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Generalized Dunkl-Williams inequality in 2-inner product spaces

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Abstract
We consider the generalized Dunkl-Williams inequality in 2-normed spaces. Also, we give necessary and sufficient conditions for having the equality case in the strictly convex 2-normed space X.

Keywords: 2-normed spaces; 2-inner product spaces; strictly convex; Dunkl-William inequality; Maligranda inequality

1 Introductions and preliminaries

In 1964, Dunkl and Williams [1] proved that if x, y are non-zero vectors in a normed linear space X, then

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \le \frac{4\|x-y\|}{\|x\| + \|y\|}. \tag{1.1}$$

Also, after that it was proved that the equality holds if and only if $x = y$.

Maligranda [2] obtained a refinement of the Dunkl-Williams inequality. He proved that if x, y are non-zero vectors in a normed linear space X, then

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \le \frac{|\|x\| - \|y\|| + \|x-y\|}{\max(\|x\|, \|y\|)}, \tag{1.2}$$

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \ge \frac{\|x-y\| - |\|x\| - \|y\||}{\min(\|x\|, \|y\|)}. \tag{1.3}$$

Also, the Maligranda inequality and its reverse in normed linear spaces were proved by Mercer in [3]. In [4] Kato *et al.* improved the triangle inequality and provided the reverse by showing that

$$\left\| \sum_{j=1}^{n} x_j \right\| + \left(n - \left\| \sum_{j=1}^{n} \frac{x_j}{\|x_j\|} \right\| \right) \min_{1\le i\le n} \|x_i\| \le \sum_{j=1}^{n} \|x_j\|, \tag{1.4}$$

$$\left\| \sum_{j=1}^{n} x_j \right\| + \left(n - \left\| \sum_{j=1}^{n} \frac{x_j}{\|x_j\|} \right\| \right) \max_{1\le i\le n} \|x_i\| \ge \sum_{j=1}^{n} \|x_j\| \tag{1.5}$$

for all non-zero elements $x_1, \ldots, x_n$ of a normed linear space.

Pečarić and Rajić [5] sharpened inequalities (1.4) and (1.5) (when $n > 2$) and generalized inequalities (1.2) and (1.3) by showing that

$$\left\|\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}\right\| \le \min_{1\le i\le n}\left\{\frac{1}{\|x_i\|}\left(\left\|\sum_{j=1}^{n}x_j\right\| + \sum_{j=1}^{n}\bigl|\|x_j\| - \|x_i\|\bigr|\right)\right\} \tag{1.6}$$

$$\left\|\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}\right\| \ge \max_{1\le i\le n}\left\{\frac{1}{\|x_i\|}\left(\left\|\sum_{j=1}^{n}x_j\right\| - \sum_{j=1}^{n}\bigl|\|x_j\| - \|x_i\|\bigr|\right)\right\} \tag{1.7}$$

for all non-zero elements $x_1,\dots,x_n$ of a normed linear space.

Dragomir [6] replaced arbitrary scalars instead of $\frac{1}{\|x_i\|}$ for $i = 1,\dots,n$ in inequalities (1.6) and (1.7) and obtained a generalization of inequalities (1.6) and (1.7). This paper contains two sections. In the first section, we will work on a generalized case of Dunkl-Williams inequality in 2-normed spaces and also give necessary and sufficient conditions for having equality. In the second part, we want to introduce a refinement of the inequality $\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\| \le \frac{2\|x-y\|}{\|x\|+\|y\|}$ in 2-inner product spaces, which was done by Mercer [3] in inner product spaces.

The concept of 2-normed spaces was introduced by Gähler [7] in 1963. After that, in 1973 and 1977, Diminnie, Gähler and White introduced the concept of 2-inner product spaces (see [8, 9]). We offer [10] to readers for more details.

