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A Mazur-Ulam problem in non-Archimedean n-normed spaces

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Abstract

In this article, we study the notions of n-isometries in non-Archimedean n-normed spaces over linear ordered non-Archimedean fields and prove the Mazur-Ulam theorem in the spaces. Furthermore, we obtain some properties for n-isometries in non-Archimedean n-normed spaces.
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1 Introduction

Let X and Y be metric spaces with metrics d_X and d_Y, respectively. A map $f : X \to Y$ is called an isometry if $d_Y(f(x), f(y)) = d_X(x, y)$ for every $x, y \in X$. Mazur and Ulam [1] treated the theory of isometry for the first time. They have proved the following theorem.

Mazur-Ulam theorem *Let f be an isometric transformation from a real normed vector space X onto a real normed vector space Y with $f(0) = 0$. Then f is linear.*

It was natural to ask if the result holds without the onto assumption. Having asked this natural question, Baker [2] answered that every isometry of a real normed linear space into a strictly convex real normed linear space is affine. The Mazur-Ulam theorem has been widely studied by [3–12].

Chu *et al.* [13] have defined the notion of a 2-isometry which is suitable to represent the concept of an area-preserving mapping in linear 2-normed spaces. In [14], Chu proved that the Mazur-Ulam theorem holds in linear 2-normed spaces under the condition that a 2-isometry preserves collinearity. Chu *et al.* [15] discussed characteristics of 2-isometries. In [16], Amyari and Sadeghi proved the Mazur-Ulam theorem in non-Archimedean 2-normed spaces under the condition of strict convexity. Recently, Choy *et al.* [17] proved the theorem on non-Archimedean 2-normed spaces over linear ordered non-Archimedean fields without the strict convexity assumption.

Misiak [18, 19] defined the concept of an n-normed space and investigated the space. Park and Rassias [20] investigated the stability of linear n-isometries in a linear n-normed Banach module. Chu *et al.* [21], in linear n-normed spaces, defined the concept of an n-isometry that is suitable to represent the notion of a volume-preserving mapping. In [22], Chu *et al.* generalized the Mazur-Ulam theorem to n-normed spaces.

In this paper, without the condition of strict convexity, we prove the (additive) Mazur-Ulam theorem on non-Archimedean n-normed spaces. Firstly, we assert that an n-

isometry f from a non-Archimedean space to a non-Archimedean space preserves the midpoint of a segment under some condition about the set of all elements of a valued field whose valuations are 1. Using the above result, we show the Mazur-Ulam theorem on non-Archimedean n-normed spaces over linear ordered non-Archimedean fields. In addition, we prove that the barycenter of a triangle in the non-Archimedean n-normed spaces is f-invariant under different conditions from those referred in previous statements. And then we also prove the (second type) Mazur-Ulam theorem in non-Archimedean n-normed spaces under some different conditions.

2 The Mazur-Ulam theorem I in non-Archimedean *n*-normed spaces

In this section, we introduce a non-Archimedean n-normed space which is a kind of generalization of a non-Archimedean 2-normed space, and we show the (additive) Mazur-Ulam theorem for an n-isometry f defined on a non-Archimedean n-normed space, that is, $f(x) - f(0)$ is additive. Firstly, we consider some definitions and lemmas which are needed to prove the theorem.

Recall that a *non-Archimedean* (or *ultrametric*) *valuation* is given by a map $|\cdot|$ from a field $\mathcal{K}$ into $[0, \infty)$ such that for all $r, s \in \mathcal{K}$,

   (i) $|r| = 0$ if and only if $r = 0$;

  (ii) $|rs| = |r||s|$;

 (iii) $|r + s| \leq \max\{|r|, |s|\}$.

If every element of $\mathcal{K}$ carries a valuation, then a field $\mathcal{K}$ is called a *valued field*; for convenience, we simply call it a field. It is obvious that $|1| = |-1| = 1$ and $|n| \leq 1$ for all $n \in \mathbb{N}$. A trivial example of a non-Archimedean valuation is the map $|\cdot|$ taking everything but 0 into 1 and $|0| = 0$ (see [23]).

