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A further remark to paper 'Convergence theorems for the common solution for a finite family of ϕ-strongly accretive operator equations'

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Abstract

In this note, we point out several gaps in Gurudwan and Sharma (Appl. Math. Comput. 217(15):6748-6754, 2011) and Yang (Appl. Math. Comput. 218(21):10367-10369, 2012) and give the main results under weaker conditions.

MSC: 47H10; 47H09; 46B20

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1 Introduction

Recently, Gurudwan, Sharma [1] and Yang [2] studied the strong convergence of the sequence, respectively, which was defined by

$$
\begin{aligned}
&x_0 \in E,\\
&x_n^1 = a_n^1 x_n + b_n^1 S_1 x_n + c_n^1 u_n^1,\\
&x_n^2 = a_n^2 x_n + b_n^2 S_2 x_n^1 + c_n^2 u_n^2,\\
&\vdots\\
&x_{n+1} = x_n^N = a_n^N x_n + b_n^N S_N x_n^{N-1} + c_n^N u_n^N, \quad n \geq 0,
\end{aligned}
$$

for approximation of a common solution of a finite family of uniformly continuous Φ-strongly accretive operator equations. Their results are as follows.

Theorem GS [1, Theorem 3.1] *Let E be an arbitrary real Banach space and let $\{A_i\}_{i=1}^N : E \to E$ be uniformly continuous ϕ-strongly accretive operators and each range of either A_i or $(I - A_i)$ be bounded. Let, for $i = 1, \dots, N$, $\{u_n^i\}_{n=1}^\infty$ be sequences in E and $\{a_n^i\}_{n=1}^\infty$, $\{b_n^i\}_{n=1}^\infty$, $\{c_n^i\}_{n=1}^\infty$ be real sequences in $[0,1]$ satisfying*

(i) $a_n^i + b_n^i + c_n^i = 1$,

(ii) $\sum_{n=0}^\infty b_n^N = \infty$,

(iii) $\sum_{n=0}^\infty c_n < \infty$,

(iv) $\lim_{n\to\infty} b_n^i = \lim_{n\to\infty} c_n^i = \lim_{n\to\infty} \frac{c_n^i}{b_n^i + c_n^i} = 0$, $\forall i = 1, \dots, N$, $n \geq 1$.

For any given $f \in E$, define $\{S_i\}_{i=1}^N : E \to E$ by $S_i x = x - A_i x + f$, $\forall i = 1, \ldots, N$, $\forall x \in E$. Then the multi-step iterative sequence with errors $\{x_n\}_{n=1}^\infty$ defined by the above converges strongly to the unique solution of the operator equations $\{A_i x\}_{i=1}^N = f$.

On the basis of the above result, Yang [2] proved the following convergence theorem.

Thoerem Yang [2, Theorem 2] *Let E be an arbitrary real Banach space and let $\{A_i\}_{i=1}^N : E \to E$ be uniformly continuous ϕ-strongly accretive operators and each range of either A_i or $(I - A_i)$ is bounded. Let for $i = 1, \ldots, N$, $\{u_n^i\}_{n=1}^\infty$ be bounded sequences in E and $\{a_n^i\}_{n=1}^\infty$, $\{b_n^i\}_{n=1}^\infty$, $\{c_n^i\}_{n=1}^\infty$ be real sequences in $[0,1]$ satisfying*

(i) $a_n^i + b_n^i + c_n^i = 1$,
(ii) $\sum_{n=0}^\infty b_n^N = \infty$,
(iii) $\lim_{n\to\infty} b_n^i = \lim_{n\to\infty} c_n^i = \lim_{n\to\infty} \frac{c_n^i}{b_n^i + c_n^i} = 0$, $\forall i = 1, \ldots, N$, $n \geq 1$.

For any given $f \in E$, define $\{S_i\}_{i=1}^N : E \to E$ by $S_i x = (I - A_i)x + f$, $\forall i = 1, \ldots, N$, $\forall x \in E$. Then the multi-step iterative sequence with errors $\{x_n\}_{n=1}^\infty$ defined by the above converges strongly to the unique solution of the operator equations $\{A_i x\}_{i=1}^N = f$.

