Ahmad et al. *Journal of Inequalities and Applications* 2013, **2013**:296
http://www.journalofinequalitiesandapplications.com/content/2013/1/296

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

An existence result for fractional differential inclusions with nonlinear integral boundary conditions

Bashir Ahmad[1], Sotiris K Ntouyas[2] and Ahmed Alsaedi[1*]

*Correspondence: aalsaedi@hotmail.com
[1]Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah, 21589, Saudi Arabia
Full list of author information is available at the end of the article

Abstract
This paper studies the existence of solutions for a fractional differential inclusion of order $q \in (2,3]$ with nonlinear integral boundary conditions by applying Bohnenblust-Karlin's fixed point theorem. Some examples are presented for the illustration of the main result.
MSC: 34A40; 34A12; 26A33

Keywords: fractional differential inclusions; integral boundary conditions; existence; Bohnenblust-Karlin's fixed point theorem

1 Introduction

In this paper, we apply the Bohnenblust-Karlin fixed point theorem to prove the existence of solutions for a fractional differential inclusion with integral boundary conditions given by

$$\begin{cases} {}^cD^q x(t) \in F(t, x(t)), \quad t \in [0,T], T > 0, 2 < q \leq 3, \\ x(0) = 0, \qquad x(T) = \mu_1 \int_0^T g(s, x(s))\,ds, \\ x'(0) - \lambda x'(T) = \mu_2 \int_0^T h(s, x(s))\,ds, \end{cases} \tag{1.1}$$

where ${}^cD^q$ denotes the Caputo fractional derivative of order q, $F : [0,T] \times \mathbb{R} \to 2^{\mathbb{R}} \setminus \{\emptyset\}$, $g, h : [0,T] \times \mathbb{R} \to \mathbb{R}$ are given continuous functions and $\lambda, \mu_1, \mu_2, \in \mathbb{R}$ with $\lambda \neq -1$.

Differential inclusions of integer order (classical case) play an important role in the mathematical modeling of various situations in economics, optimal control, *etc.* and are widely studied in literature. Motivated by an extensive study of classical differential inclusions, a significant work has also been established for fractional differential inclusions. For examples and details, see [1–10] and references therein.

2 Preliminaries

Let $C([0,T], \mathbb{R})$ denote a Banach space of continuous functions from $[0,T]$ into $\mathbb{R}$ with the norm $\|x\| = \sup_{t\in[0,T]}\{|x(t)|\}$. Let $L^1([0,T], \mathbb{R})$ be the Banach space of functions $x : [0,T] \to \mathbb{R}$ which are Lebesgue integrable and normed by $\|x\|_{L^1} = \int_0^T |x(t)|\,dt$.

Now we recall some basic definitions on multi-valued maps [11–14].

Let $(X, \|\cdot\|)$ be a Banach space. Then a multi-valued map $G : X \to 2^X$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$. The map G is bounded on bounded sets if

$G(\mathbb{B}) = \bigcup_{x\in\mathbb{B}} G(x)$ is bounded in X for any bounded set $\mathbb{B}$ of X (*i.e.*, $\sup_{x\in\mathbb{B}}\{\sup\{|y| : y \in G(x)\}\} < \infty$). G is called upper semi-continuous (u.s.c.) on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and if for each open set $\mathbb{B}$ of X containing $G(x_0)$, there exists an open neighborhood $\mathcal{N}$ of x_0 such that $G(\mathcal{N}) \subseteq \mathbb{B}$. G is said to be completely continuous if $G(\mathbb{B})$ is relatively compact for every bounded subset $\mathbb{B}$ of X. If the multi-valued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph, *i.e.*, $x_n \to x_*$, $y_n \to y_*$, $y_n \in G(x_n)$ imply $y_* \in G(x_*)$. In the following study, $BCC(X)$ denotes the set of all nonempty bounded, closed and convex subsets of X. G has a fixed point if there is $x \in X$ such that $x \in G(x)$.

Let us record some definitions on fractional calculus [15–18].

Definition 2.1 For an at least $(n-1)$-times continuously differentiable function $g : [0,\infty) \to \mathbb{R}$, the Caputo derivative of fractional order q is defined as

$$^cD^q g(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} g^{(n)}(s)\,ds, \quad n-1 < q \leq n, q > 0,$$

where Γ denotes the gamma function.

