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The improved disc theorems for the Schur complements of diagonally dominant matrices

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Abstract
The theory of Schur complement is very important in many fields such as control theory and computational mathematics. In this paper, applying the properties of Schur complement, utilizing some inequality techniques, some new estimates of diagonally dominant degree on the Schur complement of matrices are obtained, which improve some relative results. Further, as an application of these derived results, we present some distributions for the eigenvalues of the Schur complements. Finally, the numerical example is given to show the advantages of our derived results.
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1 Introduction

The Schur complement has been proved to be a useful tool in many fields such as numerical algebra and control theory. [1, 2] proposed a kind of iteration called the Schur-based iteration. Applying this method, we can solve large scale linear systems though reducing the order by the Schur complement. In addition, when utilizing the conjugate gradient method to solve large scale linear systems, if the eigenvalues of the system matrix are more concentrated, the convergent speed of the iterative method is faster (see, *e.g.*, [3, pp.312-317]). From [1, 2], it can be seen that for large scale linear systems, after applying the Schur-based iteration to reduce the order, the corresponding system matrix of linear equations is the Schur complement of the system matrix of original large scale linear systems and its eigenvalues are more concentrated than those of the original system matrix, leading to the Schur-based conjugate gradient method computing faster than the ordinary conjugate gradient method.

Hence, it is always interesting to know whether some important properties of matrices are inherited by their Schur complements. Clearly, the Schur complements of positive semidefinite matrices are positive semidefinite, the same is true for M-matrices, H-matrices and inverse M-matrices (see [4, 5]). Carlson and Markham showed that the Schur complements of strictly diagonally dominant matrices are diagonally dominant (see [6]). Li, Tsatsomeros and Ikramov independently proved the Schur complement of a strictly doubly diagonally dominant matrix is strictly doubly diagonally dominant (see [7, 8]). These properties have been repeatedly used for the convergence of iterations in numerical analysis and for deriving matrix inequalities in matrix analysis (see [3, 9, 10]). More importantly, the distribution for the eigenvalues of the Schur complement is of great significance, as shown in [1, 2, 8, 11–17]. The aim of this paper is to study the distributions for the eigenvalues of the Schur complement of some diagonally dominant matrices.

Denote by $\mathbb{C}^{n\times n}$ the set of all $n\times n$ complex matrices. Let $N=\{1,2,\dots,n\}$. For $A=(a_{ij})\in\mathbb{C}^{n\times n}$ $(n\geq 2)$, assume

$$P_i(A)=\sum_{j\in N,j\neq i}|a_{ij}|,\qquad S_i(A)=\sum_{j\in N,j\neq i}|a_{ji}|,\quad i=1,2,\dots,n.$$

Set

$$N_r(A)=\{i|i\in N,|a_{ii}|>P_i(A)\};\qquad N_c(A)=\{j|j\in N,|a_{jj}|>S_j(A)\}.$$

Let us recall that A is a (row) diagonally dominant matrix (D_n) if

$$|a_{ii}|\geq P_i(A),\quad \forall i\in N; \tag{1.1}$$

A is a doubly diagonally dominant matrix (DD_n) if

$$|a_{ii}||a_{jj}|\geq P_i(A)P_j(A),\quad \forall i\neq j,i,j\in N; \tag{1.2}$$

A is a γ-diagonally dominant matrix (D_n^γ) if there exists $\gamma\in[0,1]$ such that

$$|a_{ii}|\geq \gamma P_i(A)+(1-\gamma)Q_i(A),\quad \forall i\in N; \tag{1.3}$$

A is a product γ-diagonally dominant matrix (PD_n^γ) if there exists $\gamma\in[0,1]$ such that

$$|a_{ii}|\geq [P_i(A)]^\gamma[Q_i(A)]^{1-\gamma},\quad \forall i\in N. \tag{1.4}$$

If all inequalities in (1.1)-(1.4) are strict, then A is said to be a strictly (row) diagonally dominant matrix (SD_n), a strictly doubly diagonally dominant matrix (SDD_n), a strictly γ-diagonally dominant matrix (SD_n^γ) and a strictly product γ-diagonally dominant matrix (SPD_n^γ), respectively.

Liu and Zhang in [14] have pointed out the fact as follows. If $A\in SDD_n$ but $A\notin SD_n$, then there exists a unique index i_0 such that

$$|a_{i_0i_0}|\leq P_{i_0}(A). \tag{1.5}$$

As in [1, 2], for $1\leq i\leq n$ and $\gamma\in[0,1]$, we call $|a_{ii}|-P_i(A)$, $|a_{ii}|-\gamma P_i(A)-(1-\gamma)S_i(A)$ and $|a_{ii}|-[P_i(A)]^\gamma[S_i(A)]^{1-\gamma}$ the ith (row) dominant degree, γ-dominant degree and product γ-dominant degree of A, respectively.

The comparison matrix of A, denoted by $\mu(A)=(t_{ij})$, is defined to be

$$t_{ij}=\begin{cases}|a_{ij}|, & \text{if } i=j,\\ -|a_{ij}|, & \text{if } i\neq j.\end{cases}$$

A matrix A is an M-matrix if it can be written in the form of $A = mI - P$ with P being nonnegative and $m > \rho(P)$, where $\rho(P)$ denotes the spectral radius of P. A matrix A is a H-matrix if $\mu(A)$ is a M-matrix. We denote by $\mathbb{H}_n$ and $\mathbb{M}_n$ the sets of $n \times n$ H- and M-matrices, respectively.

For $\alpha \subseteq N$, denote by $|\alpha|$ the cardinality of α and $\alpha' = N - \alpha$. If $\alpha, \beta \subseteq N$, then $A(\alpha, \beta)$ is the submatrix of A lying in the rows indicated by α and the columns indicated by β. In particular, $A(\alpha, \alpha)$ is abbreviated to $A(\alpha)$. Assume that $A(\alpha)$ is nonsingular. Then

$$A/\alpha = A/A(\alpha) = A(\alpha') - A(\alpha', \alpha)\left[A(\alpha)\right]^{-1} A(\alpha, \alpha'),$$

is called the Schur complement of A with respect to $A(\alpha)$.

The paper is organized as follows. In Section 2, we give several new estimates of diagonally dominant degree on the Schur complement of matrices, which improve some relative results. In Section 3, as an application of these derived results, the distributions for eigenvalues are obtained. In Section 4, we give a numerical example to show the advantages of our derived results.

2 The diagonally dominant degree for the Schur complement

In this section, we give several new estimates of diagonally dominant degree on the Schur complement of matrices, which improve some relative results.

Lemma 1 [4] *If A is an H-matrix, then* $[\mu(A)]^{-1} \geq |A^{-1}|$.