However, after careful reading of their works, we discovered that there exist some problems in references [1] and [2] as follows.

Problem 1 In the proof course of Theorem 3.1 of Gurudwan and Sharma [1], which happens in line 11 of page 6751. Here, it is defective that they obtained $\|x - y\| \leq \phi_i^{-1}(\|A_i x - A_i y\|)$, that is, $\langle A_i x - A_i y, j(x - y)\rangle \geq \phi(\|x - y\|)\|x - y\| \Rightarrow \phi(\|x - y\|) \leq \|A_i x - A_i y\|$, but we cannot deduce $\|x - y\| \leq \phi_i^{-1}(\|A_i x - A_i y\|)$. The reason is that it is possible $\|A_i x - A_i y\|$ does not belong to $R(\phi)$ (range of ϕ). A counterexample is as follows. Let us define $\phi : [0, +\infty) \to [0, +\infty)$ by $\phi(\alpha) = \frac{2^\alpha - 1}{2^\alpha + 1}$; then it can be easily seen that ϕ is increasing with $\phi(0) = 0$, but $\lim_{\alpha\to+\infty} \phi(\alpha) = 1$ and $\phi^{-1}(2)$ makes no sense (see [3]).

Problem 2 In the paper of Yang [2], he referred to the mistakes of '$\|x_{n_m+j}^i - q\| < \epsilon$ for $j \geq 1$ to deduce $\|x_n - q\| \to 0$ $(n \to \infty)$' in [1] and cited an example, *i.e.*,

$$\{\nu_n\} = \{1, 0, 2, 0, 0, 3, 0, 0, 0, 4, 0, 0, 0, 0, 5, 0, 0, 0, 0, 0, 6, 0, 0, \ldots\}. \qquad (**)$$

Now, we want to clarify the fact. Let $\{\gamma_n\}$ be a real sequence, $\{\gamma_{n_m}\}$ be some infinite subsequence of $\{\gamma_n\}$ and $\{n_m\}$ be neither odd nor even sequence, then the conclusions are as follows:

(C-i) $\lim_{n\to\infty} \gamma_n = 0 \Leftrightarrow \forall \epsilon > 0$, $\exists$ nonnegative integer n_0 such that $|\gamma_{n_m+j}| < \epsilon$ for $n_m \geq n_0$, $j \geq 1$.
(C-ii) $\lim_{n\to\infty} \gamma_n = 0 \Rightarrow \lim_{m\to\infty} \gamma_{n_m} = 0$ and $\lim_{m\to\infty} \gamma_{n_m+j} = 0$ for $\forall j \geq 1$.

Indeed, the above example (**) does not satisfy the conclusion (C-i), it just illustrates the result (C-ii). Therefore, the note given by Yang [2] confused the conclusions (C-i) and (C-ii).

The aim of this paper is to generalize the results of papers [1] and [2]. For this, we need the following knowledge.

2 Preliminary

Let E be a real Banach space and E^* be its dual space. The normalized duality mapping $J : E \to 2^{E^*}$ is defined by

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\}, \quad \forall x \in E,$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. The single-valued normalized duality mapping is denoted by j.

An operator $T : E \to E$ is said to be strongly accretive if there exists a constant $k > 0$, and for $\forall x, y \in E$, $\exists j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \geq k\|x-y\|^2,$$

without loss of generality, we assume that $k \in (0,1)$. The operator T is called ϕ-strongly accretive if for any $x, y \in E$, there exist $j(x-y) \in J(x-y)$ and a strictly increasing continuous function $\phi : [0,+\infty) \to [0,+\infty)$ with $\phi(0) = 0$ such that

$$\langle Tx - Ty, j(x-y)\rangle \geq \phi(\|x-y\|)\|x-y\|.$$

It is obvious that a strongly accretive operator must be the ϕ-strongly accretive in the special case in which $\phi(t) = kt$, but the converse is not true in general. That is, the class of strongly accretive operators is a proper subclass of the class of ϕ-strongly accretive operators.