Definition 1.1 Let $\mathbb{K}$ be the symbol of the field $\mathbb{R}$ or $\mathbb{C}$ and X be a linear space on $\mathbb{K}$. Define the $\mathbb{K}$-valued function $(\cdot,\cdot|\cdot)$ on $X \times X \times X$ with the following properties:

(1) $(x,x|y) \ge 0$; $(x,x|y) = 0$ if and only if x and y are linearly dependent,
(2) $(x,x|y) = (y,y|x)$,
(3) $(x,y|z) = \overline{(y,x|z)}$,
(4) $(\alpha x,y|z) = \alpha(x,y|z)$ for any $\alpha \in \mathbb{K}$,
(5) $(x + x',y|z) = (x,y|z) + (x',y|z)$,

for all $x,x',y,z \in X$. $(\cdot,\cdot|\cdot)$ is called a *2-inner product* and $(X,(\cdot,\cdot|\cdot))$ is called a *2-inner product space*.

Lemma 1.2 [10] *Let X be a 2-inner product space. Then*

$$\bigl|(x,y|z)\bigr| \le \sqrt{(x,x|z)}\sqrt{(y,y|z)} \tag{1.8}$$

for every $x,y,z \in X$.

Definition 1.3 [10] Let X be a linear space of dimension greater than 1 on the field $\mathbb{K}$ and let $\|\cdot,\cdot\| : X \times X \to [0,+\infty)$ be a function satisfying the following conditions:

(1) $\|x,y\| = 0$ if and only if x and y are linearly dependent,
(2) $\|x,y\| = \|y,x\|$,
(3) $\|\alpha x,y\| = |\alpha|\|x,y\|$ for all $\alpha \in \mathbb{K}$,
(4) $\|x + y,z\| \le \|x,z\| + \|y,z\|$,

for all $x,y,z \in X$. $\|\cdot,\cdot\|$ is called a *2-norm* and $(X,\|\cdot,\cdot\|)$ is called a *linear 2-normed space*.

It follows from (4) that

$$\bigl|\|x,z\| - \|y,z\|\bigr| \le \|x - y,z\|$$

for all $x,y,z \in X$.

Let X be a 2-inner product space of dimension greater than 1 on the field $\mathbb{R}$. If we define $\|x,y\| = \sqrt{(x,x|y)}$ for all $x,y \in X$, then $\|\cdot,\cdot\|$ is a 2-norm on X and

$$4(x,y|z) = \|x+y,z\|^2 - \|x-y,z\|^2,$$
$$\|x+y,z\|^2 + \|x-y,z\|^2 = 2\|x,z\|^2 + 2\|y,z\|^2$$

(see Theorem 3.1.9 of [10]). In this case, (1.8) means

$$\big|(x,y|z)\big| \le \|x,z\|\,\|y,z\|.$$

A linear 2-normed space $(X,\|\cdot,\cdot\|)$ is said to be *strictly convex* if $\|x+y,z\| = \|x,z\| + \|y,z\|$, $\|x,z\| = \|y,z\| = 1$ and $z \notin \operatorname{span}\{x,y\}$, then $x = y$.

2 Main results

In this section, we establish a generalization of Dunkl-Williams inequality and its reverse in 2-normed spaces.

Theorem 2.1 *Let X be a 2-normed space on the field $\mathbb{K}$. For $x_1,\dots,x_n,z \in X$ and $a_1,\dots,a_n \in \mathbb{K}$, we have*

$$\left\|\sum_{j=1}^n a_j x_j, z\right\| \le \min_{1\le i\le n}\left\{|a_i|\left\|\sum_{j=1}^n x_j, z\right\| + \sum_{j=1}^n |a_j - a_i|\,\|x_j,z\|\right\}, \tag{2.1}$$

$$\left\|\sum_{j=1}^n a_j x_j, z\right\| \ge \max_{1\le i\le n}\left\{|a_i|\left\|\sum_{j=1}^n x_j, z\right\| - \sum_{j=1}^n |a_j - a_i|\,\|x_j,z\|\right\}. \tag{2.2}$$

