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Coefficient estimates for new subclasses of Ma-Minda bi-univalent functions

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Dedicated to Professor Hari M Srivastava

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Abstract
In this paper, we introduce and investigate two new subclasses $H_{\sigma}^{\mu}(\lambda,\varphi)$ and $M_{\sigma}^{\gamma}(\lambda,\mu,\varphi)$ of Ma-Minda bi-univalent functions defined by using subordination in the open unit disk $D=\{z\in C:|z|<1\}$. For functions belonging to these new subclasses, we obtain estimates for the initial coefficients $|a_2|$ and $|a_3|$. The results presented in this paper would generalize those in related works of several earlier authors.
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1 Introduction

Let C be a set of complex numbers and let $N=\{1,2,3,\dots\}=N_0\setminus\{0\}$ be a set of positive integers. Let A be a class of functions of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n, \tag{1.1}$$

which are analytic in the open unit disk $D=\{z\in C:|z|<1\}$. Also, let S denote a subclass of all functions in A which are univalent in D (for details, see [1, 2]).

Since univalent functions are one-to-one, they are invertible and the inverse functions need not be defined on the entire unit disk D. However, the famous Koebe one-quarter theorem [1] ensures that the image of the unit disk D under every function $f\in S$ contains a disk of radius 1/4. Thus, every univalent function $f\in S$ has an inverse f^{-1} satisfying

$$f^{-1}\bigl(f(z)\bigr)=z\quad(z\in D)$$

and

$$f\bigl(f^{-1}(w)\bigr)=w\quad\left(|w|<r_0(f);r_0(f)\geq\frac{1}{4}\right),$$

where

$$f^{-1}(w)=w-a_2w^2+\bigl(2a_2^2-a_3\bigr)w^3-\bigl(5a_2^3-5a_2a_3+a_4\bigr)w^4+\cdots. \tag{1.2}$$

A function $f \in A$ is said to be bi-univalent in D if both f and f^{-1} are univalent in D. Let σ denote the class of bi-univalent functions defined in the unit disk D. In 1967, Lewin [3] first introduced the class σ of bi-univalent functions and showed that $|a_2| \leq 1.51$ for every $f \in \sigma$. Subsequently, Branan and Clunie [4] conjectured that $|a_2| \leq \sqrt{2}$ for $f \in \sigma$. Later, Netanyahu [5] proved that $\max_{f\in\sigma} |a_2| = 3/4$. The coefficient estimate problem for each of $|a_n|$ ($n \in N \setminus \{1,2\}$) is still an open problem.

Brannan and Taha [6] (see also [7]) introduced certain subclasses of a bi-univalent function class σ similar to the familiar subclasses $S^*(\alpha)$ and $K(\alpha)$ of starlike and convex functions of order α ($0 < \alpha \leq 1$), respectively (see [8]). Thus, following Brannan and Taha [6] (see also [7]), a function $f \in A$ is in the class $S^*_\sigma[\alpha]$ of strongly bi-starlike functions of order α ($0 < \alpha \leq 1$) if both functions f and f^{-1} are strongly starlike functions of order α. The classes $S^*_\sigma(\alpha)$ and $K_\sigma(\alpha)$ of bi-starlike functions of order α and bi-convex functions of order α, corresponding (respectively) to the function classes $S^*(\alpha)$ and $K(\alpha)$, were also introduced analogously. For each of the function classes $S^*_\sigma(\alpha)$ and $K_\sigma(\alpha)$, they found non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ (for details, see [6, 7]).

An analytic function f is subordinate to an analytic function g, written $f \prec g$, if there is an analytic function w with $|w(z)| \leq |z|$ such that $f = (g(w))$. If g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(D) \subseteq g(D)$. Ma and Minda [9] unified various subclasses of starlike and convex functions for which either of the quantities $zf'(z)/f(z)$ or $1 + zf''(z)/f'(z)$ is subordinate to a more general superordinate function. For this purpose, they considered an analytic function φ with positive real part in the unit disk D, $\varphi(0) = 1$, $\varphi'(0) > 0$, and φ maps D onto a region starlike with respect to 1 and symmetric with respect to the real axis. The classes $S^*(\varphi)$ and $K(\varphi)$ of Ma-Minda starlike and Ma-Minda convex functions are respectively characterized by $zf'(z)/f(z) \prec \varphi(z)$ or $1 + zf''(z)/f'(z) \prec \varphi(z)$. A function f is bi-starlike of Ma-Minda type or bi-convex of Ma-Minda type if both f and f^{-1} are respectively Ma-Minda starlike or convex. These classes are denoted respectively by $S^*_\sigma(\varphi)$ and $K_\sigma(\varphi)$. Recently, Srivastava *et al.* [10], Frasin and Aouf [11] and Caglar *et al.* [12] introduced and investigated various subclasses of bi-univalent functions and found estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in these classes. Very recently, Ali *et al.* [13], Kumar *et al.* [14], Srivastava *et al.* [15] and Xu *et al.*[16] unified and extended some related results in [7, 10–12, 17] by generalizing their classes using subordination.