Let $\mathcal{X}$ be a vector space over a valued field $\mathcal{K}$. A *non-Archimedean norm* is a function $\|\cdot\| : \mathcal{X} \to [0, \infty)$ such that for all $r \in \mathcal{K}$ and $x, y \in \mathcal{X}$,

   (i) $\|x\| = 0$ if and only if $x = 0$;

  (ii) $\|rx\| = |r|\|x\|$;

 (iii) the strong triangle inequality

$$\|x + y\| \leq \max\{\|x\|, \|y\|\}.$$

Then we say $(\mathcal{X}, \|\cdot\|)$ is a *non-Archimedean space*.

Definition 2.1 Let $\mathcal{X}$ be a vector space with the dimension greater than $n - 1$ over a valued field $\mathcal{K}$ with a non-Archimedean valuation $|\cdot|$. A function $\|\cdot, \ldots, \cdot\| : \mathcal{X} \times \cdots \times \mathcal{X} \to [0, \infty)$ is said to be a *non-Archimedean n-norm* if

   (i) $\|x_1, \ldots, x_n\| = 0 \Leftrightarrow x_1, \ldots, x_n$ are linearly dependent;

  (ii) $\|x_1, \ldots, x_n\| = \|x_{j_1}, \ldots, x_{j_n}\|$ for every permutation $(j_1, \ldots, j_n)$ of $(1, \ldots, n)$;

 (iii) $\|\alpha x_1, \ldots, x_n\| = |\alpha|\|x_1, \ldots, x_n\|$;

 (iv) $\|x + y, x_2, \ldots, x_n\| \leq \max\{\|x, x_2, \ldots, x_n\|, \|y, x_2, \ldots, x_n\|\}$

for all $\alpha \in \mathcal{K}$ and all $x, y, x_1, \ldots, x_n \in \mathcal{X}$. Then $(\mathcal{X}, \|\cdot, \ldots, \cdot\|)$ is called a *non-Archimedean n-normed space*.

From now on, let $\mathcal{X}$ and $\mathcal{Y}$ be non-Archimedean n-normed spaces over a linear ordered non-Archimedean field $\mathcal{K}$.

Definition 2.2 Let $\mathcal{X}$ and $\mathcal{Y}$ be non-Archimedean n-normed spaces and $f : \mathcal{X} \to \mathcal{Y}$ be a mapping. We call f an *n-isometry* if

$$\|x_1 - x_0, \ldots, x_n - x_0\| = \|f(x_1) - f(x_0), \ldots, f(x_n) - f(x_0)\|$$

for all $x_0, x_1, \ldots, x_n \in \mathcal{X}$.

Definition 2.3 The points $x_0, x_1, \ldots, x_n$ of a non-Archimedean n-normed space $\mathcal{X}$ are said to be *n-collinear* if for every i, $\{x_j - x_i \mid 0 \le j \ne i \le n\}$ is linearly dependent.

The points x_0, x_1 and x_2 of a non-Archimedean n-normed space $\mathcal{X}$ are said to be 2-collinear if and only if $x_2 - x_0 = t(x_1 - x_0)$ for some element t of the non-Archimedean field $\mathcal{K}$. We denote the set of all elements of $\mathcal{K}$ whose valuations are 1 by $\mathcal{C}$, that is, $\mathcal{C} = \{\gamma \in \mathcal{K} : |\gamma| = 1\}$.

Lemma 2.4 *Let x_i be an element of a non-Archimedean n-normed space $\mathcal{X}$ for every $i \in \{1, \ldots, n\}$ and $\gamma \in \mathcal{K}$. Then*

$$\|x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n\| = \|x_1, \ldots, x_i, \ldots, x_j + \gamma x_i, \ldots, x_n\|$$

for all $1 \le i \ne j \le n$.

Proof By Definition 2.1, we have

$$\|x_1, \ldots, x_i, \ldots, x_j + \gamma x_i, \ldots, x_n\|$$
$$\le \max\{\|x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n\|, |\gamma| \|x_1, \ldots, x_i, \ldots, x_i, \ldots, x_n\|\}$$
$$= \max\{\|x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n\|, 0\}$$
$$= \|x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n\|.$$

One can easily prove the converse using similar methods. This completes the proof. □

Remark 2.5 Let $\mathcal{X}$, $\mathcal{Y}$ be non-Archimedean n-normed spaces over a linear ordered non-Archimedean field $\mathcal{K}$ and let $f : \mathcal{X} \to \mathcal{Y}$ be an n-isometry. One can show that the n-isometry f from $\mathcal{X}$ to $\mathcal{Y}$ preserves the 2-collinearity using a similar method to that in [22, Lemma 3.2].