Proof For a fixed $1 \le i \le n$, we have

$$\begin{aligned}
\left\|\sum_{j=1}^n a_j x_j, z\right\| &= \left\|\sum_{j=1}^n a_i x_j + \sum_{j=1}^n (a_j - a_i)x_j, z\right\| \\
&\le \left\|\sum_{j=1}^n a_i x_j, z\right\| + \left\|\sum_{j=1}^n (a_j - a_i)x_j, z\right\| \\
&\le \left\|\sum_{j=1}^n a_i x_j, z\right\| + \sum_{j=1}^n \big\|(a_j - a_i)x_j, z\big\| \\
&= |a_i|\left\|\sum_{j=1}^n x_j, z\right\| + \sum_{j=1}^n |a_j - a_i|\,\|x_j, z\|.
\end{aligned}$$

By taking minimum over $i = 1,\dots,n$, we obtain (2.1).

Now, we have

$$\begin{aligned}
\left\|\sum_{j=1}^n a_j x_j, z\right\| &= \left\|\sum_{j=1}^n a_i x_j - \sum_{j=1}^n (a_i - a_j)x_j, z\right\| \\
&\ge \left|\left\|\sum_{j=1}^n a_i x_j, z\right\| - \left\|\sum_{j=1}^n (a_i - a_j)x_j, z\right\|\right|
\end{aligned}$$

$$\begin{aligned}&\geq \left\| \sum_{j=1}^{n} a_i x_j, z \right\| - \sum_{j=1}^{n} \left\| (a_i - a_j) x_j, z \right\| \\ &= |a_i| \left\| \sum_{j=1}^{n} x_j, z \right\| - \sum_{j=1}^{n} |a_i - a_j| \|x_j, z\|.\end{aligned}$$

By taking maximum over $1 \leq i \leq n$, we obtain (2.2). □

Theorem 2.2 [10] *Let X be a real linear 2-normed space. The following statements are equivalent*:

(1) $(X, \|\cdot,\cdot\|)$ *is strictly convex.*
(2) *If* $\|x+y,z\| = \|x,z\| + \|y,z\|$ *and* $z \notin \operatorname{span}\{x,y\}$, *then* $x = \lambda y$ *for some* $\lambda > 0$.

Let X be a real linear 2-normed space. If $z \notin \operatorname{span}\{x,y\}$ and $|(x,y|z)| = \|x,z\|\|y,z\|$, then $x = \lambda y$ for some real λ.

Lemma 2.3 *Let X be a linear strictly convex 2-normed space with respect to a 2-inner product on the field $\mathbb{R}$. For non-zero elements $x_1,\dots,x_n \in X$ satisfying $\sum_{j=1}^{n} x_j \neq 0$, $z \notin \operatorname{span}\{x_1,\dots,x_n\}$ and non-zero elements $a_1,\dots,a_n$ of $\mathbb{R}$ such that $a_i \neq a_j$ for some i, j, the following statements are equivalent*:

(1) $\|\sum_{j=1}^{n} a_j x_j, z\| = |a_i| \|\sum_{j=1}^{n} x_j, z\| + \sum_{j=1}^{n} |a_j - a_i| \|x_j, z\|$;
(2) $\sum_{j=1}^{n} (x_j, x_k|z) a_i (a_k - a_i) = \|\sum_{j=1}^{n} x_j, z\| |a_i| |a_k - a_i| \|x_k, z\|$.

Proof Let (1) hold. For a fixed $1 \leq i \leq n$, we have

$$\begin{aligned}\left\| \sum_{j=1}^{n} a_j x_j, z \right\| &= \left\| \sum_{j=1}^{n} a_i x_j + \sum_{j=1}^{n} (a_j - a_i) x_j, z \right\| \\ &= |a_i| \left\| \sum_{j=1}^{n} x_j, z \right\| + \sum_{j=1}^{n} |a_j - a_i| \|x_j, z\|. \end{aligned} \tag{2.3}$$

From the assumption, there exists a non-empty maximal subset $\{j_1,\dots,j_m\}$ of $\{1,\dots,n\}$ for some $1 \leq m \leq n$ such that $a_{j_k} \neq a_i$ for all $1 \leq k \leq m$. Hence, (2.3) holds if and only if

$$\left\| \sum_{j=1}^{n} a_i x_j + \sum_{k=1}^{m} (a_{j_k} - a_i) x_{j_k}, z \right\| = |a_i| \left\| \sum_{j=1}^{n} x_j, z \right\| + \sum_{k=1}^{m} |a_{j_k} - a_i| \|x_{j_k}, z\|.$$