Motivated by Ali *et al.* [13] and Kumar *et al.* [14], we investigate the estimates for the initial coefficients $|a_2|$ and $|a_3|$ of bi-univalent functions of Ma-Minda type belonging to the classes $H^\mu_\sigma(\lambda,\varphi)$ and $M^\gamma_\sigma(\lambda,\mu,\varphi)$ defined in Section 2. Our results generalize several well-known results in [10–14] and these are also pointed out.

2 Coefficient estimates

Throughout this paper, we assume that φ is an analytic univalent function with positive real part in D, $\varphi(D)$ is symmetric with respect to the real axis and starlike with respect to $\varphi(0) = 1$, and $\varphi'(0) > 0$. Such a function has series expansion of the form

$$\varphi(z) = 1 + B_1 z + B_2 z_2 + B_3 z_3 + \cdots \quad (B_1 > 0). \tag{2.1}$$

With this assumption on φ, we now introduce the following subclasses of Ma-Minda bi-univalent functions.

Definition 2.1 A function $f \in \sigma$ given by (1.1) is said to be in the class $H_\sigma^\mu(\lambda,\varphi)$ if it satisfies

$$(1-\lambda)\left(\frac{f(z)}{z}\right)^\mu + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1} \prec \varphi(z) \quad (\lambda \geq 1, \mu \geq 1, z \in D) \tag{2.2}$$

and

$$(1-\lambda)\left(\frac{g(w)}{w}\right)^\mu + \lambda g'(w)\left(\frac{g(w)}{z}\right)^{\mu-1} \prec \varphi(w) \quad (\lambda \geq 1, \mu \geq 1, w \in D), \tag{2.3}$$

where the function g is given by

$$g(w) = f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right)w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right)w^4 + \cdots. \tag{2.4}$$

We note that, for suitable choices λ, μ and φ, the class $H_\sigma^\mu(\lambda,\varphi)$ reduces to the following known classes.

(1) $H_\sigma^\mu(\lambda, (\frac{1+z}{1-z})^\alpha) = H_\sigma^\mu(\lambda,\alpha)$ $(\lambda \geq 1,\ 0 < \alpha \leq 1,\ \mu \geq 0)$ (see Caglar *et al.* [12, Definition 2.1]);
(2) $H_\sigma^\mu(\lambda, \frac{1+(1-2\beta)z}{1-z}) = H_\sigma^\mu(\lambda,\beta)$ $(\lambda \geq 1,\ 0 \leq \beta < 1,\ \mu \geq 0)$ (see Caglar *et al.* [12, Definition 3.1]);
(3) $H_\sigma^1(\lambda,\varphi) = H_\sigma(\lambda,\varphi)$ $(\lambda \geq 1)$ (see Kumar *et al.* [14, Definition 1.1]);
(4) $H_\sigma^\mu(1,\varphi) = H_\sigma^\mu(\varphi)$ $(\mu \geq 0)$ (see Kumar *et al.* [14, Definition 2.1]);
(5) $H_\sigma^1(1,\varphi) = H_\sigma(\varphi)$ (see Ali *et al.* [13, p.345]);
(6) $H_\sigma^1(\lambda, (\frac{1+z}{1-z})^\alpha) = H_\sigma(\lambda,\alpha)$ $(\lambda \geq 1,\ 0 < \alpha \leq 1)$ (see Frasin and Aouf [11, Definition 2.1]);
(7) $H_\sigma^1(\lambda, \frac{1+(1-2\beta)z}{1-z}) = H_\sigma(\lambda,\beta)$ $(\lambda \geq 1,\ 0 \leq \beta < 1)$ (see Frasin and Aouf [11, Definition 3.1]);
(8) $H_\sigma^1(1, (\frac{1+z}{1-z})^\alpha) = H_\sigma(\alpha)$ $(0 < \alpha \leq 1)$ (see Srivastava *et al.* [10, Definition 1]);
(9) $H_\sigma^1(1, \frac{1+(1-2\beta)z}{1-z}) = H_\sigma(\beta)$ $(0 \leq \beta < 1)$ (see Srivastava *et al.* [10, Definition 2]).