In order to obtain the main conclusion of this paper, we need the following lemmas.

Lemma 2.1 [1] *Suppose that E is an arbitrary Banach space and $A : E \to E$ is a continuous ϕ-strongly accretive operator. Then the equation $Ax = f$ has a unique solution for any $f \in E$.*

Lemma 2.2 [4] *Let E be a real Banach space and let $J : E \to 2^{E^*}$ be a normalized duality mapping. Then*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle, \tag{2.1}$$

for all $x, y \in E$ and $j(x+y) \in J(x+y)$.

Lemma 2.3 [5] *Let $\{\delta_n\}_{n=0}^{\infty}$, $\{\lambda_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ be three nonnegative real sequences and $\phi : [0,+\infty) \to [0,+\infty)$ be a strictly increasing and continuous function with $\phi(0) = 0$ satisfying the following inequality*:

$$\delta_{n+1}^2 \leq \delta_n^2 - \lambda_n \phi(\delta_{n+1}) + \gamma_n, \quad n \geq 0, \tag{2.2}$$

where $\lambda_n \in [0,1]$ with $\sum_{n=0}^{\infty} \lambda_n = \infty$, $\gamma_n = o(\lambda_n)$. Then $\delta_n \to 0$ as $n \to \infty$.

3 Main results

Theorem 3.1 *Let E be an arbitrary real Banach space and $\{A_i\}_{i=1}^{N} : E \to E$ be N uniformly continuous ϕ-strongly accretive operators. For $i = 1, 2, \ldots, N$, let $\{u_n^i\}_{n=1}^{\infty}$ be bounded sequences in E and $\{a_n^i\}_{n=1}^{\infty}$, $\{b_n^i\}_{n=1}^{\infty}$, $\{c_n^i\}_{n=1}^{\infty}$ be real sequences in $[0,1]$ satisfying*

(i) $a_n^i + b_n^i + c_n^i = 1$, $i = 1, 2, \ldots, N$;
(ii) $\sum_{n=1}^{\infty} b_n^N = +\infty$;
(iii) $\lim_{n\to\infty} b_n^i = \lim_{n\to\infty} c_n^i = 0$, $i = 1, 2, \ldots, N$;
(iv) $c_n^N = o(b_n^N)$.

For any given $f \in E$, define $\{S_i\}_{i=1}^N : E \to E$ with $\bigcap_{i=1}^N F(S_i) \neq \emptyset$ by $S_i x = x - A_i x + f$, $\forall i = 1, 2, \ldots, N$, $\forall x \in E$, where $F(S_i) = \{x \in E : S_i x = x\}$. Then, for some $x_0 \in E$, the multi-step iterative sequence with errors $\{x_n\}_{n=1}^\infty$ defined by

$$\begin{aligned}
&x_0 \in E, \\
&x_n^1 = a_n^1 x_n + b_n^1 S_1 x_n + c_n^1 u_n^1, \\
&x_n^2 = a_n^2 x_n + b_n^2 S_2 x_n^1 + c_n^2 u_n^2, \\
&\vdots \\
&x_n^{N-1} = a_n^{N-1} x_n + b_n^{N-1} S_{N-1} x_n^{N-2} + c_n^{N-1} u_n^{N-1}, \\
&x_{n+1} = x_n^N = a_n^N x_n + b_n^N S_N x_n^{N-1} + c_n^N u_n^N, \quad n \geq 0,
\end{aligned}$$

converges strongly to the unique solution of the operator equations $\{A_i x\}_{i=1}^N = f$.