The *midpoint* of a segment with endpoints x and y in the non-Archimedean n-normed space $\mathcal{X}$ is defined by the point $\frac{x+y}{2}$.

Now, we prove the Mazur-Ulam theorem on non-Archimedean n-normed spaces. In the first step, we prove the following lemma. And then, using the lemma, we show that an n-isometry f from a non-Archimedean n-normed space $\mathcal{X}$ to a non-Archimedean n-normed space $\mathcal{Y}$ preserves the midpoint of a segment, *i.e.*, the f-image of the midpoint of a segment in $\mathcal{X}$ is also the midpoint of a corresponding segment in $\mathcal{Y}$.

Lemma 2.6 *Let $\mathcal{X}$ be a non-Archimedean n-normed space over a linear ordered non-Archimedean field $\mathcal{K}$ with $\mathcal{C} = \{2^n | n \in \mathbb{Z}\}$ and let $x_0, x_1 \in \mathcal{X}$ with $x_0 \ne x_1$. Then $u := \frac{x_0 + x_1}{2}$ is*

the unique member of $\mathcal{X}$ satisfying

$$\|x_0 - x_1, x_0 - x_2, x_0 - x_3, \ldots, x_0 - x_n\|$$
$$= \|x_0 - u, x_0 - x_2, x_0 - x_3, \ldots, x_0 - x_n\|$$
$$= \|x_1 - u, x_1 - x_2, x_1 - x_3, \ldots, x_1 - x_n\|$$

for some $x_2, \ldots, x_n \in \mathcal{X}$ with $\|x_0 - x_1, x_0 - x_2, \ldots, x_0 - x_n\| \neq 0$ and u, x_0, x_1 are 2-collinear.

Proof Let $u := \frac{x_0 + x_1}{2}$. From the assumption for the dimension of $\mathcal{X}$, there exist $n - 1$ elements $x_2, \ldots, x_n$ in $\mathcal{X}$ such that $\|x_0 - x_1, x_0 - x_2, \ldots, x_0 - x_n\| \neq 0$. One can easily prove that u satisfies the above equations and conditions. It suffices to show the uniqueness for u. Assume that there is another v satisfying

$$\|x_0 - x_1, x_0 - x_2, x_0 - x_3, \ldots, x_0 - x_n\|$$
$$= \|x_0 - v, x_0 - x_2, x_0 - x_3, \ldots, x_0 - x_n\|$$
$$= \|x_1 - v, x_1 - x_2, x_1 - x_3, \ldots, x_1 - x_n\|$$

for some elements $x_2, \ldots, x_n$ of $\mathcal{X}$ with $\|x_0 - x_1, x_0 - x_2, \ldots, x_0 - x_n\| \neq 0$ and v, x_0, x_1 are 2-collinear. Since v, x_0, x_1 are 2-collinear, $v = tx_0 + (1 - t)x_1$ for some $t \in \mathcal{K}$. Then we have

$$\|x_0 - x_1, x_0 - x_2, x_0 - x_3, \ldots, x_0 - x_n\|$$
$$= \|x_0 - v, x_0 - x_2, x_0 - x_3, \ldots, x_0 - x_n\|$$
$$= \|x_0 - tx_0 - (1 - t)x_1, x_0 - x_2, x_0 - x_3, \ldots, x_0 - x_n\|$$
$$= |1 - t|\|x_0 - x_1, x_0 - x_2, x_0 - x_3, \ldots, x_0 - x_n\|,$$
$$\|x_0 - x_1, x_0 - x_2, x_0 - x_3, \ldots, x_0 - x_n\|$$
$$= \|x_1 - v, x_1 - x_2, x_1 - x_3, \ldots, x_1 - x_n\|$$
$$= \|x_1 - tx_0 - (1 - t)x_1, x_1 - x_2, x_1 - x_3, \ldots, x_1 - x_n\|$$
$$= |t|\|x_0 - x_1, x_1 - x_2, x_1 - x_3, \ldots, x_1 - x_n\|$$
$$= |t|\|x_0 - x_1, x_0 - x_2, x_0 - x_3, \ldots, x_0 - x_n\|.$$