For functions in the class $H_\sigma^\mu(\lambda,\varphi)$, the following estimates are obtained.

Theorem 2.1 *Let the function f given by* (1.1) *be in the class $H_\sigma^\mu(\lambda,\varphi)$, $\lambda \geq 1$ and $\mu \geq 0$. Then*

$$|a_2| \leq \min\left\{\frac{B_1}{\lambda+\mu}, \sqrt{\frac{2(B_1 + |B_2 - B_1|)}{(1+\mu)(2\lambda+\mu)}}\right\} \tag{2.5}$$

and

$$|a_3| \leq \begin{cases} \min\{\frac{B_1}{2\lambda+\mu} + \frac{B_1^2}{(\lambda+\mu)^2}, \frac{2(B_1+|B_2-B_1|)}{(1+\mu)(2\lambda+\mu)}\}, & 0 \leq \mu < 1, \\ \frac{B_1}{2\lambda+\mu} + \frac{2|B_2-B_1|}{(1+\mu)(2\lambda+\mu)}, & \mu \geq 1. \end{cases} \tag{2.6}$$

Proof Since $f \in H_\sigma^\mu(\lambda,\varphi)$, there exist two analytic functions $u, v : D \to D$, with $u(0) = v(0) = 0$, such that

$$(1-\lambda)\left(\frac{f(z)}{z}\right)^\mu + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1} = \varphi\left(u(z)\right) \tag{2.7}$$

and

$$(1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu}+\lambda g'(w)\left(\frac{g(w)}{z}\right)^{\mu-1}=\varphi\bigl(v(w)\bigr). \tag{2.8}$$

Define the functions p and q by

$$\begin{aligned} p(z)&=\frac{1+u(z)}{1-u(z)}=1+p_1z+p_2z^2+\cdots \quad \text{and}\\ q(z)&=\frac{1+v(z)}{1-v(z)}=1+q_1z+q_2z^2+\cdots, \end{aligned} \tag{2.9}$$

or, equivalently,

$$u(z)=\frac{p(z)-1}{p(z)+1}=\frac{1}{2}\left(p_1z+\left(p_2-\frac{p_1^2}{2}\right)z^2+\cdots\right) \tag{2.10}$$

and

$$v(z)=\frac{q(z)-1}{q(z)+1}=\frac{1}{2}\left(q_1z+\left(q_2-\frac{q_1^2}{2}\right)z^2+\cdots\right). \tag{2.11}$$

It is clear that p and q are analytic in D and $p(0)=q(0)=1$. Since $u,v:D\to D$, the functions p and q have positive real part in D, and hence $|p_i|\le 2$ and $|q_i|\le 2$ $(i=1,2,\ldots)$. By virtue of (2.7), (2.8), (2.10) and (2.11), we have

$$(1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu}+\lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}=\varphi\left(\frac{p(z)-1}{p(z)+1}\right) \tag{2.12}$$

and

$$(1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu}+\lambda g'(w)\left(\frac{g(w)}{z}\right)^{\mu-1}=\varphi\left(\frac{q(w)-1}{q(w)+1}\right). \tag{2.13}$$

Using (2.10), (2.11), together with (2.1), we easily obtain

$$\varphi\left(\frac{p(z)-1}{p(z)+1}\right)=1+\frac{1}{2}B_1p_1z+\left(\frac{1}{2}B_1\left(p_2-\frac{1}{2}p_1^2\right)+\frac{1}{4}B_2p_1^2\right)z^2+\cdots \tag{2.14}$$

and

$$\varphi\left(\frac{q(w)-1}{q(w)+1}\right)=1+\frac{1}{2}B_1q_1w+\left(\frac{1}{2}B_1\left(q_2-\frac{1}{2}q_1^2\right)+\frac{1}{4}B_2q_1^2\right)w^2+\cdots. \tag{2.15}$$