Proof Since $\{A_i\}_{i=1}^N : E \to E$ is ϕ-strongly accretive operator, we obtain that each equation $A_i x = f$ has the unique solution by Lemma 2.1, denote q_i, *i.e.*, q_i is the unique fixed point of S_i by $S_i x = x - A_i x + f$. Since $\bigcap_{i=1}^N F(S_i) \neq \emptyset$, then $\bigcap_{i=1}^N F(S_i)$ is a single set, let q. Meanwhile, there exists a strictly increasing continuous function $\phi : [0, +\infty) \to [0, +\infty)$ with $\phi(0) = 0$ such that

$$\bigl\langle A_i x - A_i q, j(x - q)\bigr\rangle \geq \phi\bigl(\|x - q\|\bigr),$$

for $x \in E$, $q \in F(T)$, that is,

$$\bigl\langle S_i x - x, j(x - q)\bigr\rangle \leq -\phi\bigl(\|x - q\|\bigr). \tag{@}$$

Choose some $x_0 \in E$ and $x_0 \neq S_i x_0$ such that $r_0 \in R(\Phi)$, where

$$r_0 = \max\bigl\{\|x_0 - S_1 x_0\| \cdot \|x_0 - q\|, \|x_0 - S_2 x_0\| \cdot \|x_0 - q\|, \ldots, \|x_0 - S_N x_0\| \cdot \|x_0 - q\|\bigr\},$$

$R(\Phi)$ is the range of Φ. Indeed, if $\Phi(r) \to +\infty$ as $r \to +\infty$, then $r_0 \in R(\Phi)$; if $\sup\{\Phi(r) : r \in [0, +\infty)\} = r_1 < +\infty$ with $r_1 < r_0$, then for $q \in E$, there exists a sequence $\{w_n\}$ in E such that $w_n \to q$ as $n \to \infty$ with $w_n \neq q$. Since A_i is uniformly continuous, so is S_i. Furthermore, we obtain that $S_i w_n \to S_i q$ as $n \to \infty$, then $\{w_n - S_i w_n\}$ is the bounded sequence for $i = 1, 2, \ldots, N$. Hence, there exists the common natural number n_0 such that $\|w_n - S_i w_n\| \cdot \|w_n - q\| < \frac{r_1}{2}$ for $n \geq n_0$ and $i = 1, 2, \ldots, N$, then we redefine $x_0 = w_{n_0}$ and $\|x_0 - S_i x_0\| \cdot \|x_0 - q\| < \frac{r_1}{2}$. Thus, $\max_{1 \leq i \leq N}\{\|x_0 - S_i x_0\| \cdot \|x_0 - q\|\} \in R(\phi)$. It is to ensure that $\Phi^{-1}(r_0)$ is defined well.

Step I. We show that $\{x_n\}$ is a bounded sequence.

Set $R = \Phi^{-1}(r_0)$, then from the above formula (@), we obtain that $\|x_0 - q\| \leq R$. Denote

$$B_1 = \bigl\{x \in E : \|x - q\| \leq R\bigr\}, \qquad B_2 = \bigl\{x \in E : \|x - q\| \leq 2R\bigr\}.$$

Since S_i is uniformly continuous, then S_i is bounded. We let

$$M = \max_{1\le i\le N}\Bigl\{\sup_{x\in B_2}\bigl\{\|S_ix-q\|+1\bigr\}\Bigr\}+\max_{1\le i\le N}\Bigl\{\sup_n\bigl\{\|u_n^i-q\|\bigr\}\Bigr\}.$$

Next, we want to prove that $x_n \in B_1$. If $n = 0$, then $x_0 \in B_1$. Now, assume that it holds for some n, *i.e.*, $x_n \in B_1$. We prove that $x_{n+1} \in B_1$. Suppose it is not the case, then $\|x_{n+1} - q\| > R > \frac{R}{2}$. Since S_i is uniformly continuous for $i = 1, 2, \dots, N$, then for $\epsilon_0 = \frac{\Phi(\frac{R}{2})}{8R}$, there exists common $\delta > 0$ such that $\|S_ix - S_iy\| < \epsilon_0$ when $\|x - y\| < \delta$. Denote

$$\tau_0 = \min\biggl\{1, \frac{R}{M}, \frac{\Phi(\frac{R}{2})}{8R(M+2R)}, \frac{\delta}{2M+5R}\biggr\}.$$