Lemma 2 [4] *If $A \in SD_n$ or SDD_n, then $\mu(A) \in \mathbb{M}_n$, i.e., $A \in \mathbb{H}_n$.*

Lemma 3 [11] *If $A \in SD_n$ or SDD_n and $\alpha \subseteq N$, then the Schur complement of A is in $SD_{|\alpha'|}$ or $SDD_{|\alpha'|}$, where $\alpha' = N - \alpha$ is the Schur complement of α in N and $|\alpha'|$ is the cardinality of α'.*

Lemma 4 [1] *Let $a > b$, $c > b$, $b > 0$ and $0 \leq r \leq 1$. Then*

$$a^r c^{1-r} \geq (a - b)^r (c - b)^{1-r} + b.$$

Theorem 1 *Let $A \in \mathbb{C}^{n\times n}$, $\alpha = \{i_1, i_2, \ldots, i_k\} \subseteq N_r(A)$, $\alpha' = N - \alpha = \{j_1, j_2, \ldots, j_l\}$, $|\alpha| < n$ and denote $A/\alpha = (a'_{ts})$. Then for all $1 \leq t \leq l$,*

$$\left|a'_{tt}\right| - P_t(A/\alpha) \geq |a_{j_t j_t}| - P_{j_t}(A) + w_{j_t} \geq |a_{j_t j_t}| - P_{j_t}(A) \tag{2.1}$$

and

$$\left|a'_{tt}\right| + P_t(A/\alpha) \leq |a_{j_t j_t}| + P_{j_t}(A) - w_{j_t} \leq |a_{j_t j_t}| + P_{j_t}(A), \tag{2.2}$$

where

$$w_{j_t} = \min_{1\leq\omega\leq k} \frac{|a_{i_\omega i_\omega}| - P_{i_\omega}(A)}{|a_{i_\omega i_\omega}| - \sum_{\substack{u=1\\u\neq\omega}}^{k} |a_{i_\omega i_u}|} \sum_{u=1}^{k} |a_{j_t i_u}|. \tag{2.3}$$

Proof According to Lemmas 1 and 2, we have $\{\mu[A(\alpha)]\}^{-1} \geq [A(\alpha)]^{-1}$. Thus, $\forall\varepsilon > 0$ and $t = 1, 2, \ldots, l$,

$$\begin{aligned}
&\left|a'_{tt}\right| - P_t(A/\alpha) \\
&\quad = \left|a'_{tt}\right| - \sum_{\substack{s=1\\ s\neq t}}^{l} \left|a'_{ts}\right| \\
&\quad = \left| a_{j_t j_t} - (a_{j_t i_1}, \ldots, a_{j_t i_k}) \big[A(\alpha)\big]^{-1} \begin{pmatrix} a_{i_1 j_t} \\ \vdots \\ a_{i_k j_t} \end{pmatrix} \right| \\
&\qquad - \sum_{\substack{s=1\\ s\neq t}}^{l} \left| a_{j_t j_s} - (a_{j_t i_1}, \ldots, a_{j_t i_k}) \big[A(\alpha)\big]^{-1} \begin{pmatrix} a_{i_1 j_s} \\ \vdots \\ a_{i_k j_s} \end{pmatrix} \right| \\
&\quad \geq |a_{j_t j_t}| - \sum_{\substack{s=1\\ s\neq t}}^{l} |a_{j_t j_s}| - \sum_{s=1}^{l} \left| (a_{j_t i_1}, \ldots, a_{j_t i_k}) \big[A(\alpha)\big]^{-1} \begin{pmatrix} a_{i_1 j_s} \\ \vdots \\ a_{i_k j_s} \end{pmatrix} \right| \\
&\quad \geq |a_{j_t j_t}| - \sum_{\substack{s=1\\ s\neq t}}^{l} |a_{j_t j_s}| - \sum_{s=1}^{l} \left| \big(|a_{j_t i_1}|, \ldots, |a_{j_t i_k}|\big) \big\{\mu\big[A(\alpha)\big]\big\}^{-1} \begin{pmatrix} |a_{i_1 j_s}| \\ \vdots \\ |a_{i_k j_s}| \end{pmatrix} \right| \\
&\quad \geq |a_{j_t j_t}| - P_{j_t}(A) + \sum_{u=1}^{k} |a_{j_t i_u}| + (\omega_{j_t} - \varepsilon) - (\omega_{j_t} - \varepsilon) \\
&\qquad - \sum_{s=1}^{l} \left| \big(|a_{j_t i_1}|, \ldots, |a_{j_t i_k}|\big) \big\{\mu\big[A(\alpha)\big]\big\}^{-1} \begin{pmatrix} |a_{i_1 j_s}| \\ \vdots \\ |a_{i_k j_s}| \end{pmatrix} \right| \\
&\quad \geq |a_{j_t j_t}| - P_{j_t}(A) + \omega_{j_t} - \varepsilon + \sum_{u=1}^{k} |a_{j_t i_u}| - \omega_{j_t} + \varepsilon \\
&\qquad - \sum_{s=1}^{l} \left| \big(|a_{j_t i_1}|, \ldots, |a_{j_t i_k}|\big) \big\{\mu\big[A(\alpha)\big]\big\}^{-1} \begin{pmatrix} |a_{i_1 j_s}| \\ \vdots \\ |a_{i_k j_s}| \end{pmatrix} \right| \\
&\quad = |a_{j_t j_t}| - P_{j_t}(A) + \omega_{j_t} - \varepsilon \\
&\qquad + \frac{1}{\det\{\mu[A(\alpha)]\}} \\
&\qquad \times \det \begin{pmatrix} \sum_{u=1}^{k} |a_{j_t i_u}| - w_{j_t} + \varepsilon & -|a_{j_t i_1}| & \cdots & -|a_{j_t i_k}| \\ -\sum_{s=1}^{l} |a_{i_1 j_s}| & & & \\ \vdots & & \mu\big[A(\alpha)\big] & \\ -\sum_{s=1}^{l} |a_{i_k j_s}| & & & \end{pmatrix}. \qquad (2.4)
\end{aligned}$$

Denote

$$B_t \equiv \begin{pmatrix} x & -|a_{j_t i_1}| & \cdots & -|a_{j_t i_k}| \\ -\sum_{u=1}^{l} |a_{i_1 j_u}| & & & \\ \vdots & & \mu\big[A(\alpha)\big] & \\ -\sum_{u=1}^{l} |a_{i_k j_u}| & & & \end{pmatrix}. \tag{2.5}$$

If

$$x > \max_{1\le\omega\le k} \frac{\sum_{u=1}^{l} |a_{i_\omega j_u}|}{|a_{i_\omega i_\omega}| - \sum_{\substack{u=1\\ u\neq\omega}}^{k} |a_{i_\omega i_u}|} \sum_{u=1}^{k} |a_{j_t i_u}|,$$

then

$$\frac{x}{\sum_{u=1}^{k} |a_{j_t i_u}|} > \max_{1\le\omega\le k} \frac{\sum_{u=1}^{l} |a_{i_\omega j_u}|}{|a_{i_\omega i_\omega}| - \sum_{\substack{u=1\\ u\neq\omega}}^{k} |a_{i_\omega i_u}|}.$$