Definition 2.2 The Riemann-Liouville fractional integral of order q for a function g is defined as

$$I^q g(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{g(s)}{(t-s)^{1-q}}\,ds, \quad q > 0,$$

provided the right-hand side is pointwise defined on $(0,\infty)$.

To define the solution for (1.1), we consider the following lemma. We do not provide the proof of this lemma as it employs the standard arguments.

Lemma 2.3 *For a given $y \in C([0,T],\mathbb{R})$, the solution of the boundary value problem*

$$\begin{cases} ^cD^q x(t) = y(t), & t \in [0,T], T > 0, 2 < q \leq 3, \\ x(0) = 0, & x(T) = \mu_1 \int_0^T g(s,x(s))\,ds, \\ x'(0) - \lambda x'(T) = \mu_2 \int_0^T h(s,x(s))\,ds, \end{cases} \tag{2.1}$$

is given by the integral equation

$$\begin{aligned} x(t) = {} & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} y(s)\,ds + \frac{\lambda t(T-t)}{T(1+\lambda)} \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} y(s)\,ds \\ & - \frac{t[t+\lambda(2T-t)]}{T^2(1+\lambda)} \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} y(s)\,ds \\ & + \frac{\mu_1 t[t+\lambda(2T-t)]}{T^2(1+\lambda)} \int_0^T g\bigl(s,x(s)\bigr)\,ds + \frac{\mu_2 t(T-t)}{T(1+\lambda)} \int_0^T h\bigl(s,x(s)\bigr)\,ds. \end{aligned} \tag{2.2}$$

In view of Lemma 2.3, a function $x \in AC^2([0,T],\mathbb{R})$ is a solution of the problem (1.1) if there exists a function $f \in L^1([0,T],\mathbb{R})$ such that $f(t) \in F(t,x)$ a.e. on $[0,T]$ and

$$\begin{aligned} x(t) = {} & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)\,ds + \frac{\lambda t(T-t)}{T(1+\lambda)} \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f(s)\,ds \\ & - \frac{t[t+\lambda(2T-t)]}{T^2(1+\lambda)} \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f(s)\,ds \\ & + \frac{\mu_1 t[t+\lambda(2T-t)]}{T^2(1+\lambda)} \int_0^T g\bigl(s,x(s)\bigr)\,ds + \frac{\mu_2 t(T-t)}{T(1+\lambda)} \int_0^T h\bigl(s,x(s)\bigr)\,ds. \end{aligned} \tag{2.3}$$

Now we state the following lemmas which are necessary to establish the main result.

Lemma 2.4 (Bohnenblust-Karlin [19]) *Let D be a nonempty subset of a Banach space X, which is bounded, closed and convex. Suppose that $G : D \to 2^X \setminus \{0\}$ is u.s.c. with closed, convex values such that $G(D) \subset D$ and $\overline{G(D)}$ is compact. Then G has a fixed point.*

Lemma 2.5 [20] *Let I be a compact real interval. Let F be a multi-valued map satisfying* (A_1) *and let Θ be linear continuous from $L^1(I,\mathbb{R}) \to C(I)$, then the operator $\Theta \circ S_F : C(I) \to BCC(C(I))$, $x \mapsto (\Theta \circ S_F)(x) = \Theta(S_{F,x})$ is a closed graph operator in $C(I) \times C(I)$.*

For the forthcoming analysis, we need the following assumptions:

- (A_1) Let $F : [0,T] \times \mathbb{R} \to BCC(\mathbb{R})$; $(t,x) \to F(t,x)$ be measurable with respect to t for each $x \in \mathbb{R}$, u.s.c. with respect to x for a.e. $t \in [0,T]$, and for each fixed $x \in \mathbb{R}$, the set $S_{F,x} := \{f \in L^1([0,T],\mathbb{R}) : f(t) \in F(t,x)$ for a.e. $t \in [0,T]\}$ is nonempty.
- (A_2) For each $r > 0$, there exists a function $m_r, p_r, \overline{p}_r \in L^1([0,T],\mathbb{R}_+)$ such that $\|F(t,x)\| = \sup\{|v| : v(t) \in F(t,x)\} \le m_r(t)$, $\|g(t,x)\| \le p_r(t)$, $\|h(t,x)\| \le \overline{p}_r(t)$ for each $(t,x) \in [0,T] \times \mathbb{R}$ with $|x| \le r$, and

$$\liminf_{r \to +\infty} \left(\frac{\zeta_r}{r}\right) = \gamma < \infty, \tag{2.4}$$

where $\zeta_r = \max\{\|m_r\|_{L^1}, \|p_r\|_{L^1}, \|\overline{p}_r\|_{L^1}\}$.