Using Theorem 2.2, we deduce that there exists $\beta_{j_k} > 0$ such that $\sum_{j=1}^{n} a_i x_j = \beta_{j_k} (a_{j_k} - a_i) x_{j_k}$ which is equivalent to

$$\left(\sum_{j=1}^{n} a_i x_j, (a_{j_k} - a_i) x_{j_k} \middle| z \right) = \left\| \sum_{j=1}^{n} a_i x_j, z \right\| \left\| (a_{j_k} - a_i) x_{j_k}, z \right\|.$$

So, (1) and (2) are equivalent. □

Lemma 2.4 *Let X be a linear strictly convex 2-normed space with respect to a 2-inner product on $\mathbb{R}$. For non-zero elements $x_1,\dots,x_n \in X$ satisfying $\sum_{j=1}^{n} x_j = 0$, $z \notin \operatorname{span}\{x_1,\dots,x_n\}$*

and non-zero elements $a_1,\dots,a_n$ of $\mathbb{R}$ such that $a_i \neq a_j$ for some i,j, the following statements are equivalent:

(1) $\|\sum_{j=1}^{n} a_j x_j, z\| = \sum_{j=1}^{n} |a_j - a_i| \|x_j, z\|$.

(2) *There exist* $1 \le i, l \le n$ *such that* $a_l \neq a_i$ *and* $(x_k, x_l|z)(a_k - a_i)(a_l - a_i) = \|x_k, z\| \|x_l, z\| |a_k - a_i||a_l - a_i|$ *for all* k.

Proof Let i $(1 \le i \le n)$ be fixed. By $\sum_{j=1}^{n} x_j = 0$, we get

$$\sum_{j=1}^{n} a_j x_j = \sum_{j=1}^{n} (a_j - a_i) x_j.$$

Now, by the above equality and (1), we get

$$\left\| \sum_{j=1}^{n} (a_j - a_i) x_j, z \right\| = \left\| \sum_{j=1}^{n} a_j x_j, z \right\| = \sum_{j=1}^{n} |a_j - a_i| \|x_j, z\|.$$

Let $\{j_1,\dots,j_m\}$ be as in the proof of Lemma 2.3. So,

$$\left\| \sum_{k=1}^{m} (a_{j_k} - a_i) x_{j_k}, z \right\| = \sum_{k=1}^{m} |a_{j_k} - a_i| \|x_{j_k}, z\|.$$

Now, let $1 \le l \le m$. Then we get

$$\left\| \sum_{\substack{k=1 \\ k \neq l}}^{m} (a_{j_k} - a_i) x_{j_k} + (a_{j_l} - a_i) x_{j_l}, z \right\| = \sum_{\substack{k=1 \\ k \neq l}}^{m} |a_{j_k} - a_i| \|x_{j_k}, z\| + |a_{j_l} - a_i| \|x_{j_l}, z\|.$$

Hence, for some $\beta_{jk} > 0$, $(a_{j_k} - a_i) x_{j_k} = \beta_{jk} (a_{j_l} - a_i) x_{j_l}$. This is equivalent to

$$(x_{j_k}, x_{j_l}|z)(a_{j_k} - a_i)(a_{j_l} - a_i) = \|x_{j_k}, z\| \|x_{j_l}, z\| |a_{j_k} - a_i||a_{j_l} - a_i|,$$

as desired. □

As an application of Lemmas 2.3 and 2.4, we offer the following theorem.