Choose $\varepsilon_t \in \mathbb{R}^+$ such that

$$\frac{x}{\sum_{u=1}^{k} |a_{j_t i_u}|} > \varepsilon_t > \max_{1\le\omega\le k} \frac{\sum_{u=1}^{l} |a_{i_\omega j_u}|}{|a_{i_\omega i_\omega}| - \sum_{\substack{u=1\\ u\neq\omega}}^{k} |a_{i_\omega i_u}|},$$

where we denote $\frac{x}{\sum_{u=1}^{k} |a_{j_t i_u}|} = \infty$ if $\sum_{u=1}^{k} |a_{j_t i_u}| = 0$. Set $D = \operatorname{diag}(y_1, y_2, \ldots, y_{k+1})$, where

$$y_i = \begin{cases} 1, & i = 1, \\ \varepsilon_t, & i = 2, 3, \ldots, k+1. \end{cases}$$

Denote $C_t = B_t D = (c_{sv})$. If $s = 1$, then

$$|c_{ss}| - \sum_{v\neq i} |c_{sv}| = |c_{11}| - \sum_{s=2}^{k+1} |c_{1v}| = x - \sum_{v=1}^{k} \varepsilon_t |a_{j_t i_v}| = x - \varepsilon_t \sum_{v=1}^{k} |a_{j_t i_v}| > 0;$$

otherwise,

$$\begin{aligned} |c_{ss}| - \sum_{s\neq v}^{k+1} |c_{sv}| &= \varepsilon_t |a_{i_\omega i_\omega}| - \sum_{v\neq\omega}^{k} \varepsilon_t |a_{i_\omega i_v}| - \sum_{u=1}^{l} |a_{i_\omega j_u}| \\ &= \varepsilon_t \left(|a_{i_\omega i_\omega}| - \sum_{v\neq\omega}^{k} |a_{i_\omega i_v}| \right) - \sum_{u=1}^{l} |a_{i_\omega j_u}| \\ &> \frac{\sum_{u=1}^{l} |a_{i_\omega j_u}|}{|a_{i_\omega i_\omega}| - \sum_{\substack{u=1\\ u\neq\omega}}^{k} |a_{i_\omega i_u}|} \left(|a_{i_\omega i_\omega}| - \sum_{\substack{u=1\\ u\neq\omega}}^{k} |a_{i_\omega i_u}| \right) - \sum_{u=1}^{l} |a_{i_\omega j_u}| = 0. \end{aligned}$$

Therefore, we have $C_t \in SD_{k+1}$, and so $B_t \in \mathbb{H}_{k+1}$. Note that $B_t = \mu(B_t)$. So,

$$\det B_t > 0. \tag{2.6}$$

Take $x = \sum_{u=1}^{k} |a_{j_t i_u}| - w_{j_t} + \varepsilon$ in (2.5), then

$$\begin{aligned}
&\sum_{u=1}^{k} |a_{j_t i_u}| - w_{j_t} + \varepsilon - \max_{1\le\omega\le k} \frac{\sum_{u=1}^{l} |a_{i_\omega j_u}|}{|a_{i_\omega i_\omega}| - \sum_{\substack{u=1\\u\ne\omega}}^{k} |a_{i_\omega i_u}|} \sum_{u=1}^{k} |a_{j_t i_u}| \\
&\quad\ge \sum_{u=1}^{k} |a_{j_t i_u}| - \min_{1\le\omega\le k} \frac{|a_{i_\omega i_\omega}| - P_{i_\omega}(A)}{|a_{i_\omega i_\omega}| - \sum_{\substack{u=1\\u\ne\omega}}^{k} |a_{i_\omega i_u}|} \sum_{u=1}^{k} |a_{j_t i_u}| + \varepsilon \\
&\qquad - \max_{1\le\omega\le k} \frac{\sum_{u=1}^{l} |a_{i_\omega j_u}|}{|a_{i_\omega i_\omega}| - \sum_{\substack{u=1\\u\ne\omega}}^{k} |a_{i_\omega i_u}|} \sum_{u=1}^{k} |a_{j_t i_u}| > 0.
\end{aligned}$$

Noting that $\det\{\mu[A(\alpha)]\} > 0$, by (2.4) and (2.6), we have

$$\left|a'_{tt}\right| - P_t(A/\alpha) \ge |a_{j_t j_t}| - P_{j_t}(A) + \omega_{j_t} - \varepsilon.$$

Let $\varepsilon \to 0$, thus we easily get (2.1). Similarly, we obtain (2.2). □

Remark 1 Observe that

$$\frac{P_{i_w}(A)}{|a_{i_w i_w}|} \ge \frac{\sum_{u=1}^{l} |a_{i_\omega j_u}|}{|a_{i_\omega i_\omega}| - \sum_{\substack{u=1\\u\ne\omega}}^{k} |a_{i_\omega i_u}|}.$$

This means that Theorem 1 improves Theorem 1 of [14].

Theorem 2 *Let* $A \in SDD_n$, $\alpha = \{i_0, i_1, i_2, \dots, i_k\}$ *with the index* i_0 *satisfying* (1.5), $\alpha' = N - \alpha = \{j_1, j_2, \dots, j_l\}$, $|\alpha| < n$ *and denote* $A/\alpha = (a'_{ts})$. *Then for all* $1 \le t \le l$,

$$\begin{aligned}
\left|a'_{tt}\right| - P_t(A/\alpha) &\ge |a_{j_t j_t}| - P_{j_t}(A) + \left(1 - \frac{\sum_{u=1}^{l} |a_{i_0 j_u}|}{|a_{i_0 i_0}| - \sum_{j\ne i_0}^{j\in\alpha} |a_{i_0 j}|}\right) \sum_{v=1}^{k} |a_{j_t i_v}| \\
&\ge |a_{j_t j_t}| - \frac{\sum_{u=1}^{l} |a_{i_0 j_u}|}{|a_{i_0 i_0}| - \sum_{j\ne i_0}^{j\in\alpha} |a_{i_0 j}|} P_{j_t}(A)
\end{aligned} \tag{2.7}$$

and

$$\begin{aligned}
\left|a'_{tt}\right| + P_t(A/\alpha) &\le |a_{j_t j_t}| + P_{j_t}(A) - \left(1 - \frac{\sum_{u=1}^{l} |a_{i_0 j_u}|}{|a_{i_0 i_0}| - \sum_{j\ne i_0}^{j\in\alpha} |a_{i_0 j}|}\right) \sum_{v=1}^{k} |a_{j_t i_v}| \\
&\le |a_{j_t j_t}| + \frac{\sum_{u=1}^{l} |a_{i_0 j_u}|}{|a_{i_0 i_0}| - \sum_{j\ne i_0}^{j\in\alpha} |a_{i_0 j}|} P_{j_t}(A).
\end{aligned} \tag{2.8}$$