Furthermore, we set

$$\begin{aligned} & \max_{t \in [0,T]} \left| \frac{t[t+\lambda(2T-t)]}{T^2(1+\lambda)} \right| = \frac{|1-\lambda| + 2|\lambda|}{|1+\lambda|} := \delta_1, \\ & \max_{t \in [0,T]} \left| \frac{t(T-t)}{T(1+\lambda)} \right| = \frac{T}{|1+\lambda|} := \delta_2, \\ & \Lambda_2 = \frac{T^{q-1}}{\Gamma(q)} \bigl\{1 + |\lambda|\delta_2(q-1)T^{-1} + \delta_1\bigr\}. \end{aligned} \tag{2.5}$$

3 Main result

Theorem 3.1 *Suppose that the assumptions* (A_1) *and* (A_2) *are satisfied, and*

$$\gamma < \Lambda, \tag{3.1}$$

where γ is given by (2.4) *and*

$$\left(\Lambda_2 + |\mu_1|\delta_1 + |\mu_2|\delta_2 T\right)^{-1} = \Lambda.$$

Then the boundary value problem (1.1) *has at least one solution on* $[0, T]$.

Proof In order to transform the problem (1.1) into a fixed point problem, we define a multi-valued map $N : C([0, T], \mathbb{R}) \to 2^{C([0,T],\mathbb{R})}$ as

$$N(x) = \left\{ h \in C\big([0, T], \mathbb{R}\big) : h(t) = \left\{ \begin{array}{l} \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)\,ds + \frac{\lambda t(T-t)}{T(1+\lambda)} \int_0^T \frac{(t-s)^{q-2}}{\Gamma(q-1)} f(s)\,ds \\ \quad - \frac{t[t+\lambda(2T-t)]}{T^2(1+\lambda)} \int_0^T \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)\,ds \\ \quad + \frac{\mu_1 t[t+\lambda(2T-t)]}{T^2(1+\lambda)} \int_0^T g(s, x(s))\,ds \\ \quad + \frac{\mu_2 t(T-t)}{T(1+\lambda)} \int_0^T h(s, x(s))\,ds, \quad t \in [0, T], f \in S_{F,x} \end{array} \right. \right\}.$$

Now we prove that the multi-valued map N satisfies all the assumptions of Lemma 2.4, and thus N has a fixed point which is a solution of the problem (1.1). In the first step, we show that $N(x)$ is convex for each $x \in C([0, T], \mathbb{R})$. For that, let $h_1, h_2 \in N(x)$. Then there exist $f_1, f_2 \in S_{F,x}$ such that for each $t \in [0, T]$, we have

$$\begin{aligned} h_i(t) = {} & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f_i(s)\,ds + \frac{\lambda t(T-t)}{T(1+\lambda)} \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f_i(s)\,ds \\ & - \frac{t[t+\lambda(2T-t)]}{T^2(1+\lambda)} \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f_i(s)\,ds + \frac{\mu_1 t[t+\lambda(2T-t)]}{T^2(1+\lambda)} \int_0^T g\big(s, x(s)\big)\,ds \\ & + \frac{\mu_2 t(T-t)}{T(1+\lambda)} \int_0^T h\big(s, x(s)\big)\,ds, \quad i = 1, 2. \end{aligned}$$

Let $0 \le \vartheta \le 1$. Then, for each $t \in [0, T]$, we have

$$\begin{aligned} & \big[\vartheta h_1 + (1 - \vartheta h_2\big](t) \\ & \quad = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \big[\vartheta f_1(s) + (1-\vartheta) f_2(s)\big]\,ds \\ & \qquad + \frac{\lambda t(T-t)}{T(1+\lambda)} \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} \big[\vartheta f_1(s) + (1-\vartheta) f_2(s)\big]\,ds \\ & \qquad - \frac{t[t+\lambda(2T-t)]}{T^2(1+\lambda)} \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} \big[\vartheta f_1(s) + (1-\vartheta) f_2(s)\big]\,ds \\ & \qquad + \frac{\mu_1 t[t+\lambda(2T-t)]}{T^2(1+\lambda)} \int_0^T g\big(s, x(s)\big)\,ds \\ & \qquad + \frac{\mu_2 t(T-t)}{T(1+\lambda)} \int_0^T h\big(s, x(s)\big)\,ds. \end{aligned}$$

Since $S_{F,x}$ is convex (F has convex values), therefore it follows that $\lambda h_1 + (1-\vartheta)h_2 \in N(x)$.