Since $\|x_0 - x_1, x_0 - x_2, \ldots, x_0 - x_n\| \neq 0$, we have two equations $|1 - t| = 1$ and $|t| = 1$. So, there are two integers k_1, k_2 such that $1 - t = 2^{k_1}$, $t = 2^{k_2}$. Since $2^{k_1} + 2^{k_2} = 1$, $k_i < 0$ for all $i = 1, 2$. Thus, we may assume that $1 - t = 2^{-n_1}$, $t = 2^{-n_2}$ and $n_1 \geq n_2 \in \mathbb{N}$ without loss of generality. If $n_1 \gneq n_2$, then $1 = 2^{-n_1} + 2^{-n_2} = 2^{-n_1}(1 + 2^{n_1 - n_2})$, that is, $2^{n_1} = 1 + 2^{n_1 - n_2}$. This is a contradiction because the left-hand side of the equation is a multiple of 2 but the right-hand side of the equation is not. Thus, $n_1 = n_2 = 1$ and hence $v = \frac{1}{2}x_0 + \frac{1}{2}x_1 = u$. □

Theorem 2.7 *Let $\mathcal{X}$, $\mathcal{Y}$ be non-Archimedean n-normed spaces over a linear ordered non-Archimedean field $\mathcal{K}$ with $\mathcal{C} = \{2^n | n \in \mathbb{Z}\}$ and $f : \mathcal{X} \to \mathcal{Y}$ be an n-isometry. Then the midpoint of a segment is f-invariant, i.e., for every $x_0, x_1 \in \mathcal{X}$ with $x_0 \neq x_1$, $f(\frac{x_0 + x_1}{2})$ is also the midpoint of a segment with endpoints $f(x_0)$ and $f(x_1)$ in $\mathcal{Y}$.*

Proof Let $x_0, x_1 \in \mathcal{X}$ with $x_0 \neq x_1$. Since the dimension of $\mathcal{X}$ is greater than $n - 1$, there exist $n - 1$ elements $x_2, \ldots, x_n$ of $\mathcal{X}$ satisfying $\|x_0 - x_1, x_0 - x_2, \ldots, x_0 - x_n\| \neq 0$. Since x_0, x_1 and their midpoint $\frac{x_0 + x_1}{2}$ are 2-collinear in $\mathcal{X}, f(x_0), f(x_1), f(\frac{x_0 + x_1}{2})$ are also 2-collinear in $\mathcal{Y}$ by Remark 2.5. Since f is an n-isometry, we have the following:

$$\left\| f(x_0) - f\left(\frac{x_0 + x_1}{2}\right), f(x_0) - f(x_2), \ldots, f(x_0) - f(x_n) \right\|$$

$$= \left\| x_0 - \frac{x_0 + x_1}{2}, x_0 - x_2, \ldots, x_0 - x_n \right\|$$

$$= \left| \frac{1}{2} \right| \|x_0 - x_1, x_0 - x_2, \ldots, x_0 - x_n\|$$

$$= \left\| f(x_0) - f(x_1), f(x_0) - f(x_2), \ldots, f(x_0) - f(x_n) \right\|,$$

$$\left\| f(x_1) - f\left(\frac{x_0 + x_1}{2}\right), f(x_1) - f(x_2), \ldots, f(x_1) - f(x_n) \right\|$$

$$= \left\| x_1 - \frac{x_0 + x_1}{2}, x_1 - x_2, \ldots, x_1 - x_n \right\|$$

$$= \left| \frac{1}{2} \right| \|x_1 - x_0, x_1 - x_2, \ldots, x_1 - x_n\|$$

$$= \left\| f(x_0) - f(x_1), f(x_0) - f(x_2), \ldots, f(x_0) - f(x_n) \right\|.$$

By Lemma 2.6, we obtain that $f(\frac{x_0 + x_1}{2}) = \frac{f(x_0) + f(x_1)}{2}$ for all $x_0, x_1 \in \mathcal{X}$ with $x_0 \neq x_1$. This completes the proof. □

Lemma 2.8 *Let $\mathcal{X}$ and $\mathcal{Y}$ be non-Archimedean n-normed spaces over a linear ordered non-Archimedean field $\mathcal{K}$ and $f : \mathcal{X} \to \mathcal{Y}$ be an n-isometry. Then the following conditions are equivalent.*