Proof For all $1 \le t \le l$, we have

$$\begin{aligned}
|a'_{tt}| - P_t(A/\alpha) &= |a'_{tt}| - \sum_{\substack{s=1\\ s\neq t}}^{l} |a'_{ts}| \\
&= \left| a_{j_t j_t} - (a_{j_t i_1}, \ldots, a_{j_t i_k}) [A(\alpha)]^{-1} \begin{pmatrix} a_{i_1 j_t} \\ \vdots \\ a_{i_k j_t} \end{pmatrix} \right| \\
&\quad - \sum_{\substack{s=1\\ s\neq t}}^{l} \left| a_{j_t j_s} - (a_{j_t i_1}, \ldots, a_{j_t i_k}) [A(\alpha)]^{-1} \begin{pmatrix} a_{i_1 j_s} \\ \vdots \\ a_{i_k j_s} \end{pmatrix} \right| \\
&\ge |a_{j_t j_t}| - P_{j_t}(A) + \left(1 - \frac{\sum_{u=1}^{l} |a_{i_0 j_u}|}{|a_{i_0 i_0}| - \sum_{j\neq i_0}^{j\in\alpha} |a_{i_0 j}|}\right) \sum_{v=1}^{k} |a_{j_t i_v}| \\
&\quad + \frac{\sum_{u=1}^{l} |a_{i_0 j_u}|}{|a_{i_0 i_0}| - \sum_{j\neq i_0}^{j\in\alpha} |a_{i_0 j}|} \sum_{v=1}^{k} |a_{j_t i_v}| \\
&\quad - \sum_{s=1}^{l} \left| (|a_{j_t i_1}|, \ldots, |a_{j_t i_k}|) \{\mu[A(\alpha)]\}^{-1} \begin{pmatrix} |a_{i_1 j_s}| \\ \vdots \\ |a_{i_k j_s}| \end{pmatrix} \right|.
\end{aligned}$$

Thus,

$$\begin{aligned}
&|a'_{tt}| - P_t(A/\alpha) \\
&\quad \ge |a_{j_t j_t}| - P_{j_t}(A) + \left(1 - \frac{\sum_{u=1}^{l} |a_{i_0 j_u}|}{|a_{i_0 i_0}| - \sum_{j\neq i_0}^{j\in\alpha} |a_{i_0 j}|}\right) \sum_{v=1}^{k} |a_{j_t i_v}| \\
&\qquad + \frac{1}{\det\{\mu[A(\alpha)]\}} \det \begin{pmatrix} \frac{\sum_{u=1}^{l} |a_{i_0 j_u}|}{|a_{i_0 i_0}| - \sum_{j\neq i_0}^{j\in\alpha} |a_{i_0 j}|} \sum_{v=1}^{k} |a_{j_t i_v}| & -|a_{j_t i_1}| \quad \cdots \quad -|a_{j_t i_k}| \\ -\sum_{s=1}^{l} |a_{i_1 j_s}| & \\ \vdots & \mu[A(\alpha)] \\ -\sum_{s=1}^{l} |a_{i_k j_s}| & \end{pmatrix}.
\end{aligned}$$

Take $x = \frac{\sum_{u=1}^{l} |a_{i_0 j_u}|}{|a_{i_0 i_0}| - \sum_{j\neq i_0}^{j\in\alpha} |a_{i_0 j}|} \sum_{v=1}^{k} |a_{j_t i_v}|$ in (2.5). By (2.6), thus it is not difficult to get that (2.7) follows. Similarly, we obtain (2.8). □

It is known that the Schur complements of diagonally dominant matrices are diagonally dominant (see [12, 13]). However, this property is not always true for γ-diagonally dominant matrices and for product γ-diagonally dominant matrices, as shown in [1].

In the sequel, we obtain some disc separations for the γ-diagonally and product γ-diagonally dominant degree of the Schur complement, from which we provide that the Schur complement of the γ-diagonally and product γ-diagonally dominant matrices is also γ-diagonally dominant and product γ-diagonally under some restrictive conditions.

Theorem 3 *Let $A \in \mathbb{C}^{n\times n}$, $\alpha = \{i_1, i_2, \ldots, i_k\} \subseteq N_r(A) \cap N_c(A) \neq \emptyset$, $\alpha' = N - \alpha = \{j_1, j_2, \ldots, j_l\}$, $|\alpha| < n$ and denote $A/\alpha = (a'_{ts})$. Then for all $1 \le t \le l$,*

$$\begin{aligned} \left|a'_{tt}\right| - P_t^{\gamma}(A/\alpha)S_t^{(1-\gamma)}(A/\alpha) &> |a_{j_tj_t}| - \left(P_{j_t}(A) - w_t\right)^{\gamma}\left(S_{j_t}(A) - w_t^T\right)^{1-\gamma} \\ &> |a_{j_tj_t}| - P_{j_t}^{\gamma}(A)S_{j_t}^{1-\gamma}(A) \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} \left|a'_{tt}\right| + P_t^{\gamma}(A/\alpha)S_t^{(1-\gamma)}(A/\alpha) &< |a_{j_tj_t}| + \left(P_{j_t}(A) - w_t\right)^{\gamma}\left(S_{j_t}(A) - w_t^T\right)^{1-\gamma} \\ &< |a_{j_tj_t}| + P_{j_t}^{\gamma}(A)S_{j_t}^{1-\gamma}(A), \end{aligned} \tag{2.10}$$

where

$$w_t = \min_{1\le v\le k} \frac{|a_{i_vi_v}| - P_{i_v}(A)}{|a_{i_vi_v}| - \sum_{u=1, u\neq v}^{k}|a_{i_vi_u}|}\sum_{u=1}^{k}|a_{j_ti_u}|;$$

$$w_t^T = \min_{1\le v\le k} \frac{|a_{i_vi_v}| - S_{i_v}(A)}{|a_{i_vi_v}| - \sum_{u=1, u\neq v}^{k}|a_{i_ui_v}|}\sum_{u=1}^{k}|a_{i_uj_t}|.$$

Proof For all $1 \le t \le l$, we have

$$\begin{aligned} &\left|a'_{tt}\right| - P_t^{\gamma}(A/\alpha)S_t^{(1-\gamma)}(A/\alpha) \\ &\quad = \left|a'_{tt}\right| - \left(\sum_{\substack{s=1\\ s\neq t}}^{l}\left|a'_{ts}\right|\right)^{\gamma}\left(\sum_{\substack{s=1\\ s\neq t}}^{l}\left|a'_{st}\right|\right)^{1-\gamma} \\ &\quad = \left|a_{j_tj_t} - (a_{j_ti_1}, \ldots, a_{j_ti_k})\left[A(\alpha)\right]^{-1}\begin{pmatrix} a_{i_1j_t} \\ \vdots \\ a_{i_kj_t}\end{pmatrix}\right| \\ &\qquad - \left[\sum_{\substack{s=1\\ s\neq t}}^{l}\left|a_{j_tj_s} - (a_{j_ti_1}, \ldots, a_{j_ti_k})\left[A(\alpha)\right]^{-1}\begin{pmatrix} a_{i_1j_s} \\ \vdots \\ a_{i_kj_s}\end{pmatrix}\right|\right]^{\gamma} \\ &\qquad \times \left[\sum_{\substack{s=1\\ s\neq t}}^{l}\left|a_{j_sj_t} - (a_{j_si_1}, \ldots, a_{j_si_k})\left[A(\alpha)\right]^{-1}\begin{pmatrix} a_{i_1j_t} \\ \vdots \\ a_{i_kj_t}\end{pmatrix}\right|\right]^{(1-\gamma)} \\ &\quad \ge |a_{j_tj_t}| - \left|(a_{j_ti_1}, \ldots, a_{j_ti_k})\left[A(\alpha)\right]^{-1}\begin{pmatrix} a_{i_1j_t} \\ \vdots \\ a_{i_kj_t}\end{pmatrix}\right| \\ &\qquad - \left(\sum_{\substack{s=1\\ s\neq t}}^{l}\left[|a_{j_tj_s}| + \left|(a_{j_ti_1}, \ldots, a_{j_ti_k})\left[A(\alpha)\right]^{-1}\begin{pmatrix} a_{i_1j_s} \\ \vdots \\ a_{i_kj_s}\end{pmatrix}\right|\right]\right)^{\gamma} \end{aligned}$$