Next it will be shown that there exists a positive number r such that $N(B_r) \subseteq B_r$, where $B_r = \{x \in C([0, T]) : \|x\| \le r\}$. Clearly B_r is a bounded closed convex set in $C([0, T])$ for each positive constant r. If it is not true, then for each positive number r, there exists a function $x_r \in B_r$, $h_r \in N(x_r)$ with $\|N(x_r)\| > r$, and

$$\begin{aligned} h_r(t) = {} & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f_r(s)\,ds + \frac{\lambda t(T-t)}{T(1+\lambda)} \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f_r(s)\,ds \\ & - \frac{t[t+\lambda(2T-t)]}{T^2(1+\lambda)} \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f_r(s)\,ds + \frac{\mu_1 t[t+\lambda(2T-t)]}{T^2(1+\lambda)} \int_0^T g\bigl(s, x_r(s)\bigr)\,ds \\ & + \frac{\mu_2 t(T-t)}{T(1+\lambda)} \int_0^T h\bigl(s, x_r(s)\bigr)\,ds \quad \text{for some } f_r \in S_{F,x_r}. \end{aligned}$$

On the other hand, using (A_2), we have

$$\begin{aligned} r & < \bigl\|N(x_r)\bigr\| \\ & \le \left\{ \frac{T^{q-1}}{\Gamma(q)} + |\lambda|\delta_2 \frac{T^{q-2}}{\Gamma(q-1)} + \delta_1 \frac{T^{q-1}}{\Gamma(q)} \right\} \int_0^T m_r(s)\,ds \\ & \quad + |\mu_1|\delta_1 \int_0^T p_r(s)\,ds + |\mu_2|\delta_2 T \int_0^T \overline{p}_r(s)\,ds \\ & \le \Lambda_2 \|m_r\|_{L^1} + |\mu_1|\delta_1 \|p_r\|_{L^1} + |\mu_2|\delta_2 T \|\overline{p}_r\|_{L^1} \\ & \le \zeta_r\bigl(\Lambda_2 + |\mu_1|\delta_1 + |\mu_2|\delta_2 T\bigr). \end{aligned}$$

Dividing both sides by r and taking the lower limit as $r \to \infty$, we find that

$$\gamma \ge \bigl(\Lambda_2 + |\mu_1|\delta_1 + |\mu_2|\delta_2 T\bigr)^{-1} = \Lambda,$$

which contradicts (3.1). Hence there exists a positive number r' such that $N(Br') \subseteq Br'$.

Now we show that $N(B_{r'})$ is equi-continuous. Let $t', t'' \in [0, T]$ with $t' < t''$. Let $x \in B_{r'}$ and $h \in N(x)$, then there exists $f \in S_{F,x}$ such that for each $t \in [0, T]$, we have

$$\begin{aligned} h(t) = {} & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)\,ds + \frac{\lambda t(T-t)}{T(1+\lambda)} \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f(s)\,ds \\ & - \frac{t[t+\lambda(2T-t)]}{T^2(1+\lambda)} \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f(s)\,ds + \frac{\mu_1 t[t+\lambda(2T-t)]}{T^2(1+\lambda)} \int_0^T g\bigl(s, x(s)\bigr)\,ds \\ & + \frac{\mu_2 t(T-t)}{T(1+\lambda)} \int_0^T h\bigl(s, x(s)\bigr)\,ds, \end{aligned}$$