Since $b_n^i, c_n^i \to 0$ as $n \to \infty$ for $i = 1, 2, \dots, p$. Without loss of generality, we let $0 \le b_n^i, c_n^i \le \tau_0$ for any $n \ge 0$ and $i = 1, 2, \dots, N$. Since $c_n^N = o(b_n^N)$, let $c_n^N < b_n^N\tau_0$. Now, estimate $\|x_n^i - q\|$ for $i = 1, 2, \dots, N$. From the multi-step iteration, we have

$$\begin{aligned}
&\|x_n^1-q\| \\
&\quad\le \bigl(1-b_n^1-c_n^1\bigr)\|x_n-q\|+b_n^1\|S_1x_n-q\|+c_n^1\|u_n^1-q\| \\
&\quad\le R+\tau_0M \\
&\quad\le 2R,
\end{aligned}\tag{3.1}$$

then $x_n^1 \in B_2$. Similarly, we have

$$\begin{aligned}
&\|x_n^2-q\| \\
&\quad\le \bigl(1-b_n^2-c_n^2\bigr)\|x_n-q\|+b_n^2\|S_2x_n^1-q\|+c_n^2\|u_n^2-q\| \\
&\quad\le R+\tau_0M \\
&\quad\le 2R,
\end{aligned}\tag{3.2}$$

then $x_n^2 \in B_2$. $\cdots\cdots$, we have

$$\begin{aligned}
&\|x_n^{N-1}-q\| \\
&\quad\le \bigl(1-b_n^{N-1}-c_n^{N-1}\bigr)\|x_n-q\|+b_n^{N-1}\|S_{N-1}x_n^{N-2}-q\|+c_n^{N-1}\|u_n^{N-1}-q\| \\
&\quad\le R+\tau_0M \\
&\quad\le 2R,
\end{aligned}\tag{3.3}$$

then $x_n^{N-1} \in B_2$. Therefore, we get

$$\begin{aligned}
&\|x_{n+1}-q\| \\
&\quad\le \bigl(1-b_n^N-c_n^N\bigr)\|x_n-q\|+b_n^N\|S_Nx_n^{N-1}-q\|+c_n^N\|u_n^N-q\| \\
&\quad\le R+\tau_0M \\
&\quad\le 2R.
\end{aligned}\tag{3.4}$$

And we also have

$$\begin{aligned}
&\|x_{n+1} - x_n\| \\
&\quad \le b_n^N \|S_N x_n^{N-1} - x_n\| + c_n^N \|u_n^N - x_n\| \\
&\quad \le b_n^N (\|S_N x_n^{N-1} - q\| + \|x_n - q\|) + c_n^N (\|u_n^N - q\| + \|x_n - q\|) \\
&\quad \le \tau_0 [(\|S_N x^{N-1} - q\| + \|u_n^N - q\|) + 2\|x_n - q\|] \\
&\quad \le \tau_0 (M + 2R) \\
&\quad \le \frac{\Phi(\frac{R}{2})}{8R},
\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}
&\|x_{n+1} - x_n^{N-1}\| \\
&\quad \le b_n^N \|S_N x_n^{N-1} - x_n\| + c_n^N \|u_n^N - x_n\| + b_n^{N-1} \|S_{N-1} x_n^{N-2} - x_n\| + c_n^{N-1} \|u_n^{N-1} - x_n\| \\
&\quad \le b_n^N (\|S_N x_n^{N-1} - q\| + \|x_n - q\|) + c_n^N (\|u_n^N - q\| + \|x_n - q\|) \\
&\qquad + b_n^{N-1} (\|S_{N-1} x_n^{N-2} - q\| + \|x_n - q\|) + c_n^{N-1} (\|u_n^{N-1} - q\| + \|x_n - q\|) \\
&\quad \le \tau_0 [(\|S_N x_n^{N-1} - q\| + \|u_n^N - q\| + 2\|x_n - q\|) \\
&\qquad + (\|S_{N-1} x_n^{N-2} - q\| + \|u_n^{N-1} - q\| + 2\|x_n - q\|)] \\
&\quad \le \tau_0 (2M + 4R) \\
&\quad < \delta.
\end{aligned} \tag{3.6}$$