$$\times\left(\sum_{\substack{s=1\\s\neq t}}^{l}\left[|a_{j_sj_t}|+\left|(a_{j_si_1},\ldots,a_{j_si_k})\left[A(\alpha)\right]^{-1}\begin{pmatrix}a_{i_1j_t}\\\vdots\\a_{i_kj_t}\end{pmatrix}\right|\right]\right)^{1-\gamma}$$

$$\geq |a_{j_tj_t}|-\left(|a_{j_ti_1}|,\ldots,|a_{j_ti_k}|\right)\left\{\mu\left[A(\alpha)\right]\right\}^{-1}\begin{pmatrix}|a_{i_1j_t}|\\\vdots\\|a_{i_kj_t}|\end{pmatrix}$$

$$-\left(\sum_{\substack{s=1\\s\neq t}}^{l}\left[|a_{j_tj_s}|+\left(|a_{j_ti_1}|,\ldots,|a_{j_ti_k}|\right)\left\{\mu\left[A(\alpha)\right]\right\}^{-1}\begin{pmatrix}|a_{i_1j_s}|\\\vdots\\|a_{i_kj_s}|\end{pmatrix}\right]\right)^{\gamma}$$

$$\times\left(\sum_{\substack{s=1\\s\neq t}}^{l}\left[|a_{j_sj_t}|+\left(|a_{j_si_1}|,\ldots,|a_{j_si_k}|\right)\left\{\mu\left[A(\alpha)\right]\right\}^{-1}\begin{pmatrix}|a_{i_1j_t}|\\\vdots\\|a_{i_kj_t}|\end{pmatrix}\right]\right)^{1-\gamma}. \tag{2.11}$$

Similar as in the proof of Theorem 1, we easily obtain

$$\sum_{\substack{s=1\\s\neq t}}^{l}\left[|a_{j_tj_s}|+\left(|a_{j_ti_1}|,\ldots,|a_{j_ti_k}|\right)\left\{\mu\left[A(\alpha)\right]\right\}^{-1}\begin{pmatrix}|a_{i_1j_s}|\\\vdots\\|a_{i_kj_s}|\end{pmatrix}\right]$$

$$< P_{j_t}(A)-w_t-\left(|a_{j_ti_1}|,\ldots,|a_{j_ti_k}|\right)\left\{\mu\left[A(\alpha)\right]\right\}^{-1}\begin{pmatrix}|a_{i_1j_t}|\\\vdots\\|a_{i_kj_t}|\end{pmatrix}. \tag{2.12}$$

Similarly,

$$\sum_{\substack{s=1\\s\neq t}}^{l}\left[|a_{j_sj_t}|+\left(|a_{j_si_1}|,\ldots,|a_{j_si_k}|\right)\left\{\mu\left[A(\alpha)\right]\right\}^{-1}\begin{pmatrix}|a_{i_1j_t}|\\\vdots\\|a_{i_kj_t}|\end{pmatrix}\right]$$

$$< S_{j_t}(A)-w_t^T-\left(|a_{j_ti_1}|,\ldots,|a_{j_ti_k}|\right)\left\{\mu\left[A(\alpha)\right]\right\}^{-1}\begin{pmatrix}|a_{i_1j_t}|\\\vdots\\|a_{i_kj_t}|\end{pmatrix}. \tag{2.13}$$

Set

$$h=\left(|a_{j_ti_1}|,\ldots,|a_{j_ti_k}|\right)\left\{\mu\left[A(\alpha)\right]\right\}^{-1}\begin{pmatrix}|a_{i_1j_t}|\\\vdots\\|a_{i_kj_t}|\end{pmatrix}.$$

From (2.11), (2.12), (2.13), using Lemma 4, we have

$$\left|a'_{tt}\right|-P_t^{\gamma}(A/\alpha)S_t^{(1-\gamma)}(A/\alpha)$$
$$> |a_{j_tj_t}|-h-\left(P_{j_t}(A)-w_t-h\right)^{\gamma}\left(S_{j_t}(A)-w_t^T-h\right)^{1-\gamma}$$

$$> |a_{j_tj_t}| - h - \left[\left(P_{j_t}(A) - w_t\right)^{\gamma}\left(S_{j_t}(A) - w_t^T\right)^{1-\gamma} - h\right]$$
$$= |a_{j_tj_t}| - \left(P_{j_t}(A) - w_t\right)^{\gamma}\left(S_{j_t}(A) - w_t^T\right)^{1-\gamma}.$$

Thus we get (2.9). Similarly, we have (2.10). □

Remark 2 Observe that

$$w_{j_t} = \min_{1\le v\le k} \frac{|a_{i_vi_v}| - P_{i_v}(A)}{|a_{i_vi_v}| - \sum_{\substack{u=1\\u\neq v}}^{k} |a_{i_vi_u}|} \sum_{u=1}^{k} |a_{j_ti_u}| \ge \min_{1\le v\le k} \frac{|a_{i_vi_v}| - P_{i_v}(A)}{|a_{i_vi_v}|} \sum_{u=1}^{k} |a_{j_ti_u}|,$$

$$w_{j_t}^T = \min_{1\le v\le k} \frac{|a_{i_vi_v}| - S_{i_v}(A)}{|a_{i_vi_v}| - \sum_{\substack{u=1\\u\neq v}}^{k} |a_{i_ui_v}|} \sum_{u=1}^{k} |a_{i_uj_t}| \ge \min_{1\le v\le k} \frac{|a_{i_vi_v}| - S_{i_v}(A)}{|a_{i_vi_v}|} \sum_{u=1}^{k} |a_{i_uj_t}|.$$

This means that Theorem 3 improves Theorem 2 of [1].

In a similar way to the proof of Theorem 3, we get the following theorem immediately.