Since $f\in\sigma$ has the Maclaurin series given by (1.1), a computation shows that its inverse $g=f^{-1}$ has the expansion given by (1.2). Also, since

$$\begin{aligned} f'(z)&=1+2a_2z+3a_3z^2+\cdots \quad \text{and}\\ g'(w)&=1-2a_2w+3(2a_2-a_3)w^2-\cdots, \end{aligned}$$

it follows from (2.12)-(2.15) that

$$(\lambda + \mu)a_2 = \frac{1}{2}B_1p_1, \tag{2.16}$$

$$(2\lambda + \mu)a_3 + \frac{(\mu - 1)(2\lambda + \mu)}{2}a_2^2 = \frac{1}{2}B_1\left(p_2 - \frac{1}{2}p_1^2\right) + \frac{1}{4}B_2p_1^2, \tag{2.17}$$

$$-(\lambda + \mu)a_2 = \frac{1}{2}B_1q_1 \tag{2.18}$$

and

$$-(2\lambda + \mu)a_3 + \frac{(3 + \mu)(2\lambda + \mu)}{2}a_2^2 = \frac{1}{2}B_1\left(q_2 - \frac{1}{2}q_1^2\right) + \frac{1}{4}B_2q_1^2. \tag{2.19}$$

From (2.16) and (2.18), we get

$$p_1 = -q_1 \tag{2.20}$$

and

$$8(\lambda + \mu)^2a_2^2 = B_1^2\left(p_1^2 + q_1^2\right). \tag{2.21}$$

Also, from (2.17) and (2.19), we obtain

$$(1 + \mu)(2\lambda + \mu)a_2^2 = \frac{1}{2}B_1(p_2 + q_2) + \frac{1}{4}(B_2 - B_1)\left(p_1^2 + q_1^2\right),$$

or

$$a_2^2 = \frac{2B_1(p_2 + q_2) + (B_2 - B_1)(p_1^2 + q_1^2)}{4(1 + \mu)(2\lambda + \mu)}. \tag{2.22}$$

Since $|p_i| \leq 2$ and $|q_i| \leq 2$ $(i = 1, 2)$, it follows from (2.21) and (2.22) that

$$|a_2| \leq \frac{B_1}{\lambda + \mu} \tag{2.23}$$

and

$$|a_2| \leq \sqrt{\frac{2(B_1 + |B_2 - B_1|)}{(1 + \mu)(2\lambda + \mu)}}, \tag{2.24}$$

which yields the desired estimate on $|a_2|$ as asserted in (2.5).

Next, in order to find the bound on $|a_3|$, by subtracting (2.19) from (2.17), we get

$$2(2\lambda + \mu)\left(a_3 - a_2^2\right) = \frac{1}{2}B_1(p_2 - q_2) + \frac{1}{4}(B_2 - B_1)\left(p_1^2 - q_1^2\right). \tag{2.25}$$

Using (2.20) and (2.21) in (2.25), we have

$$a_3 = \frac{1}{4(2\lambda + \mu)}B_1(p_2 - q_2) + \frac{1}{4(\lambda + \mu)^2}B_1^2p_1^2,$$

which evidently yields

$$|a_3| \leq \frac{B_1}{2\lambda + \mu} + \frac{B_1^2}{(\lambda + \mu)^2}. \tag{2.26}$$

On the other hand, by using (2.20) and (2.22) in (2.25), we obtain

$$a_3 = \frac{B_1[(\mu + 3)p_2 + (1 - \mu)q_2] + (B_2 - B_1)(p_1^2 + q_1^2)}{4(1 + \mu)(2\lambda + \mu)}, \tag{2.27}$$

and applying $|p_i| \leq 2$ and $|q_i| \leq 2$ $(i = 1, 2)$ for (2.27), we get

$$|a_3| \leq \frac{B_1}{2(2\lambda + \mu)}\left[\frac{\mu + 3}{1 + \mu} + \frac{|1 - \mu|}{1 + \mu}\right] + \frac{2|B_2 - B_1|}{(1 + \mu)(2\lambda + \mu)}. \tag{2.28}$$

Now, we consider the bounds on $|a_3|$ according to μ.