Theorem 2.5 *Let X be a linear strictly convex 2-normed space with respect to a 2-inner product on $\mathbb{R}$. Also, let $x_1,\dots,x_n \in X$ be non-zero elements, $z \notin \operatorname{span}\{x_1,\dots,x_n\}$ and $a_1,\dots,a_n \in \mathbb{R}$ be non-zero elements such that $a_i \neq a_j$ for some i,j.*

(1) *If* $\sum_{j=1}^{n} x_j \neq 0$, *then*

$$\left\| \sum_{j=1}^{n} a_j x_j, z \right\| = \min_{1 \le k \le n} \left\{ |a_i| \left\| \sum_{j=1}^{n} x_j, z \right\| + \sum_{j=1}^{n} |a_j - a_i| \|x_j, z\| \right\}$$

if and only if there exists $1 \le i \le n$ *such that*

$$\sum_{j=1}^{n} (x_j, x_k|z) a_i (a_k - a_i) = \left\| \sum_{j=1}^{n} x_j, z \right\| |a_i||a_k - a_i| \|x_k, z\|.$$

(2) *If* $\sum_{j=1}^{n} x_j = 0$, *then*

$$\left\| \sum_{j=1}^{n} a_j x_j, z \right\| = \min_{1 \le k \le n} \sum_{j=1}^{n} |a_j - a_i| \|x_j, z\|$$

if and only if there exist $1 \le i, l \le n$ *such that* $a_l \ne a_i$ *and*

$$(x_k, x_l | z)(a_k - a_i)(a_l - a_i) = \|x_k, z\| \|x_l, z\| |a_k - a_i| |a_l - a_i|$$

for all k.

3 Improvement of Dunkl-Williams inequality with two elements

In [1], Dunkl and Williams proved that in inequality (1.1) the constant 4 is the best choice in normed linear spaces. Moreover, they proved that in an inner product space, the constant 4 can be replaced by 2; that is,

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \le \frac{2\|x - y\|}{\|x\| + \|y\|}. \tag{3.1}$$

In addition, the equality in (3.1) holds if and only if $\|x\| = \|y\|$.

In 2-inner product spaces, we have the following theorem.

Theorem 3.1 [11] *For non-zero vectors* x, y, z *in a* 2-*inner product space* X *with* $z \notin \operatorname{span}\{x, y\}$,

$$\left\| \frac{x}{\|x, z\|} - \frac{y}{\|y, z\|}, z \right\| \le \frac{2\|x - y, z\|}{\|x, z\| + \|y, z\|}. \tag{3.2}$$

If X is a linear 2-normed space in Theorem 3.1, then we have the following inequality:

$$\left\| \frac{x}{\|x, z\|} - \frac{y}{\|y, z\|}, z \right\| \le \frac{4\|x - y, z\|}{\|x, z\| + \|y, z\|}.$$

A refinement of (3.1) has been obtained by Mercer [3]. Now, we use Mercer's inequality and give a refinement of (3.2).

Theorem 3.2 *Let* x, y, z *be non-zero vectors in a* 2-*inner product space* X *with* $z \notin \operatorname{span}\{x, y\}$. *Then we have*

$$\begin{aligned}
&\frac{(\|x, z\| - \|y, z\|)^2}{(\|x, z\| + \|y, z\|)^2} - \sqrt{\frac{4\|x - y, z\|^2}{(\|x, z\| + \|y, z\|)^2} + \frac{(\|x, z\| - \|y, z\|)^4}{(\|x, z\| + \|y, z\|)^4} - 4\frac{(\|x, z\| - \|y, z\|)^2}{(\|x, z\| + \|y, z\|)^2}} \\
&\quad \le \left\| \frac{x}{\|x, z\|} - \frac{y}{\|y, z\|}, z \right\| \\
&\quad \le \frac{(\|x, z\| - \|y, z\|)^2}{(\|x, z\| + \|y, z\|)^2} \\
&\qquad + \sqrt{\frac{4\|x - y, z\|^2}{(\|x, z\| + \|y, z\|)^2} + \frac{(\|x, z\| - \|y, z\|)^4}{(\|x, z\| + \|y, z\|)^4} - 4\frac{(\|x, z\| - \|y, z\|)^2}{(\|x, z\| + \|y, z\|)^2}}.
\end{aligned} \tag{3.3}$$