Theorem 4 *Let $A \in \mathbb{C}^{n\times n}$, $\alpha = \{i_1, i_2, \dots, i_k\} \subseteq N_r(A) \cap N_c(A) \neq \emptyset$, $\alpha' = N - \alpha = \{j_1, j_2, \dots, j_l\}$, $|\alpha| < n$ and denote $A/\alpha = (a'_{ts})$. Then for all $1 \le t \le l$,*

$$\begin{aligned}
&\left|a'_{tt}\right| - \gamma P_t(A/\alpha) - (1-\gamma)S_t(A/\alpha) \\
&\quad > |a_{j_tj_t}| - \gamma P_{j_t}(A) - (1-\gamma)S_{j_t}(A) + \gamma w_t + (1-\gamma)w_t^T \\
&\quad > |a_{j_tj_t}| - \gamma P_{j_t}(A) - (1-\gamma)S_{j_t}(A)
\end{aligned} \tag{2.14}$$

and

$$\begin{aligned}
&\left|a'_{tt}\right| + \gamma P_t(A/\alpha) + (1-\gamma)S_t(A/\alpha) \\
&\quad < |a_{j_tj_t}| + \gamma P_{j_t}(A) + (1-\gamma)S_{j_t}(A) - \gamma w_t - (1-\gamma)w_t^T \\
&\quad < |a_{j_tj_t}| + \gamma P_{j_t}(A) + (1-\gamma)S_{j_t}(A).
\end{aligned} \tag{2.15}$$

Corollary 1 *Let $A \in D^{\gamma}$ and $N_r(A) \cap N_c(A) \neq \emptyset$. Then for any $\alpha \subseteq N_r(A) \cap N_c(A)$ with $|\alpha| < n$,*

$$A/\alpha \in SD_{n-|\alpha|}^{\gamma}.$$

Proof By (2.14), we have

$$\left|a'_{tt}\right| - \gamma P_t(A/\alpha) - (1-\gamma)S_t(A/\alpha) > |a_{j_tj_t}| - \gamma P_{j_t}(A) - (1-\gamma)S_{j_t}(A) \ge 0.$$ □

Corollary 2 *Let $A \in PD_n^{\gamma}$ and $N_r(A) \cap N_c(A) \neq \emptyset$. Then for any $\alpha \subseteq N_r(A) \cap N_c(A)$ with $|\alpha| < n$,*

$$A/\alpha \in SPD_{n-|\alpha|}^{\gamma}.$$

3 Distribution for eigenvalues

In this section, as an application of our results in Section 2, we present some locations for the eigenvalues of the Schur complements.

Theorem 5 *Let $A \in \mathbb{C}^{n\times n}$, $\alpha = \{i_1, i_2, \ldots, i_k\} \subset N_r \neq \emptyset$, $\alpha' = N - \alpha = \{j_1, j_2, \ldots, j_l\}$. Then for each eigenvalue λ of A/α, there exists $1 \le t \le l$ such that*

$$|\lambda - a_{j_t j_t}| < P_{j_t}(A) - w_{j_t}. \tag{3.1}$$

Proof Set $A/\alpha = (a'_{st})$. Using the *Gerschgorin circle theorem*, we know there exists $1 \le t \le l$ such that

$$\left|\lambda - a'_{tt}\right| \le P_t(A/\alpha).$$

Thus

$$\begin{aligned}
0 &\ge \left|\lambda - a'_{tt}\right| - P_t(A/\alpha) \\
&= \left|\lambda - a_{j_t j_t} + (a_{j_t i_1}, \ldots, a_{j_t i_k})\left[A(\alpha)\right]^{-1}\begin{pmatrix} a_{i_1 j_t} \\ \vdots \\ a_{i_k j_t}\end{pmatrix}\right| \\
&\quad - \sum_{\substack{s=1\\ s\neq t}}^{l}\left|a_{j_t j_s} - (a_{j_t i_1}, \ldots, a_{j_t i_k})\left[A(\alpha)\right]^{-1}\begin{pmatrix} a_{i_1 j_s} \\ \vdots \\ a_{i_k j_s}\end{pmatrix}\right| \\
&\ge |\lambda - a_{j_t j_t}| - \sum_{\substack{s=1\\ s\neq t}}^{l}|a_{j_t j_s}| - \sum_{s=1}^{l}\left|(a_{j_t i_1}, \ldots, a_{j_t i_k})\left[A(\alpha)\right]^{-1}\begin{pmatrix} a_{i_1 j_s} \\ \vdots \\ a_{i_k j_s}\end{pmatrix}\right| \\
&\ge |\lambda - a_{j_t j_t}| - \sum_{\substack{s=1\\ s\neq t}}^{l}|a_{j_t j_s}| - \sum_{s=1}^{l}\left(|a_{j_t i_1}|, \ldots, |a_{j_t i_k}|\right)\left\{\mu\left[A(\alpha)\right]\right\}^{-1}\begin{pmatrix} |a_{i_1 j_s}| \\ \vdots \\ |a_{i_k j_s}|\end{pmatrix} \\
&\quad \text{(by Lemmas 1 and 2)} \\
&= |\lambda - a_{j_t j_t}| - P_{j_t}(A) + \sum_{u=1}^{k}|a_{j_t i_u}| + w_{j_t} - w_{j_t} \\
&\quad - \sum_{s=1}^{l}\left(|a_{j_t i_1}|, \ldots, |a_{j_t i_k}|\right)\left\{\mu\left[A(\alpha)\right]\right\}^{-1}\begin{pmatrix} |a_{i_1 j_s}| \\ \vdots \\ |a_{i_k j_s}|\end{pmatrix} \\
&= |\lambda - a_{j_t j_t}| - P_{j_t}(A) + w_{j_t} \\
&\quad + \sum_{u=1}^{k}|a_{j_t i_u}| - w_{j_t} - \left(|a_{j_t i_1}|, \ldots, |a_{j_t i_k}|\right)\left\{\mu\left[A(\alpha)\right]\right\}^{-1}\begin{pmatrix} \sum_{s=1}^{l}|a_{i_1 j_s}| \\ \vdots \\ \sum_{s=1}^{l}|a_{i_k j_s}|\end{pmatrix} \\
&= |\lambda - a_{j_t j_t}| - P_{j_t}(A) + w_{j_t}
\end{aligned}$$

$$+\frac{1}{\det\{\mu[A(\alpha)]\}}\det\begin{pmatrix} \sum_{u=1}^{k}|a_{j_t i_u}|-w_{j_t} & -|a_{j_t i_1}| & \cdots & -|a_{j_t i_k}| \\ -\sum_{s=1}^{l}|a_{i_1 j_s}| & & & \\ \vdots & & \mu[A(\alpha)] & \\ -\sum_{s=1}^{l}|a_{i_k j_s}| & & & \end{pmatrix}.$$

Hence, it is not difficult to get by (2.6) that

$$0 \ge \left|\lambda - a'_{tt}\right| - P_t(A/\alpha) > |\lambda - a_{j_t j_t}| - P_{j_t}(A) + w_{j_t},$$

i.e.,

$$|\lambda - a_{j_t j_t}| < P_{j_t}(A) - w_{j_t}.$$ □

Remark 3 By Remark 2, it is obvious that Theorem 5 improves Theorem 3 of [1].

Corollary 3 *Let $A \in \mathbb{C}^{n\times n}$, $\alpha = \{i_1, i_2, \ldots, i_k\} \subset N_r \neq \emptyset$, $\alpha' = N - \alpha = \{j_1, j_2, \ldots, j_l\}$. Then for each eigenvalue λ of A/α, there exists $1 \le t \le l$ such that*

$$|\lambda - a_{j_t j_t}| < P_{j_t}(A).$$

In a similar way to the proof of Theorem 5, we obtain the following theorem according to Theorem 2.