(i) The n-isometry f preserves the midpoint of a segment in $\mathcal{X}$, i.e., $f(\frac{x_0 + x_1}{2}) = \frac{f(x_0) + f(x_1)}{2}$ for all $x_0, x_1 \in \mathcal{X}$ with $x_0 \neq x_1$;

(ii) The n-isometry f preserves the barycenter of a triangle in $\mathcal{X}$, i.e., $f(\frac{x_0 + x_1 + x_2}{3}) = \frac{f(x_0) + f(x_1) + f(x_2)}{3}$ for all $x_0, x_1, x_2 \in \mathcal{X}$ satisfying that x_0, x_1, x_2 are not 2-collinear.

Proof Assume that the n-isometry f preserves the barycenter of a triangle in $\mathcal{X}$. Let x_0, x_1 be in $\mathcal{X}$ with $x_0 \neq x_1$. Since the n-isometry f preserves the 2-collinearity, $f(x_0), f(\frac{x_0 + x_1}{2}), f(x_1)$ are 2-collinear. So,

$$f\left(\frac{x_0 + x_1}{2}\right) - f(x_0) = s\left(f(x_1) - f(x_0)\right) \tag{2.1}$$

for some element s of $\mathcal{K}$. By the hypothesis for the dimension of $\mathcal{X}$, we can choose the element x_2 of $\mathcal{X}$ satisfying that x_0, x_1 and x_2 are not 2-collinear. Since $x_2, \frac{x_0 + x_1 + x_2}{3}, \frac{x_0 + x_1}{2}$ are 2-collinear, we have that $f(x_2), f(\frac{x_0 + x_1 + x_2}{3}), f(\frac{x_0 + x_1}{2})$ are also 2-collinear by Remark 2.5. So, we obtain that

$$f\left(\frac{x_0 + x_1 + x_2}{3}\right) - f(x_2) = t\left(f\left(\frac{x_0 + x_1}{2}\right) - f(x_2)\right) \tag{2.2}$$

for some element t of the non-Archimedean field $\mathcal{K}$. By the equations (2.1), (2.2) and the barycenter-preserving property for the n-isometry f, we have

$$\frac{f(x_0) + f(x_1) + f(x_2)}{3} - f(x_2) = t\big(f(x_0) + sf(x_1) - sf(x_0) - f(x_2)\big).$$

Thus, we get

$$\frac{f(x_0) + f(x_1) - 2f(x_2)}{3} = t(1 - s)f(x_0) + tsf(x_1) - tf(x_2).$$

So, we have the following equation:

$$\frac{2}{3}\big(f(x_0) - f(x_2)\big) - \frac{1}{3}\big(f(x_0) - f(x_1)\big) = t\big(f(x_0) - f(x_2)\big) - ts\big(f(x_0) - f(x_1)\big).$$

By a calculation, we obtain

$$\left(\frac{2}{3} - t\right)\big(f(x_0) - f(x_2)\big) + \left(-\frac{1}{3} + ts\right)\big(f(x_0) - f(x_1)\big) = 0. \tag{2.3}$$

Since x_0, x_1, x_2 are not 2-collinear, $x_0 - x_1, x_0 - x_2$ are linearly independent. Since $\dim \mathcal{X} \geq n$, there are $x_3, \ldots, x_n \in \mathcal{X}$ such that $\|x_0 - x_1, x_0 - x_2, x_0 - x_3, \ldots, x_0 - x_n\| \neq 0$. Since f is an n-isometry,

$$\big\|f(x_0) - f(x_1), f(x_0) - f(x_2), f(x_0) - f(x_3), \ldots, f(x_0) - f(x_n)\big\|$$

$$= \|x_0 - x_1, x_0 - x_2, x_0 - x_3, \ldots, x_0 - x_n\| \neq 0.$$

So, $f(x_0) - f(x_1)$ and $f(x_0) - f(x_2)$ are linearly independent. Hence, from equation (2.3), we have $\frac{2}{3} - t = 0$ and $-\frac{1}{3} + ts = 0$, *i.e.*, we obtain $t = \frac{2}{3}$, $s = \frac{1}{2}$, which imply the equation

$$f\left(\frac{x_0 + x_1}{2}\right) = \frac{f(x_0) + f(x_1)}{2}$$

for all $x_0, x_1 \in \mathcal{X}$ with $x_0 \neq x_1$.