By the uniform continuity of S_N, we have

$$\|S_N x_{n+1} - S_N x_n^{N-1}\| < \frac{\Phi(\frac{R}{2})}{8R}.$$

Using Lemma 2.2 and the above formulas, we have

$$\begin{aligned}
&\|x_{n+1} - q\|^2 \\
&\quad = \|(x_n - q) + b_n^N (S_N x_n^{N-1} - x_n) + c_n^N (u_n^N - x_n)\|^2 \\
&\quad \le \|x_n - q\|^2 + 2b_n^N \langle S_N x_n^{N-1} - x_n, j(x_{n+1} - q)\rangle + 2c_n^N \langle u_n^N - x_n, j(x_{n+1} - q)\rangle \\
&\quad \le \|x_n - q\|^2 + 2b_n^N \langle S_N x_{n+1} - x_{n+1} + x_{n+1} - x_n - S_N x_{n+1} + S_N x_n^{N-1}, j(x_{n+1} - q)\rangle \\
&\qquad + 2c_n^N \|u_n^N - x_n\| \cdot \|x_{n+1} - q\| \\
&\quad \le \|x_n - q\|^2 - 2b_n^N \Phi(\|x_{n+1} - q\|) + 2b_n^N \|x_{n+1} - x_n\| \cdot \|x_{n+1} - q\| \\
&\qquad + 2b_n^N \|S_N x_{n+1} - S_N x_n^{N-1}\| \cdot \|x_{n+1} - q\| + 2c_n^N (\|u_n^N - q\| + \|x_n - q\|)\|x_{n+1} - q\| \\
&\quad \le \|x_n - q\|^2 - 2b_n^N \Phi\left(\frac{R}{2}\right) + 2b_n^N \frac{\Phi(\frac{R}{2})}{8R} \cdot 2R + 2b_n^N \frac{\Phi(\frac{R}{2})}{8R} \cdot 2R + 2b_n^N \tau_0 (R + M)2R \\
&\quad \le \|x_n - q\|^2 - b_n^N \Phi\left(\frac{R}{2}\right) + 2b_n^N \frac{\Phi(\frac{R}{2})}{8R(M + 2R)}(R + M)2R
\end{aligned}$$

$$\begin{aligned}
&\le \|x_n - q\|^2 - \frac{b_n^N}{2}\Phi\left(\frac{R}{2}\right) \\
&\le R^2, \qquad (3.7)
\end{aligned}$$

which is a contradiction. So, $x_{n+1} \in B_1$, *i.e.*, $\{x_n\}$ is a bounded sequence, from which it follows that $\{x_n^1\}, \{x_n^2\}, \dots, \{x_n^{N-1}\}$ are all bounded sequences as well.

Step II. We want to prove $\|x_n - q\| \to 0$ as $n \to \infty$.

Since $b_n^i, c_n^i \to 0$ as $n \to \infty$ for $i = 1, 2, \dots, N$ and $\{x_n\}$, $\{x_n^{N-1}\}$ are bounded. From (3.5) and (3.6), we obtain

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0, \qquad \lim_{n\to\infty} \left\|x_{n+1} - x_n^{N-1}\right\| = 0, \qquad \lim_{n\to\infty} \left\|S_N x_{n+1} - S_N x_n^{N-1}\right\| = 0.$$

