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On a half-discrete Hilbert-type inequality similar to Mulholland's inequality

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Abstract

By using the way of weight functions and Hadamard's inequality, a half-discrete Hilbert-type inequality similar to Mulholland's inequality with a best constant factor is given. The extension with multi-parameters, the equivalent forms as well as the operator expressions are also considered.

MSC: 26D15

Keywords: Hilbert-type inequality; weight function; equivalent form

1 Introduction

Assuming that $f,g \in L^2(R_+)$, $\|f\| = \{\int_0^\infty f^2(x)\,dx\}^{\frac{1}{2}} > 0$, $\|g\| > 0$, we have the following Hilbert integral inequality (*cf.* [1]):

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}\,dx\,dy < \pi \|f\|\|g\|, \tag{1}$$

where the constant factor π is the best possible. If $a = \{a_n\}_{n=1}^\infty$, $b = \{b_n\}_{n=1}^\infty \in l^2$, $\|a\| = \{\sum_{n=1}^\infty a_n^2\}^{\frac{1}{2}} > 0$, $\|b\| > 0$, then we still have the following discrete Hilbert inequality:

$$\sum_{m=1}^\infty \sum_{n=1}^\infty \frac{a_m b_n}{m+n} < \pi \|a\|\|b\|, \tag{2}$$

with the same best constant factor π. Inequalities (1) and (2) are important in analysis and its applications (*cf.* [2–4]). Also we have the following Mulholland inequality with the same best constant factor (*cf.* [1, 5]):

$$\sum_{m=2}^\infty \sum_{n=2}^\infty \frac{a_m b_n}{\ln mn} < \pi \left\{ \sum_{m=2}^\infty m a_m^2 \sum_{n=2}^\infty n b_n^2 \right\}^{\frac{1}{2}}. \tag{3}$$

In 1998, by introducing an independent parameter $\lambda \in (0,1]$, Yang [6] gave an extension of (1). By generalizing the results from [6], Yang [7] gave some best extensions of (1) and (2) as follows: If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda_1 + \lambda_2 = \lambda$, $k_\lambda(x,y)$ is a non-negative homogeneous function of degree $-\lambda$ with $k(\lambda_1) = \int_0^\infty k_\lambda(t,1)t^{\lambda_1-1}\,dt \in R_+$, $\phi(x) = x^{p(1-\lambda_1)-1}$, $\psi(x) = x^{q(1-\lambda_2)-1}$, $f\ (\geq 0) \in L_{p,\phi}(R_+) = \{f \mid \|f\|_{p,\phi} := \{\int_0^\infty \phi(x)|f(x)|^p\,dx\}^{\frac{1}{p}} < \infty\}$, $g\ (\geq 0) \in L_{q,\psi}(R_+)$,

$\|f\|_{p,\phi}, \|g\|_{q,\psi} > 0$, then

$$\int_0^\infty \int_0^\infty k_\lambda(x,y)f(x)g(y)\,dx\,dy < k(\lambda_1)\|f\|_{p,\phi}\|g\|_{q,\psi}, \tag{4}$$

where the constant factor $k(\lambda_1)$ is the best possible. Moreover, if $k_\lambda(x,y)$ is finite and $k_\lambda(x,y)x^{\lambda_1-1}$ ($k_\lambda(x,y)y^{\lambda_2-1}$) is decreasing for $x > 0$ ($y > 0$), then for $a_m, b_n \geq 0$, $a = \{a_m\}_{m=1}^\infty \in l_{p,\phi} = \{a \mid \|a\|_{p,\phi} := \{\sum_{n=1}^\infty \phi(n)|a_n|^p\}^{\frac{1}{p}} < \infty\}$, $b = \{b_n\}_{n=1}^\infty \in l_{q,\psi}$, $\|a\|_{p,\phi}, \|b\|_{q,\psi} > 0$, we have

$$\sum_{m=1}^\infty \sum_{n=1}^\infty k_\lambda(m,n)a_m b_n < k(\lambda_1)\|a\|_{p,\phi}\|b\|_{q,\psi}, \tag{5}$$

with the same best constant factor $k(\lambda_1)$. Clearly, for $p = q = 2$, $\lambda = 1$, $k_1(x,y) = \frac{1}{x+y}$, $\lambda_1 = \lambda_2 = \frac{1}{2}$, (4) reduces to (1), while (5) reduces to (2). Some other results about Hilbert-type inequalities are provided by [5, 8–16].