Case 1. If $0 \leq \mu < 1$, then from (2.28)

$$|a_3| \leq \frac{2(B_1 + |B_2 - B_1|)}{(1 + \mu)(2\lambda + \mu)}. \tag{2.29}$$

Case 2. If $\mu \geq 1$, then from (2.28)

$$|a_3| \leq \frac{B_1}{2\lambda + \mu} + \frac{2|B_2 - B_1|}{(1 + \mu)(2\lambda + \mu)}. \tag{2.30}$$

Thus, from (2.26), (2.29) and (2.30), we obtain the desired estimate on $|a_3|$ given in (2.6). This completes the proof of Theorem 2.1. □

Putting $\mu = 1$ and $\lambda = \mu = 1$ in Theorem 2.1, we respectively get the following Corollaries 2.1 and 2.2.

Corollary 2.1 *If $f \in \mathcal{H}_\sigma(\lambda, \varphi)$ $(\lambda \geq 1)$, then*

$$|a_2| \leq \min\left\{\frac{B_1}{\lambda + 1}, \sqrt{\frac{B_1 + |B_2 - B_1|}{2\lambda + 1}}\right\}$$

and

$$|a_3| \leq \begin{cases} \min\{\frac{B_1}{2\lambda+1} + \frac{B_1^2}{(\lambda+1)^2}, \frac{B_1+|B_2-B_1|}{2\lambda+1}\}, & 0 \leq \mu < 1, \\ \frac{B_1+|B_2-B_1|}{2\lambda+1}, & \mu \geq 1. \end{cases}$$

Corollary 2.2 *If $f \in \mathcal{H}_\sigma(\varphi)$, then*

$$|a_2| \leq \min\left\{\frac{B_1}{2}, \sqrt{\frac{B_1 + |B_2 - B_1|}{3}}\right\}$$

and

$$|a_3| \leq \begin{cases} \min\{\frac{B_1}{3} + \frac{B_1^2}{4}, \frac{B_1+|B_2-B_1|}{3}\}, & 0 \leq \mu < 1, \\ \frac{B_1+|B_2-B_1|}{3}, & \mu \geq 1. \end{cases}$$

Remark 2.1 The estimates of the coefficients $|a_2|$ and $|a_3|$ of Corollaries 2.1 and 2.2 are the improvement of the estimates obtained in [14, Theorem 2.1] and [13, Theorem 2.1], respectively.

Remark 2.2 If we set

$$\varphi(z) = \frac{1+(1-2\beta)z}{1-z} = 1 + 2(1-\beta)z + 2(1-\beta)z^2 + \cdots \quad (0 \le \beta < 1)$$

in Corollaries 2.1 and 2.2, the results obtained improve the results in [11, Theorem 3.2, inequalities (3.3) and (3.4)] and [10, Theorem 2, inequality (3.3)], respectively.

Definition 2.2 Let $\gamma \in C^* = C \setminus \{0\}$, $\lambda \ge 0$ and $\mu \ge 0$. A function $f \in \sigma$ given by (1.1) is said to be in the class $M_\sigma^\gamma(\lambda,\mu,\varphi)$, if the following subordinations hold:

$$1 + \frac{1}{\gamma}\left(\frac{zf'(z) + (2\lambda\mu+\lambda-\mu)z^2f''(z) + \lambda\mu z^3 f'''(z)}{(1-\lambda+\mu)f(z) + (\lambda-\mu)zf'(z) + \lambda\mu z^2 f''(z)} - 1\right) \prec \varphi(z)$$

and

$$1 + \frac{1}{\gamma}\left(\frac{wg'(w) + (2\lambda\mu+\lambda-\mu)w^2g''(w) + \lambda\mu w^3 g'''(w)}{(1-\lambda+\mu)g(w) + (\lambda-\mu)wg'(w) + \lambda\mu w^2 g''(w)} - 1\right) \prec \varphi(w),$$

where the function g is defined by (2.4).

We note that, by choosing appropriate values for λ, μ, γ and φ, the class $M_\sigma^\gamma(\lambda,\mu,\varphi)$ reduces to several earlier known classes.

(1) $M_\sigma^\gamma(\lambda,0,\varphi) = N_{\sigma,\gamma}^\lambda(\varphi)$ ($\lambda \ge 0$, $\gamma \in C^*$) (see Kumar *et al.* [14, Definition 2.2]);
(2) $M_\sigma^1(0,0,\frac{1+(1-2\beta)z}{1-z}) = S_\sigma^*(\beta)$ ($0 \le \beta < 1$) (see Brannan and Taha [6, Definition 3.1]);
(3) $M_\sigma^1(1,0,\frac{1+(1-2\beta)z}{1-z}) = K_\sigma(\beta)$ ($0 \le \beta < 1$) (see Brannan and Taha [6, Definition 4.1]);
(4) $M_\sigma^1(0,0,(\frac{1+z}{1-z})^\alpha) = S_\sigma^*(\alpha)$ ($0 < \alpha \le 1$) (see Taha [7]).