By (3.7), we have

$$\begin{aligned}
&\|x_{n+1} - q\|^2 \\
&\quad = \left\|(x_n - q) + b_n^N\left(S_N x_n^{N-1} - x_n\right) + c_n^N\left(u_n^N - x_n\right)\right\|^2 \\
&\quad \le \|x_n - q\|^2 + 2b_n^N\left\langle S_N x_n^{N-1} - x_n, j(x_{n+1} - q)\right\rangle + 2c_n^N\left\langle u_n^N - x_n, j(x_{n+1} - q)\right\rangle \\
&\quad \le \|x_n - q\|^2 + 2b_n^N\left\langle S_N x_{n+1} - x_{n+1} + x_{n+1} - x_n - S_N x_{n+1} + S_N x_n^{N-1}, j(x_{n+1} - q)\right\rangle \\
&\qquad + 2c_n^N\left\|u_n^N - x_n\right\| \cdot \|x_{n+1} - q\| \\
&\quad \le \|x_n - q\|^2 - 2b_n^N\Phi\left(\|x_{n+1} - q\|\right) + 2b_n^N\|x_{n+1} - x_n\| \cdot \|x_{n+1} - q\| \\
&\qquad + 2b_n^N\left\|S_N x_{n+1} - S_N x_n^{N-1}\right\| \cdot \|x_{n+1} - q\| + 2c_n^N\left\|u_n^N - x_n\right\| \cdot \|x_{n+1} - q\| \\
&\quad = \|x_n - q\|^2 - 2b_n^N\Phi\left(\|x_{n+1} - q\|\right) + o\left(b_n^N\right), \qquad (3.8)
\end{aligned}$$

where

$$\begin{aligned}
&2b_n^N\|x_{n+1} - x_n\| \cdot \|x_{n+1} - q\| + 2b_n^N\left\|S_N x_{n+1} - S_N x_n^{N-1}\right\| \\
&\quad \cdot \|x_{n+1} - q\| + 2c_n^N\left\|u_n^N - x_n\right\| \cdot \|x_{n+1} - q\| = o\left(b_n^N\right).
\end{aligned}$$

By Lemma 2.3, we obtain $\lim_{n\to\infty} \|x_n - q\| = 0$. This completes the proof. □

Remark 3.2 Theorem 3.1 generalizes Theorem 3.1 of [1] and Theorem 2 of [2] in the following cases:

(a) It is not necessary for each range of A_i or $I - A_i$ to be bounded in [1] and [2].

(b) The condition of $\{c_n^i\}$ is weakened to $c_n^N = o(b_n^N)$ from $\lim_{n\to\infty} \frac{c_n^i}{b_n^i + c_n^i} = 0$ $(i = 1, 2, \dots, N)$.

(c) The proof method of our theorem differs from that of [1] and [2].

Theorem 3.3 *Let E, $\{u_n^i\}$, $\{a_n^i\}$, $\{b_n^i\}$, $\{c_n^i\}$ $(i = 1, 2, \dots, N)$ be as in Theorem 3.1 and let $\{T_i\}_{i=1}^N : E \to E$ be N uniformly continuous ϕ-strongly pseudocontractive mappings. Then, for some $x_0 \in E$, the multi-step iterative sequence with errors $\{x_n\}_{n=1}^\infty$ defined by*

$$\begin{aligned}
&x_0 \in E, \\
&x_n^1 = a_n^1 x_n + b_n^1 T_1 x_n + c_n^1 u_n^1,
\end{aligned}$$

$$
\begin{aligned}
&x_n^2 = a_n^2 x_n + b_n^2 T_2 x_n^1 + c_n^2 u_n^2,\\
&\vdots\\
&x_n^{N-1} = a_n^{N-1} x_n + b_n^{N-1} T_{N-1} x_n^{N-2} + c_n^{N-1} u_n^{N-1},\\
&x_{n+1} = x_n^N = a_n^N x_n + b_n^N T_N x_n^{N-1} + c_n^N u_n^N, \quad n \geq 0,
\end{aligned}
$$

converges strongly to the unique common fixed point of $\{T_i\}_{i=1}^N$.

Proof See [1]. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally in writing this paper, and read and approved the final manuscript.

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