8}$$

In the case

$$v(x) \not\equiv v_{\lambda_0}(x) \quad \text{on } \Sigma_{\lambda_0},$$

by Lemma 3.1, we have in fact $u(x) < u_{\lambda_0}(x)$ in the interior of Σ_{λ_0}. Let

$$\overline{\Sigma^u_{\lambda_0}} = \bigl\{x \in \Sigma_{\lambda_0} | u(x) \ge u_{\lambda_0}(x)\bigr\} \quad \text{and} \quad \overline{\Sigma^v_{\lambda_0}} = \bigl\{x \in \Sigma_{\lambda_0} | v(x) \ge v_{\lambda_0}(x\bigr\}.$$

Then, obviously, $\overline{\Sigma^u_{\lambda_0}}$ has measure zero and $\lim_{\lambda\to\lambda_0} \Sigma^u_\lambda \subset \overline{\Sigma^u_{\lambda_0}}$. The same is true for that of v. From (3.5) and (3.6), we deduce

$$\|u_\lambda - u\|_{L^{p+1}(\Sigma^u_\lambda)} \le C\|v\|^{q-1}_{L^{q+1}(\Sigma^v_\lambda)}\|u\|^{p-1}_{L^{p+1}(\Sigma^u_\lambda)}\|u_\lambda - u\|_{L^{p+1}(\Sigma^u_\lambda)}. \tag{3.9}$$

Again, the conditions that $u \in L^{p+1}(R^n_+)$ and $v \in L^{q+1}(R^n_+)$ ensure that one can choose ϵ sufficiently small, so that for all λ in $[\lambda_0, \lambda_0 + \epsilon)$,

$$C\|v\|^{q-1}_{L^{q+1}(\Sigma^v_\lambda)}\|u\|^{p-1}_{L^{p+1}(\Sigma^u_\lambda)} \le \frac{1}{2}.$$

Now, by (3.9), we have $\|u_\lambda - u\|_{L^{p+1}(\Sigma^u_\lambda)} = 0$, therefore Σ^u_λ must be measure zero. Similarly, Σ^v_λ must also be measure zero. This verifies (3.8), therefore both $u(x)$ and $v(x)$ are symmetric about the plane T_{λ_0}. Also, the monotonicity easily follows from the argument. This completes the proof of Theorem 1.4. □

Proof of Theorem 1.5 To prove the theorem, firstly we will show that the plane cannot stop at $x_n = \lambda_0$ for some $\lambda_0 < +\infty$, that is, we will prove that $\lambda_0 = +\infty$.

Suppose that $\lambda_0 < +\infty$, the process of Theorem 1.4 shows that the plane $x_n = 2\lambda_0$ is the symmetric points of the boundary ∂R^n_+ with respect to the plane T_{λ_0}, and we derive that

$u(x) = 0$ and $v(x) = 0$ when x is on the plane $x_n = 2\lambda_0$. This contradicts the pair of positive solutions $(u(x), v(x))$ of (1.2), thus $\lambda_0 = +\infty$.

Besides, we know that both $u(x)$ and $v(x)$ of positive solutions of (1.2) are strictly monotonically increasing in the positive direction of x_n-axis, but $u \in L^{p+1}(R_+^n)$ and $v \in L^{q+1}(R_+^n)$, so we come to the conclusion that the pair of positive solutions $(u(x), v(x))$ of (1.2) does not exist.

This completes the proof of Theorem 1.5. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
CL participated in the method of moving planes studies. DZ carried out the applications of inequalities and drafted the manuscript. Both authors read and approved the final manuscript.

Author details
[1]College of Mathematics and Information Science, Henan Normal University, Xinxiang, Henan 453007, P.R. China.
[2]Department of Computer Science, Henan Normal University, Xinxiang, Henan 453007, P.R. China.

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