from which we obtain

$$\begin{aligned} \bigl|h\bigl(t''\bigr) - h\bigl(t'\bigr)\bigr| \le {} & \left| \int_0^{t'} \left(\frac{(t''-s)^{q-1} - (t'-s)^{q-1}}{\Gamma(q)} \right) \bigl|f(s)\bigr|\,ds + \int_{t'}^{t''} \frac{(t''-s)^{q-1}}{\Gamma(q)} \bigl|f(s)\bigr|\,ds \right| \\ & + \frac{|\lambda||t''-t'||T-t''-t'|}{T|1+\lambda|} \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} \bigl|f(s)\bigr|\,ds \\ & + \frac{|t''-t'||(1-\lambda)t'' + (1-\lambda)t' + 2\lambda T|}{T^2|1+\lambda|} \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \bigl|f(s)\bigr|\,ds \\ & + \frac{|t''-t'||(1-\lambda)t'' + (1-\lambda)t' + 2\lambda T|}{T^2|1+\lambda|} \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} \bigl|g\bigl(s, x(s)\bigr)\bigr|\,ds \\ & + \frac{|\lambda||t''-t'||T-t''-t'|}{T|1+\lambda|} \int_0^T \bigl|h\bigl(s, x(s)\bigr)\bigr|\,ds \end{aligned}$$

$$
\begin{aligned}
&\leq \left| \int_0^{t'} \left(\frac{(t''-s)^{q-1} - (t'-s)^{q-1}}{\Gamma(q)} \right) m_{r'}\, ds + \int_{t'}^{t''} \frac{(t''-s)^{q-1}}{\Gamma(q)} m_{r'}\, ds \right| \\
&\quad + \frac{|\lambda||t''-t'||T-t''-t'|}{T|1+\lambda|} \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} m_{r'}(s)\, ds \\
&\quad + \frac{|t''-t'||(1-\lambda)t'' + (1-\lambda)t' + 2\lambda T|}{T^2|1+\lambda|} \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} m_{r'}\, ds \\
&\quad + \frac{|t''-t'||(1-\lambda)t'' + (1-\lambda)t' + 2\lambda T|}{T^2|1+\lambda|} \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} p_{r'}(s)\, ds \\
&\quad + \frac{|\lambda||t''-t'||T-t''-t'|}{T|1+\lambda|} \int_0^T \overline{p}_{r'}(s)\, ds.
\end{aligned}
$$

Obviously, the right-hand side of the above inequality tends to zero independently of $x_r \in B_{r'}$ as $t'' \to t'$. Thus, N is equi-continuous.

As N satisfies the above three assumptions, therefore it follows by the Ascoli-Arzelá theorem that N is a compact multi-valued map.

In our next step, we show that N *has a closed graph*. Let $x_n \to x_*$, $h_n \in N(x_n)$ and $h_n \to h_*$. Then we need to show that $h_* \in N(x_*)$. Associated with $h_n \in N(x_n)$, there exists $f_n \in S_{F,x_n}$ such that for each $t \in [0,T]$,

$$
\begin{aligned}
h_n(t) = {} & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f_n(s)\, ds + \frac{\lambda t(T-t)}{T(1+\lambda)} \int_0^T \frac{(t-s)^{q-2}}{\Gamma(q-1)} f_n(s)\, ds \\
& - \frac{t[t+\lambda(2T-t)]}{T^2(1+\lambda)} \int_0^T \frac{(t-s)^{q-1}}{\Gamma(q)} f_n(s)\, ds \\
& + \frac{\mu_1 t[t+\lambda(2T-t)]}{T^2(1+\lambda)} \int_0^T g\bigl(s, x_n(s)\bigr)\, ds + \frac{\mu_2 t(T-t)}{T(1+\lambda)} \int_0^T h\bigl(s, x_n(s)\bigr)\, ds.
\end{aligned}
$$

Thus we have to show that there exists $f_* \in S_{F,x_*}$ such that for each $t \in [0,T]$,

$$
\begin{aligned}
h_*(t) = {} & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f_*(s)\, ds + \frac{\lambda t(T-t)}{T(1+\lambda)} \int_0^T \frac{(t-s)^{q-2}}{\Gamma(q-1)} f_*(s)\, ds \\
& - \frac{t[t+\lambda(2T-t)]}{T^2(1+\lambda)} \int_0^T \frac{(t-s)^{q-1}}{\Gamma(q)} f_*(s)\, ds \\
& + \frac{\mu_1 t[t+\lambda(2T-t)]}{T^2(1+\lambda)} \int_0^T g\bigl(s, x_*(s)\bigr)\, ds + \frac{\mu_2 t(T-t)}{T(1+\lambda)} \int_0^T h\bigl(s, x_*(s)\bigr)\, ds.
\end{aligned}
$$