Conversely, (i) trivially implies (ii). This completes the proof of this lemma. □

Remark 2.9 One can prove that the above lemma also holds in the case of linear n-normed spaces.

Theorem 2.10 *Let $\mathcal{X}$ and $\mathcal{Y}$ be non-Archimedean n-normed spaces over a linear ordered non-Archimedean field $\mathcal{K}$ with $\mathcal{C} = \{2^n | n \in \mathbb{Z}\}$. If $f : \mathcal{X} \to \mathcal{Y}$ is an n-isometry, then $f(x) - f(0)$ is additive.*

Proof Let $g(x) := f(x) - f(0)$. Then it is clear that $g(0) = 0$ and g is also an n-isometry.

From Theorem 2.7, for $x_0, x_1 \in \mathcal{X}$ ($x_0 \neq x_1$), we have

$$g\left(\frac{x_0 + x_1}{2}\right) = \frac{g(x_0) + g(x_1)}{2}.$$

Since $g(0) = 0$, we obtain that g is additive which completes the proof. □

3 The Mazur-Ulam theorem II in non-Archimedean n-normed spaces

In this section, under different conditions from those previously referred in Theorem 2.10, we also prove the (second type) Mazur-Ulam theorem on a non-Archimedean n-normed space. Firstly, we show that an n-isometry f from a non-Archimedean n-normed space $\mathcal{X}$ to a non-Archimedean n-normed space $\mathcal{Y}$ preserves the barycenter of a triangle, *i.e.*, the f-image of the barycenter of a triangle is also the barycenter of a corresponding triangle. Then, using Lemma 2.8, we also prove the Mazur-Ulam theorem (a non-Archimedean n-normed space version) under some different conditions.

Lemma 3.1 *Let $\mathcal{X}$ be a non-Archimedean n-normed space over a linear ordered non-Archimedean field $\mathcal{K}$ with $\mathcal{C} = \{3^n | n \in \mathbb{Z}\}$ and let x_0, x_1, x_2 be elements of $\mathcal{X}$ such that x_0, x_1, x_2 are not 2-collinear. Then $u := \frac{x_0 + x_1 + x_2}{3}$ is the unique member of $\mathcal{X}$ satisfying*

$$\|x_0 - x_1, x_0 - x_2, x_0 - x_3, \ldots, x_0 - x_n\|$$
$$= \|x_0 - x_1, x_0 - u, x_0 - x_3, \ldots, x_0 - x_n\|$$
$$= \|x_1 - x_2, x_1 - u, x_1 - x_3, \ldots, x_1 - x_n\|$$
$$= \|x_2 - x_0, x_2 - u, x_2 - x_3, \ldots, x_2 - x_n\|$$

for some $x_3, \ldots, x_n \in \mathcal{X}$ with $\|x_0 - x_1, x_0 - x_2, x_0 - x_3, \ldots, x_0 - x_n\| \neq 0$ and u is an interior point of $\triangle_{x_0 x_1 x_2}$.

Proof Let $u := \frac{x_0 + x_1 + x_2}{3}$. Thus u is an interior point of $\triangle_{x_0 x_1 x_2}$. Since $\dim \mathcal{X} > n - 1$, there are $n - 2$ elements $x_3, \ldots, x_n$ of $\mathcal{X}$ such that $\|x_0 - x_1, x_0 - x_2, x_0 - x_3, \ldots, x_0 - x_n\| \neq 0$. Applying Lemma 2.4, we have that

$$\|x_0 - x_1, x_0 - u, x_0 - x_3, \ldots, x_0 - x_n\|$$
$$= \left\| x_0 - x_1, x_0 - \frac{x_0 + x_1 + x_2}{3}, x_0 - x_3, \ldots, x_0 - x_n \right\|$$
$$= \left| \frac{1}{3} \right| \|x_0 - x_1, x_0 - x_1 + x_0 - x_2, x_0 - x_3, \ldots, x_0 - x_n\|$$
$$= \|x_0 - x_1, x_0 - x_2, x_0 - x_3, \ldots, x_0 - x_n\|.$$