Theorem 6 *Let $A \in SDD_n$, $\alpha = \{i_0, i_1, i_2, \ldots, i_k\}$ with the index i_0 satisfying* (1.5), *$\alpha' = N - \alpha = \{j_1, j_2, \ldots, j_l\}$. Then for each eigenvalue λ of A/α, there exists $1 \le t \le l$ such that*

$$|\lambda - a_{j_t j_t}| < P_{j_t}(A) - \left(1 - \frac{\sum_{u=1}^{l}|a_{i_0 j_u}|}{|a_{i_0 i_0}| - \sum_{j\neq i_0}^{j\in\alpha}|a_{i_0 j}|}\right)\sum_{v=1}^{k}|a_{j_t i_v}|. \tag{3.2}$$

Next, we obtain some distributions for the eigenvalues of the Schur complements of matrices under the conditions such as PD_n^{γ} degree.

Lemma 5 [1] *Let $A \in \mathbb{C}^{n\times n}$ and $0 \le \gamma \le 1$. Then for every eigenvalue λ of A, there exists $1 \le i \le n$ such that*

$$|\lambda - a_{ii}| \le P_i^{\gamma}(A) S_i^{1-\gamma}(A). \tag{3.3}$$

Theorem 7 *Let $A \in \mathbb{C}^{n\times n}$, $\alpha = \{i_1, i_2, \ldots, i_k\} \subseteq N_r(A) \cap N_c(A) \neq \emptyset$, $\alpha' = N - \alpha = \{j_1, j_2, \ldots, j_l\}$, $|\alpha| < n$. Then for each eigenvalue λ of A/α, there exists $1 \le t \le l$ such that*

$$|\lambda - a_{j_t j_t}| < \left(P_{j_t}(A) - w_t\right)^{\gamma}\left(S_{j_t}(A) - w_t^T\right)^{1-\gamma}. \tag{3.4}$$

Proof Set $A/\alpha = (a'_{ts})$. From Lemma 5, we know that for each eigenvalue λ of A/α, there exists $1 \le t \le l$ such that

$$\left|\lambda - a'_{tt}\right| \le P_t^{\gamma}(A/\alpha) S_t^{1-\gamma}(A/\alpha). \tag{3.5}$$

Hence,

$$\begin{aligned}
0 &\geq \left|\lambda - a'_{tt}\right| - P_t^{\gamma}(A/\alpha) S_t^{(1-\gamma)}(A/\alpha) \\
&= \left|\lambda - a'_{tt}\right| - \left(\sum_{\substack{s=1 \\ s\neq t}}^{l} \left|a'_{ts}\right|\right)^{\gamma} \left(\sum_{\substack{s=1 \\ s\neq t}}^{l} \left|a'_{st}\right|\right)^{1-\gamma} \\
&= \left|\lambda - a_{j_t j_t} + (a_{j_t i_1}, \ldots, a_{j_t i_k}) \bigl[A(\alpha)\bigr]^{-1} \begin{pmatrix} a_{i_1 j_t} \\ \vdots \\ a_{i_k j_t} \end{pmatrix}\right| \\
&\quad - \left[\sum_{\substack{s=1 \\ s\neq t}}^{l} \left| a_{j_t j_s} - (a_{j_t i_1}, \ldots, a_{j_t i_k}) \bigl[A(\alpha)\bigr]^{-1} \begin{pmatrix} a_{i_1 j_s} \\ \vdots \\ a_{i_k j_s} \end{pmatrix}\right|\right]^{\gamma} \\
&\quad \times \left[\sum_{\substack{s=1 \\ s\neq t}}^{l} \left| a_{j_s j_t} - (a_{j_s i_1}, \ldots, a_{j_s i_k}) \bigl[A(\alpha)\bigr]^{-1} \begin{pmatrix} a_{i_1 j_t} \\ \vdots \\ a_{i_k j_t} \end{pmatrix}\right|\right]^{(1-\gamma)} \\
&\geq |\lambda - a_{j_t j_t}| - \left|(a_{j_t i_1}, \ldots, a_{j_t i_k}) \bigl[A(\alpha)\bigr]^{-1} \begin{pmatrix} a_{i_1 j_t} \\ \vdots \\ a_{i_k j_t} \end{pmatrix}\right| \\
&\quad - \left(\sum_{\substack{s=1 \\ s\neq t}}^{l} \left[|a_{j_t j_s}| + \left|(a_{j_t i_1}, \ldots, a_{j_t i_k}) \bigl[A(\alpha)\bigr]^{-1} \begin{pmatrix} a_{i_1 j_s} \\ \vdots \\ a_{i_k j_s} \end{pmatrix}\right|\right]\right)^{\gamma} \\
&\quad \times \left(\sum_{\substack{s=1 \\ s\neq t}}^{l} \left[|a_{j_s j_t}| + \left|(a_{j_s i_1}, \ldots, a_{j_s i_k}) \bigl[A(\alpha)\bigr]^{-1} \begin{pmatrix} a_{i_1 j_t} \\ \vdots \\ a_{i_k j_t} \end{pmatrix}\right|\right]\right)^{1-\gamma}. \qquad (3.6)
\end{aligned}$$

From the proof of Theorem 3, we know

$$\begin{aligned}
&\left|(a_{j_t i_1}, \ldots, a_{j_t i_k}) \bigl[A(\alpha)\bigr]^{-1} \begin{pmatrix} a_{i_1 j_t} \\ \vdots \\ a_{i_k j_t} \end{pmatrix}\right| \\
&\quad + \left(\sum_{\substack{s=1 \\ s\neq t}}^{l} \left[|a_{j_t j_s}| + \left|(a_{j_t i_1}, \ldots, a_{j_t i_k}) \bigl[A(\alpha)\bigr]^{-1} \begin{pmatrix} a_{i_1 j_s} \\ \vdots \\ a_{i_k j_s} \end{pmatrix}\right|\right]\right)^{\gamma} \\
&\quad \times \left(\sum_{\substack{s=1 \\ s\neq t}}^{l} \left[|a_{j_s j_t}| + \left|(a_{j_s i_1}, \ldots, a_{j_s i_k}) \bigl[A(\alpha)\bigr]^{-1} \begin{pmatrix} a_{i_1 j_t} \\ \vdots \\ a_{i_k j_t} \end{pmatrix}\right|\right]\right)^{1-\gamma} \\
&< \bigl(P_{j_t}(A) - w_t\bigr)^{\gamma} \bigl(S_{j_t}(A) - w_t^T\bigr)^{1-\gamma}.
\end{aligned}$$

Therefore, from (3.6) we obtain

$$\begin{aligned}0 &\geq \left|\lambda - a'_{tt}\right| - P_t^{\gamma}(A/\alpha)S_t^{1-\gamma}(A/\alpha)\\ &> |\lambda - a_{j_tj_t}| - \left(P_{j_t}(A) - w_t\right)^{\gamma}\left(S_{j_t}(A) - w_t^T\right)^{1-\gamma}.\end{aligned}$$

Thus (3.4) holds. □

Remark 4 By Remark 2, it is obvious that Theorem 7 improves Theorem 4 of [1].