Let us consider the continuous linear operator $\Theta : L^1([0,T],\mathbb{R}) \to C([0,T],\mathbb{R})$ given by

$$
\begin{aligned}
& f \mapsto \Theta(f)(t) \\
& \quad = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)\, ds + \frac{\lambda t(T-t)}{T(1+\lambda)} \int_0^T \frac{(t-s)^{q-2}}{\Gamma(q-1)} f(s)\, ds \\
& \qquad - \frac{t[t+\lambda(2T-t)]}{T^2(1+\lambda)} \int_0^T \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)\, ds \\
& \qquad + \frac{\mu_1 t[t+\lambda(2T-t)]}{T^2(1+\lambda)} \int_0^T g\bigl(s, x(s)\bigr)\, ds + \frac{\mu_2 t(T-t)}{T(1+\lambda)} \int_0^T h\bigl(s, x(s)\bigr)\, ds.
\end{aligned}
$$

Observe that

$$\begin{aligned}
&\|h_n(t) - h_*(t)\| \\
&= \bigg\| \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \big(f_n(s) - f_*(s)\big)\,ds + \frac{\lambda t(T-t)}{T(1+\lambda)} \int_0^T \frac{(t-s)^{q-2}}{\Gamma(q-1)} f\big(s, x(s)\big)\big(f_n(s) - f_*(s)\big)\,ds \\
&\quad - \frac{t[t+\lambda(2T-t)]}{T^2(1+\lambda)} \int_0^T \frac{(t-s)^{q-1}}{\Gamma(q)} \big(f_n(s) - f_*(s)\big)\,ds \\
&\quad + \frac{\mu_1 t[t+\lambda(2T-t)]}{T^2(1+\lambda)} \int_0^T \big(g\big(s, x_n(s)\big) - g\big(s, x_*(s)\big)\big)\,ds \\
&\quad + \frac{\mu_2 t(T-t)}{T(1+\lambda)} \int_0^T \big(h\big(s, x_n(s)\big) - h\big(s, x_*(s)\big)\big)\,ds \bigg\| \to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

Thus, it follows by Lemma 2.5 that $\Theta \circ S_F$ is a closed graph operator. Further, we have $h_n(t) \in \Theta(S_{F,x_n})$. Since $x_n \to x_*$, therefore, Lemma 2.5 yields

$$\begin{aligned}
h_*(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f_*(s)\,ds + \frac{\lambda t(T-t)}{T(1+\lambda)} \int_0^T \frac{(t-s)^{q-2}}{\Gamma(q-1)} f_*(s)\,ds \\
&\quad - \frac{t[t+\lambda(2T-t)]}{T^2(1+\lambda)} \int_0^T \frac{(t-s)^{q-1}}{\Gamma(q)} f_*(s)\,ds \\
&\quad + \frac{\mu_1 t[t+\lambda(2T-t)]}{T^2(1+\lambda)} \int_0^T g\big(s, x_*(s)\big)\,ds + \frac{\mu_2 t(T-t)}{T(1+\lambda)} \int_0^T h\big(s, x_*(s)\big)\,ds
\end{aligned}$$

for some $f_* \in S_{F,x_*}$.

Hence, we conclude that N is a compact multi-valued map, u.s.c. with convex closed values. Thus, all the assumptions of Lemma 2.4 are satisfied. Hence the conclusion of Lemma 2.4 applies and, in consequence, N has a fixed point x which is a solution of the problem (1.1). This completes the proof. □

Special cases

By fixing the parameters in the boundary conditions of (1.1), we obtain some new results. As the first case, by taking $\mu_1 = 0$, $\lambda = 0$, $\mu_2 = 0$, our main result with $\Lambda = \Gamma(q)/2T^{q-1}$ corresponds to the problem

$$\begin{cases} {}^cD^q x(t) \in F(t, x(t)), & t \in [0, T], T > 0, 2 < q \le 3, \\ x(0) = 0, \qquad x'(0) = 0, \qquad x(T) = 0. \end{cases}$$

In case we fix $\mu_1 = 0$, $\lambda = 1$, $\mu_2 = 0$, we obtain a new result for the problem

$$\begin{cases} {}^cD^q x(t) \in F(t, x(t)), & t \in [0, T], T > 0, 2 < q \le 3, \\ x(0) = 0, \qquad x(T) = 0, \qquad x'(0) = x'(T), \end{cases}$$

with $\Lambda = 2\Gamma(q)/(3+q)T^{q-1}$.