And we can also obtain that

$$\|x_0 - x_1, x_0 - x_2, x_0 - x_3, \ldots, x_0 - x_n\|$$
$$= \|x_1 - x_2, x_1 - u, x_1 - x_3, \ldots, x_1 - x_n\|$$
$$= \|x_2 - x_0, x_2 - u, x_2 - x_3, \ldots, x_2 - x_n\|.$$

For the proof of uniqueness, let v be another interior point of $\triangle_{x_0 x_1 x_2}$ satisfying

$$\|x_0 - x_1, x_0 - x_2, x_0 - x_3, \ldots, x_0 - x_n\|$$
$$= \|x_0 - x_1, x_0 - v, x_0 - x_3, \ldots, x_0 - x_n\|$$

$$= \|x_1 - x_2, x_1 - v, x_1 - x_3, \ldots, x_1 - x_n\|$$

$$= \|x_2 - x_0, x_2 - v, x_2 - x_3, \ldots, x_2 - x_n\|$$

with $\|x_0 - x_1, x_0 - x_2, x_0 - x_3, \ldots, x_0 - x_n\| \neq 0$. Since v is an element of the set $\{t_0 x_0 + t_1 x_1 + t_2 x_2 | t_0 + t_1 + t_2 = 1, t_i \in \mathcal{K}, t_i > 0 \text{ for all } i\}$, there are elements s_0, s_1, s_2 of $\mathcal{K}$ with $s_0 + s_1 + s_2 = 1$, $s_i > 0$ such that $v = s_0 x_0 + s_1 x_1 + s_2 x_2$. Then we have

$$\|x_0 - x_1, x_0 - x_2, x_0 - x_3, \ldots, x_0 - x_n\|$$

$$= \|x_0 - x_1, x_0 - v, x_0 - x_3, \ldots, x_0 - x_n\|$$

$$= \|x_0 - x_1, x_0 - s_0 x_0 - s_1 x_1 - s_2 x_2, x_0 - x_3, \ldots, x_0 - x_n\|$$

$$= \|x_0 - x_1, (s_0 - 1)x_0 + s_1 x_1 + (1 - s_0 - s_1)x_2, x_0 - x_3, \ldots, x_0 - x_n\|$$

$$= \|x_0 - x_1, (s_0 + s_1 - 1)x_0 + (1 - s_0 - s_1)x_2, x_0 - x_3, \ldots, x_0 - x_n\|$$

$$= |s_0 + s_1 - 1| \|x_0 - x_1, x_0 - x_2, x_0 - x_3, \ldots, x_0 - x_n\|$$

$$= |s_2| \|x_0 - x_1, x_0 - x_2, x_0 - x_3, \ldots, x_0 - x_n\|$$

and hence $|s_2| = 1$ since $\|x_0 - x_1, x_0 - x_2, x_0 - x_3, \ldots, x_0 - x_n\| \neq 0$. Similarly, we obtain $|s_0| = |s_1| = 1$. By the hypothesis of $\mathcal{C}$, there are integers k_0, k_1, k_2 such that $s_0 = 3^{k_0}$, $s_1 = 3^{k_2}$, $s_2 = 3^{k_2}$. Since $s_0 + s_1 + s_2 = 1$, every k_i is less than 0. So, one may let $s_0 = 3^{-n_0}$, $s_1 = 3^{-n_1}$, $s_2 = 3^{-n_2}$ and $n_0 \geq n_1 \geq n_2 \in \mathbb{N}$. Assume that one of the above inequalities holds. Then $1 = s_0 + s_1 + s_2 = 3^{-n_0}(1 + 3^{n_0 - n_1} + 3^{n_0 - n_2})$, *i.e.*, $3^{n_0} = 1 + 3^{n_0 - n_1} + 3^{n_0 - n_2}$. This is a contradiction, because the left-hand side is a multiple of 3 whereas the right-hand side is not. Thus, $n_0 = n_1 = n_2$. Consequently, $s_0 = s_1 = s_2 = \frac{1}{3}$. This means that u is unique. □

Theorem 3.2 *Let $\mathcal{X}$, $\mathcal{Y}$ be non-Archimedean n-normed spaces over a linear ordered non-Archimedean field $\mathcal{K}$ with $\mathcal{C} = \{3^n | n \in \mathbb{Z}\}$ and $f : \mathcal{X} \to \mathcal{Y}$ be an interior-preserving n-isometry. Then the barycenter of a triangle is f-invariant.*