For functions in the class $M_\sigma^\gamma(\lambda,\mu,\varphi)$, the following estimates are derived.

Theorem 2.2 *Let* $\gamma \in C^*$, $\lambda \ge 0$ *and* $\mu \ge 0$. *If* $f \in M_\sigma^\gamma(\lambda,\mu,\varphi)$, *then*

$$|a_2| \le \frac{|\gamma|B_1\sqrt{B_1}}{\sqrt{|[2(6\lambda\mu+2\lambda-2\mu+1)-(2\lambda\mu+\lambda-\mu+1)^2]B_1^2\gamma + 2(2\lambda\mu+\lambda-\mu+1)^2(B_1-B_2)|}} \tag{2.31}$$

and

$$|a_3| \le \frac{|\gamma|(B_1+|B_2-B_1|)}{|2(6\lambda\mu+2\lambda-2\mu+1)-(2\lambda\mu+\lambda-\mu+1)^2|}. \tag{2.32}$$

Proof If $f \in M_\sigma^\gamma(\lambda,\mu,\varphi)$, then there are analytic functions $u, v : D \to D$, with $u(0) = v(0) = 0$, satisfying

$$1 + \frac{1}{\gamma}\left(\frac{zf'(z) + (2\lambda\mu+\lambda-\mu)z^2f''(z) + \lambda\mu z^3 f'''(z)}{(1-\lambda+\mu)f(z) + (\lambda-\mu)zf'(z) + \lambda\mu z^2 f''(z)} - 1\right) = \varphi\big(u(z)\big) \tag{2.33}$$

and

$$1+\frac{1}{\gamma}\left(\frac{wg'(w)+(2\lambda\mu+\lambda-\mu)w^2g''(w)+\lambda\mu w^3g'''(w)}{(1-\lambda+\mu)g(w)+(\lambda-\mu)wg'(w)+\lambda\mu w^2g''(w)}-1\right)=\varphi\bigl(\nu(w)\bigr). \tag{2.34}$$

Let p and q be defined as in (2.8), then it is clear from (2.33), (2.34), (2.9) and (2.10) that

$$\begin{aligned}&1+\frac{1}{\gamma}\left(\frac{zf'(z)+(2\lambda\mu+\lambda-\mu)z^2f''(z)+\lambda\mu z^3f'''(z)}{(1-\lambda+\mu)f(z)+(\lambda-\mu)zf'(z)+\lambda\mu z^2f''(z)}-1\right)\\&\quad=\varphi\left(\frac{p(z)-1}{p(z)+1}\right)\end{aligned} \tag{2.35}$$

and

$$\begin{aligned}&1+\frac{1}{\gamma}\left(\frac{wg'(w)+(2\lambda\mu+\lambda-\mu)w^2g''(w)+\lambda\mu w^3g'''(w)}{(1-\lambda+\mu)g(w)+(\lambda-\mu)wg'(w)+\lambda\mu w^2g''(w)}-1\right)\\&\quad=\varphi\left(\frac{q(w)-1}{q(w)+1}\right).\end{aligned} \tag{2.36}$$

It follows from (2.35), (2.36), (2.14) and (2.15) that

$$(2\lambda\mu+\lambda-\mu+1)a_2=\frac{1}{2}B_1p_1\gamma, \tag{2.37}$$

$$\begin{aligned}&-(2\lambda\mu+\lambda-\mu+1)^2a_2^2+2(6\lambda\mu+2\lambda-2\mu+1)a_3\\&\quad=\gamma\left[\frac{1}{2}B_1\left(p_2-\frac{1}{2}p_1^2\right)+\frac{1}{4}B_2p_1^2\right],\end{aligned} \tag{2.38}$$

$$-(2\lambda\mu+\lambda-\mu+1)a_2=\frac{1}{2}B_1q_1\gamma \tag{2.39}$$

and

$$\begin{aligned}&\bigl[4(6\lambda\mu+2\lambda-2\mu+1)-(2\lambda\mu+\lambda-\mu+1)^2\bigr]a_2^2-2(6\lambda\mu+2\lambda-2\mu+1)a_3\\&\quad=\gamma\left[\frac{1}{2}B_1\left(q_2-\frac{1}{2}q_1^2\right)+\frac{1}{4}B_2q_1^2\right].\end{aligned} \tag{2.40}$$