Proof Let $\alpha = \|\frac{x}{\|x,z\|} - \frac{y}{\|y,z\|}, z\|$. By using (2.2) for two elements x, $-y$ with constants $\frac{1}{\|x,z\|}$, $\frac{1}{\|y,z\|}$ respectively, we obtain

$$\alpha \geq \frac{\|x-y,z\| - |\|x,z\| - \|y,z\||}{\min(\|x,z\|, \|y,z\|)}.$$

Clearly, we have

$$\begin{aligned}
&\bigl|\|y,z\| - \|x,z\|\bigr| - \|x-y,z\| + 2\min\bigl(\|x,z\|, \|y,z\|\bigr) \\
&\quad = \|y,z\| + \|x,z\| - \|x-y,z\|.
\end{aligned}$$

So,

$$\|y,z\| + \|x,z\| - \|x-y,z\| \geq (2-\alpha)\min\bigl(\|x,z\|, \|y,z\|\bigr).$$

A simple computation shows that

$$\frac{\operatorname{Re}(x,y|z)}{\|x,z\|\|y,z\|} = 1 - \frac{1}{2}\alpha^2, \qquad \alpha^2 = \frac{\|x-y,z\|^2 - (\|y,z\| - \|x,z\|)^2}{\|x,z\|\|y,z\|}.$$

Therefore,

$$\begin{aligned}
&\|x-y,z\|^2 - \left(\frac{\|y,z\| + \|x,z\|}{2}\right)^2 \alpha^2 \\
&\quad = \frac{(\|y,z\| - \|x,z\|)^2}{4\|x,z\|\|y,z\|}\bigl(\bigl(\|y,z\| + \|x,z\|\bigr)^2 - \|x-y,z\|^2\bigr) \\
&\quad = \frac{(\|y,z\| - \|x,z\|)^2}{4\|x,z\|\|y,z\|}\bigl(\|y,z\| + \|x,z\| - \|x-y,z\|\bigr)\bigl(\|y,z\| + \|x,z\| + \|x-y,z\|\bigr) \\
&\quad \geq \frac{(\|y,z\| - \|x,z\|)^2}{4\|x,z\|\|y,z\|}\bigl(\|y,z\| + \|x,z\| - \|x-y,z\|\bigr)\bigl(\|y,z\| + \|x,z\| + \bigl|\|x,z\| - \|y,z\|\bigr|\bigr) \\
&\quad = \frac{(\|y,z\| - \|x,z\|)^2}{2\|x,z\|\|y,z\|}\bigl(\|y,z\| + \|x,z\| - \|x-y,z\|\bigr)\max\bigl(\|x,z\|, \|y,z\|\bigr) \\
&\quad \geq \frac{(\|y,z\| - \|x,z\|)^2}{2\|x,z\|\|y,z\|}(2-\alpha)\min\bigl(\|x,z\|, \|y,z\|\bigr)\max\bigl(\|x,z\|, \|y,z\|\bigr) \\
&\quad = \frac{(\|y,z\| - \|x,z\|)^2}{2}(2-\alpha).
\end{aligned}$$

Hence,

$$\|x-y,z\|^2 - \left(\frac{\|y,z\| + \|x,z\|}{2}\right)^2 \alpha^2 \geq \frac{(\|y,z\| - \|x,z\|)^2}{2}(2-\alpha).$$

Therefore,

$$\alpha^2 - 2\alpha\left(\frac{\|y,z\| - \|x,z\|}{\|y,z\| + \|x,z\|}\right)^2 + 4\frac{(\|y,z\| - \|x,z\|)^2 - \|x-y,z\|^2}{(\|y,z\| + \|x,z\|)^2} \leq 0.$$

So, α is between two roots of the quadratic equation

$$\lambda^2 - 2\lambda\left(\frac{\|y,z\| - \|x,z\|}{\|y,z\| + \|x,z\|}\right)^2 + 4\frac{(\|y,z\| - \|x,z\|)^2 - \|x-y,z\|^2}{(\|y,z\| + \|x,z\|)^2} = 0.$$

Hence, we get (3.3). $\square$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors jointly worked on the results and they read and approved the final manuscript.

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