Proof Let x_0, x_1 and x_2 be elements of $\mathcal{X}$ satisfying that x_0, x_1 and x_2 are not 2-collinear. It is obvious that the barycenter $\frac{x_0 + x_1 + x_2}{3}$ of a triangle $\triangle_{x_0 x_1 x_2}$ is an interior point of the triangle. By the assumption, $f(\frac{x_0 + x_1 + x_2}{3})$ is also the interior point of a triangle $\triangle_{f(x_0)f(x_1)f(x_2)}$. Since $\dim \mathcal{X} > n - 1$, there exist $n - 2$ elements $x_3, \ldots, x_n$ in $\mathcal{X}$ such that $\|x_0 - x_1, x_0 - x_2, x_0 - x_3, \ldots, x_0 - x_n\|$ is not zero. Since f is an n-isometry, we have

$$\left\| f(x_0) - f(x_1), f(x_0) - f\left(\frac{x_0 + x_1 + x_2}{3}\right), f(x_0) - f(x_3), \ldots, f(x_0) - f(x_n) \right\|$$

$$= \left\| x_0 - x_1, x_0 - \frac{x_0 + x_1 + x_2}{3}, x_0 - x_3, \ldots, x_0 - x_n \right\|$$

$$= \left| \frac{1}{3} \right| \|x_0 - x_1, x_0 - x_1 + x_0 - x_2, x_0 - x_3, \ldots, x_0 - x_n\|$$

$$= \|x_0 - x_1, x_0 - x_2, x_0 - x_3, \ldots, x_0 - x_n\|$$

$$= \left\| f(x_0) - f(x_1), f(x_0) - f(x_2), f(x_0) - f(x_3), \ldots, f(x_0) - f(x_n) \right\|.$$

Similarly, we obtain

$$\left\| f(x_1) - f(x_2), f(x_1) - f\left(\frac{x_0 + x_1 + x_2}{3}\right), f(x_1) - f(x_3), \ldots, f(x_1) - f(x_n) \right\|$$

$$= \left\| f(x_2) - f(x_1), f(x_2) - f\left(\frac{x_0 + x_1 + x_2}{3}\right), f(x_2) - f(x_3), \ldots, f(x_2) - f(x_n) \right\|$$

$$= \left\| f(x_0) - f(x_1), f(x_0) - f(x_2), f(x_0) - f(x_3), \ldots, f(x_0) - f(x_n) \right\|.$$

From Lemma 3.1, we get

$$f\left(\frac{x_0 + x_1 + x_2}{3}\right) = \frac{f(x_0) + f(x_1) + f(x_2)}{3}$$

for all $x_0, x_1, x_2 \in \mathcal{X}$ satisfying that x_0, x_1, x_2 are not 2-collinear. □

The next theorem is the Mazur-Ulam theorem II in non-Archimedean n-normed spaces over a linear ordered non-Archimedean field $\mathcal{K}$. The assumptions of this theorem are different from that of Theorem 2.10. In particular, in the proof of the theorem, we use the f-preserving property for the barycenter of a triangle.

Theorem 3.3 *Let $\mathcal{X}$ and $\mathcal{Y}$ be non-Archimedean n-normed spaces over a linear ordered non-Archimedean field $\mathcal{K}$ with $\mathcal{C} = \{3^n | n \in \mathbb{Z}\}$. If $f : \mathcal{X} \to \mathcal{Y}$ is an interior-preserving n-isometry, then $f(x) - f(0)$ is additive.*

Proof Let $g(x) := f(x) - f(0)$. One can easily check that $g(0) = 0$ and g is also an n-isometry. Using a similar method in [17, Theorem 2.4], we can easily prove that g is also an interior-preserving mapping.

Now, let x_0, x_1, x_2 be elements of $\mathcal{X}$ satisfying that x_0, x_1, x_2 are not 2-collinear. Since g is an interior-preserving n-isometry, by Theorem 3.2,

$$g\left(\frac{x_0 + x_1 + x_2}{3}\right) = \frac{g(x_0) + g(x_1) + g(x_2)}{3}$$

for any $x_0, x_1, x_2 \in \mathcal{X}$ satisfying that x_0, x_1, x_2 are not 2-collinear. Using Lemma 2.8 and the property $g(0) = 0$, we obtain that the interior-preserving n-isometry g is additive, which completes the proof. □

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