Discussion

As an application of Theorem 3.1, we discuss two cases for nonlinearities $F(t,x)$, $g(t,x)$, $h(t,x)$: (a) sub-linear growth in the second variable of the nonlinearities; (b) linear growth in the second variable (state variable). In case of sub-linear growth, there exist functions $\eta_i(t), \rho_i(t) \in L^1([0,T],\mathbb{R}_+)$, $\mu_i \in [0,1)$ with $i = 1,2,3$ such that $\|F(t,x)\| \le \eta_1(t)|x|^{\mu_1} + \rho_1(t)$, $\|g(t,x)\| \le \eta_2(t)|x|^{\mu_2} + \rho_2(t)$, $\|h(t,x)\| \le \eta_3(t)|x|^{\mu_3} + \rho_3(t)$ for each $(t,x) \in [0,T] \times \mathbb{R}$. In this case, $m_r(t) = \eta_1(t)r^{\mu_1} + \rho_1(t)$, $p_r(t) = \eta_2(t)r^{\mu_2} + \rho_2(t)$, $\overline{p}_r(t) = \eta_3(t)r^{\mu_3} + \rho_3(t)$, and the condition (3.1) is $0 < \Lambda$. For the linear growth, the nonlinearities F, g, h satisfy the relation $\|F(t,x)\| \le \eta_1(t)|x| + \rho_1(t)$, $p_r(t) = \eta_2(t)|x| + \rho_2(t)$, $\overline{p}_r(t) = \eta_3(t)|x| + \rho_3(t)$ for each $(t,x) \in [0,T] \times \mathbb{R}$. In this case $m_r(t) = \eta_1(t)r + \rho_1(t)$, $p_r(t) = \eta_2(t)r + \rho_2(t)$, $\overline{p}_r(t) = \eta_3(t)r + \rho_3(t)$, and the condition (3.1) becomes $\max\{\|\eta_1\|_{L^1}, \|\eta_2\|_{L^1}, \|\eta_3\|_{L^1}\} < \Lambda$. In both cases, the boundary value problem (1.1) has at least one solution on $[0,T]$.

Example 3.2 (linear growth case) Consider the following problem:

$$\begin{cases} {}^cD^{5/2}x(t) \in F(t,x(t)), \quad t \in [0,1], \\ x(0) = 0, \qquad x(1) = \int_0^1 g(s,x(s))\,ds, \qquad x'(0) - \frac{1}{2}x'(T) = \frac{1}{2}\int_0^1 h(s,x(s))\,ds, \end{cases} \tag{3.2}$$

where $q = 5/2$, $T = 1$, $\mu_1 = 1$, $\lambda = 1/2$, $\mu_2 = 1/2$, and

$$\|F(t,x)\| \le \frac{1}{2(1+t)^2}|x| + e^{-t}, \qquad \|g(t,x)\| \le \frac{1}{4(1+t)}|x| + 1,$$
$$\|h(t,x)\| \le \frac{e^t}{(1+4e^t)}|x| + t + 1.$$

With the given data, $\delta_1 = 1$, $\delta_2 = 2/3$, $\Lambda_2 = 10/3\sqrt{\pi}$,

$$\gamma = \max\{\|\eta_1\|_{L^1}, \|\eta_2\|_{L^1}, \|\eta_3\|_{L^1}\} = \max\left\{\frac{1}{4}, \frac{1}{4}\ln 2, \frac{1}{4}\big(\ln(1+4e) - \ln 5\big)\right\} = \frac{1}{4},$$
$$\Lambda = \big(\Lambda_2 + |\mu_1|\delta_1 + |\mu_2|\delta_2 T\big)^{-1} = \frac{3\sqrt{\pi}}{2(5+2\sqrt{\pi})}.$$

Clearly, $\gamma < \Lambda$. Thus, by Theorem 3.1, the problem (3.2) has at least one solution on $[0,1]$.