Corollary 4 *Let* $A \in \mathbb{C}^{n\times n}$, $\alpha = \{i_1, i_2, \dots, i_k\} \subseteq N_r(A) \cap N_c(A) \neq \emptyset$, $\alpha' = N - \alpha = \{j_1, j_2, \dots, j_l\}$, $|\alpha| < n$. *Then for each eigenvalue* λ *of* A/α, *there exists* $1 \leq t \leq l$ *such that*

$$|\lambda - a_{j_tj_t}| < P_{j_t}^{\gamma}(A)S_{j_t}^{1-\gamma} \leq \gamma P_{j_t}(A) + (1-\gamma)S_{j_t}(A).$$

4 A numerical example

In this section, we show how to estimate the bounds for eigenvalues of the Schur complement with the elements of the original matrix to show the advantages of our results.

Example Let

$$A = \begin{pmatrix} 330 & 323 & 2 & 3 & 1\\ 321 & 328 & 3 & 1 & 2\\ 3 & 2 & 6 & 7 & 2\\ 2 & 1 & 1 & 2 & 6\\ 2 & 1 & 2 & 3 & 4\end{pmatrix}, \qquad \alpha = \{1,2\}.$$

If we estimate the bounds for eigenvalues of A/α by the elements of A/α, there would be great computations to do. However, as

$$P_1(A) = 329; \quad P_2(A) = 327; \quad P_3(A) = 14; \quad P_4(A) = 10; \quad P_5(A) = 8;$$

$$S_1(A) = 328; \quad S_2(A) = 327; \quad S_3(A) = 8; \quad S_4(A) = 14; \quad S_5(A) = 11;$$

$$w_3 = \min\left\{\frac{|a_{11}| - P_1(A)}{|a_{11}| - |a_{12}|}, \frac{|a_{22}| - P_2(A)}{|a_{22}| - |a_{21}|}\right\}\sum_{i=1}^{2}|a_{3i}| = \frac{5}{7};$$

$$w_4 = \min\left\{\frac{|a_{11}| - P_1(A)}{|a_{11}| - |a_{12}|}, \frac{|a_{22}| - P_2(A)}{|a_{22}| - |a_{21}|}\right\}\sum_{i=1}^{2}|a_{4i}| = \frac{3}{7};$$

$$w_5 = \min\left\{\frac{|a_{11}| - P_1(A)}{|a_{11}| - |a_{12}|}, \frac{|a_{22}| - P_2(A)}{|a_{22}| - |a_{21}|}\right\}\sum_{i=1}^{2}|a_{5i}| = \frac{3}{7};$$

$$w_3^T = \min\left\{\frac{|a_{11}| - S_1(A)}{|a_{11}| - |a_{21}|}, \frac{|a_{22}| - S_2(A)}{|a_{22}| - |a_{12}|}\right\}\sum_{i=1}^{2}|a_{i3}| = 1;$$

$$w_4^T = \min\left\{\frac{|a_{11}| - S_1(A)}{|a_{11}| - |a_{21}|}, \frac{|a_{22}| - S_2(A)}{|a_{22}| - |a_{12}|}\right\}\sum_{i=1}^{2}|a_{i4}| = \frac{4}{5};$$

$$w_5^T = \min\left\{\frac{|a_{11}| - S_1(A)}{|a_{11}| - |a_{21}|}, \frac{|a_{22}| - S_2(A)}{|a_{22}| - |a_{12}|}\right\}\sum_{i=1}^{2}|a_{i5}| = \frac{3}{5}.$$

Since $\alpha \in N_r(A)$, according to Theorem 5, the eigenvalue z of A/α satisfies

$$z \in \{z \mid |z-6| \le 13.29\} \cup \{z \mid |z-2| \le 9.58\} \cup \{z \mid |z-4| \le 7.58\} \equiv G_1. \tag{4.1}$$

According to Theorem 3 in [1], the eigenvalue z of A/α satisfies

$$z \in \{z \mid |z-6| \le 13.98\} \cup \{z \mid |z-2| \le 9.99\} \cup \{z \mid |z-4| \le 7.99\} \equiv G_1'. \tag{4.2}$$

Further, we use Figure 1 to illustrate (4.1) and (4.2).

It is clear that $G_1 \subset G_1'$ from both (4.1), (4.2) and Figure 1.

In addition, since $\alpha \in N_r(A) \cap N_c(A)$, by taking $\gamma = \frac{1}{2}$ in Theorem 7, the eigenvalue z of A/α satisfies

$$z \in \{z \mid |z-6| \le 9.64\} \cup \{z \mid |z-2| \le 11.24\} \cup \{z \mid |z-4| \le 8.87\} \equiv G_2. \tag{4.3}$$

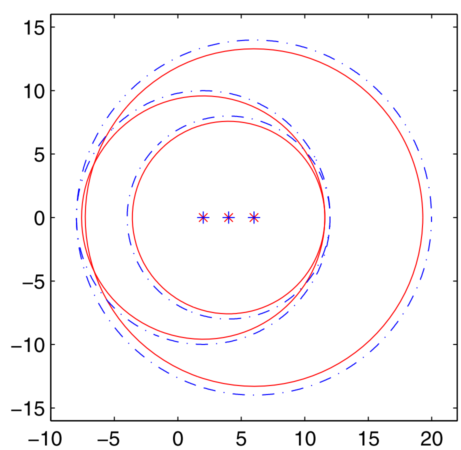

Figure 1 The red dotted line and blue dashed line denote the corresponding discs of G_1 and G_1', respectively.

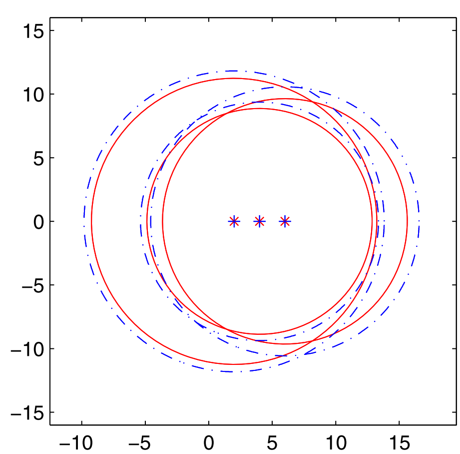

Figure 2 The red dotted line and blue dashed line denote the corresponding discs of G_2 and G_2', respectively.

According to Theorem 4 in [1], the eigenvalue z of A/α satisfies

$$z \in \{z \mid |z-6| \le 10.57\} \cup \{z \mid |z-2| \le 11.82\} \cup \{z \mid |z-4| \le 9.37\} \equiv G_2'. \tag{4.4}$$

Further, we use Figure 2 to illustrate (4.3) and (4.4).

It is clear that $G_2 \subset G_2'$ from both (4.3), (4.4) and Figure 2.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

JZ and JZL carried out the preparation, participated in the sequence alignment and drafted the manuscript. GT participated in its design and coordination. All authors read and approved the final manuscript.

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