Equations (2.37) and (2.39) yield

$$p_1=-q_1 \tag{2.41}$$

and

$$8(2\lambda\mu+\lambda-\mu+1)^2a_2^2=B_1^2\gamma^2\bigl(p_1^2+q_1^2\bigr). \tag{2.42}$$

From (2.38), (2.40), (2.41) and (2.42), it follows that

$$a_2^2=\frac{\gamma^2B_1^3(p_2+q_2)}{4[(2(6\lambda\mu+2\lambda-2\mu+1)-(2\lambda\mu+\lambda-\mu+1)^2)B_1^2\gamma+(2\lambda\mu+\lambda-\mu+1)^2(B_1-B_2)]}$$

which yields the desired estimate on $|a_2|$ as described in (2.31).

Similarly, it can be obtained from (2.38), (2.40) and (2.41) that

$$a_3 = \frac{\gamma B_1[p_2(4(6\lambda\mu + 2\lambda - 2\mu + 1) - (2\lambda\mu + \lambda - \mu + 1)^2) + q_2(2\lambda\mu + \lambda - \mu + 1)^2]}{8[2(6\lambda\mu + 2\lambda - 2\mu + 1) - (2\lambda\mu + \lambda - \mu + 1)^2](6\lambda\mu + 2\lambda - 2\mu + 1)} + \frac{2\gamma(B_2 - B_1)(6\lambda\mu + 2\lambda - 2\mu + 1)p_1^2}{8[2(6\lambda\mu + 2\lambda - 2\mu + 1) - (2\lambda\mu + \lambda - \mu + 1)^2](6\lambda\mu + 2\lambda - 2\mu + 1)}$$

which easily leads to the desired estimate (2.32) on $|a_3|$. □

Taking $\mu = 0$ in Theorem 2.2, we obtain the following corollary.

Corollary 2.3 [14, Theorem 2.3] *If $f \in N^{\lambda}_{\sigma,\gamma}(\varphi)$, then*

$$|a_2| \leq \frac{|\gamma| B_1\sqrt{B_1}}{\sqrt{|(1 + 2\lambda - \lambda^2)B_1^2\gamma + (1 + \lambda)^2(B_1 - B_2)|}} \quad \textit{and} \quad |a_3| \leq \frac{|\gamma|(B_1 + |B_2 - B_1|)}{|1 + 2\lambda - \lambda^2|}.$$

Further, for $\gamma = 1$, putting $\lambda = 0$ and $\lambda = 1$ in Corollary 2.3, respectively, we have the following Corollaries 2.4 and 2.5.

Corollary 2.4 [13, Corollary 2.1] *If $f \in M^1_\sigma(0, 0, \varphi) = ST_\sigma(\varphi)$, then*

$$|a_2| \leq \frac{B_1\sqrt{B_1}}{\sqrt{|B_1^2 + B_1 - B_2|}} \quad \textit{and} \quad |a_3| \leq B_1 + |B_2 - B_1|.$$

Corollary 2.5 [13, Corollary 2.2] *If $f \in M^1_\sigma(1, 0, \varphi) = CV_\sigma(\varphi)$, then*

$$|a_2| \leq \frac{B_1\sqrt{B_1}}{\sqrt{2|B_1^2 + 2B_1 - 2B_2|}} \quad \textit{and} \quad |a_3| \leq \frac{1}{2}\bigl(B_1 + |B_2 - B_1|\bigr).$$

Remark 2.3 If we set

$$\varphi(z) = \left(\frac{1 + z}{1 - z}\right)^\alpha = 1 + 2\alpha z + 2\alpha^2 z^2 + \cdots \quad (0 < \alpha \leq 1)$$

and

$$\varphi(z) = \frac{1 + (1 - 2\beta)z}{1 - z} = 1 + 2(1 - \beta)z + 2(1 - \beta)z^2 + \cdots \quad (0 \leq \beta < 1)$$

in Corollaries 2.4 and 2.5, we obtain the results of Brannan and Taha [6, Theorems 2.1, 3.1 and 4.1, respectively].

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors jointly worked on the results and they read and approved the final manuscript.

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