Example 3.3 (sub-linear growth case) Letting $\|F(t,x)\| \le \frac{1}{2(1+t)^2}|x|^{1/3} + e^{-t}$, $\|g(t,x)\| \le \frac{1}{4(1+t)}|x|^{1/4} + 1$, $\|h(t,x)\| \le \frac{e^t}{(1+4e^t)}|x|^{1/2} + t + 1$ in Example 3.2, we find that $0 = \gamma < \Lambda$. Hence there exits a solution for the sub-linear case of the problem (3.2) by Theorem 3.1.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Each of the authors, BA, SKN and AA contributed to each part of this work equally and read and approved the final version of the manuscript.

Author details
[1]Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah, 21589, Saudi Arabia. [2]Department of Mathematics, University of Ioannina, Ioannina, 451 10, Greece.

Acknowledgements
This research was partially supported by Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah, Saudi Arabia.

Received: 25 November 2012 Accepted: 26 April 2013 Published: 19 June 2013

References

1. Ouahab, A: Some results for fractional boundary value problem of differential inclusions. Nonlinear Anal. **69**, 3877-3896 (2008)
2. Henderson, J, Ouahab, A: Fractional functional differential inclusions with finite delay. Nonlinear Anal. **70**, 2091-2105 (2009)
3. Ahmad, B, Otero-Espinar, V: Existence of solutions for fractional differential inclusions with anti-periodic boundary conditions. Bound. Value Probl. **2009**, Art. ID 625347 (2009)
4. Agarwal, RP, Belmekki, M, Benchohra, M: A survey on semilinear differential equations and inclusions involving Riemann-Liouville fractional derivative. Adv. Differ. Equ. **2009**, Art. ID 981728 (2009)
5. Chang, Y-K, Nieto, JJ: Some new existence results for fractional differential inclusions with boundary conditions. Math. Comput. Model. **49**, 605-609 (2009)
6. Ahmad, B, Ntouyas, SK: Some existence results for boundary value problems of fractional differential inclusions with non-separated boundary conditions. Electron. J. Qual. Theory Differ. Equ. **2010**, 71 (2010)
7. Ahmad, B: Existence results for fractional differential inclusions with separated boundary conditions. Bull. Korean Math. Soc. **47**, 805-813 (2010)
8. Cernea, A: On the existence of solutions for nonconvex fractional hyperbolic differential inclusions. Commun. Math. Anal. **9**(1), 109-120 (2010)
9. Agarwal, RP, Ahmad, B: Existence theory for anti-periodic boundary value problems of fractional differential equations and inclusions. Comput. Math. Appl. **62**, 1200-1214 (2011)
10. Ahmad, B, Nieto, JJ, Pimentel, J: Some boundary value problems of fractional differential equations and inclusions. Comput. Math. Appl. **62**, 1238-1250 (2011)
11. Deimling, K: Multivalued Differential Equations. de Gruyter, Berlin (1992)
12. Hu, S, Papageorgiou, N: Handbook of Multivalued Analysis, Volume I: Theory. Kluwer Academic, Dordrecht (1997)
13. Kisielewicz, M: Differential Inclusions and Optimal Control. Kluwer Academic, Dordrecht (1991)
14. Smirnov, GV: Introduction to the Theory of Differential Inclusions. Am. Math. Soc., Providence (2002)
15. Samko, SG, Kilbas, AA, Marichev, OI: Fractional Integrals and Derivatives: Theory and Applications. Gordon & Breach, Yverdon (1993)
16. Podlubny, I: Fractional Differential Equations. Academic Press, San Diego (1999)
17. Kilbas, AA, Srivastava, HM, Trujillo, JJ: Theory and Applications of Fractional Differential Equations. North-Holland Mathematics Studies, vol. 204. Elsevier, Amsterdam (2006)
18. Lakshmikantham, V, Leela, S, Vasundhara Devi, J: Theory of Fractional Dynamic Systems. Cambridge Academic Publishers, Cambridge (2009)
19. Bohnenblust, HF, Karlin, S: On a theorem of Ville. In: Contributions to the Theory of Games, vol. I, pp. 155-160. Princeton University Press, Princeton (1950)
20. Lasota, A, Opial, Z: An application of the Kakutani-Ky Fan theorem in the theory of ordinary differential equations. Bull. Acad. Pol. Sci., Sér. Sci. Math. Astron. Phys. **13**, 781-786 (1965)

doi:10.1186/1029-242X-2013-296
Cite this article as: Ahmad et al.: **An existence result for fractional differential inclusions with nonlinear integral boundary conditions.** *Journal of Inequalities and Applications* 2013 **2013**:296.