On the topic of half-discrete Hilbert-type inequalities with the general non-homogeneous kernels, Hardy *et al.* provided a few results in Theorem 351 of [1]. But they did not prove that the constant factors in the inequalities are the best possible. Moreover, Yang [17] gave an inequality with the particular kernel $\frac{1}{(1+nx)^\lambda}$ and an interval variable, and proved that the constant factor is the best possible. Recently, [18] and [19] gave the following half-discrete Hilbert inequality with the best constant factor π:

$$\int_0^\infty f(x)\sum_{n=1}^\infty \frac{a_n}{(x+n)^\lambda}\,dx < \pi\|f\|\|a\|. \tag{6}$$

In this paper, by using the way of weight functions and Hadamard's inequality, a half-discrete Hilbert-type inequality similar to (3) and (6) with the best constant factor is given as follows:

$$\int_0^\infty f(x)\sum_{n=1}^\infty \frac{a_n}{\ln e(n+\frac{1}{2})^x}\,dx < \pi\|f\|\left\{\sum_{n=1}^\infty \left(n+\frac{1}{2}\right)a_n^2\right\}^{\frac{1}{2}}. \tag{7}$$

Moreover, the best extension of (7) with multi-parameters, some equivalent forms as well as the operator expressions are considered.

2 Some lemmas

Lemma 1 *If* $0 < \lambda \leq 2$, $\alpha \geq \frac{1}{2}$, *setting weight functions* $\omega(n)$ *and* $\varpi(x)$ *as follows*:

$$\omega(n) := \ln^{\frac{\lambda}{2}}(n+\alpha)\int_0^\infty \frac{x^{\frac{\lambda}{2}-1}}{\ln^\lambda e(n+\alpha)^x}\,dx, \quad n \in \mathbf{N}, \tag{8}$$

$$\varpi(x) := x^{\frac{\lambda}{2}}\sum_{n=1}^\infty \frac{\ln^{\frac{\lambda}{2}-1}(n+\alpha)}{(n+\alpha)\ln^\lambda e(n+\alpha)^x}, \quad x \in (0,\infty), \tag{9}$$

we have

$$\varpi(x) < \omega(n) = B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right). \tag{10}$$

Proof Substitution of $t = x\ln(n+\alpha)$ in (8), by calculation, yields

$$\omega(n) = \int_0^\infty \frac{1}{(1+t)^\lambda} t^{\frac{\lambda}{2}-1}\,dt = B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right).$$

Since, for fixed $x > 0$ and in view of the conditions,

$$\begin{aligned} h(x,y) &:= \frac{\ln^{\frac{\lambda}{2}-1}(y+\alpha)}{(y+\alpha)\ln^\lambda e(y+\alpha)^x} \\ &= \frac{\ln^{\frac{\lambda}{2}-1}(y+\alpha)}{(y+\alpha)[1+x\ln(y+\alpha)]^\lambda} \end{aligned}$$

is decreasing and strictly convex for $y \in (\frac{1}{2},\infty)$, then by Hadamard's inequality (*cf.* [20]), we find

$$\begin{aligned} \varpi(x) &< x^{\frac{\lambda}{2}} \int_{\frac{1}{2}}^\infty \frac{\ln^{\frac{\lambda}{2}-1}(y+\alpha)}{(y+\alpha)[1+x\ln(y+\alpha)]^\lambda}\,dy \\ &\overset{t=x\ln(y+\alpha)}{=} \int_{x\ln(\frac{1}{2}+\alpha)}^\infty \frac{t^{\frac{\lambda}{2}-1}}{(1+t)^\lambda}\,dt \le B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right), \end{aligned}$$

namely, (10) follows. □

Lemma 2 *Let the assumptions of Lemma* 1 *be fulfilled and, additionally, let* $p > 1$, $\frac{1}{p}+\frac{1}{q} = 1$, $a_n \ge 0$, $n \in \mathbf{N}$, $f(x)$ *be a non-negative measurable function in* $(0,\infty)$. *Then we have the following inequalities*:

$$\begin{aligned} J &:= \left\{\sum_{n=1}^\infty \frac{\ln^{\frac{p\lambda}{2}-1}(n+\alpha)}{n+\alpha}\left[\int_0^\infty \frac{f(x)}{\ln^\lambda e(n+\alpha)^x}\,dx\right]^p\right\}^{\frac{1}{p}} \\ &\le \left[B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\right]^{\frac{1}{q}}\left\{\int_0^\infty \varpi(x)x^{p(1-\frac{\lambda}{2})-1}f^p(x)\,dx\right\}^{\frac{1}{p}}, \qquad (11) \\ L_1 &:= \left\{\int_0^\infty \frac{x^{\frac{q\lambda}{2}-1}}{[\varpi(x)]^{q-1}}\left[\sum_{n=1}^\infty \frac{a_n}{\ln^\lambda e(n+\alpha)^x}\right]^q dx\right\}^{\frac{1}{q}} \\ &\le \left\{B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\sum_{n=1}^\infty (n+\alpha)^{q-1}\ln^{q(1-\frac{\lambda}{2})-1}(n+\alpha)a_n^q\right\}^{\frac{1}{q}}. \qquad (12) \end{aligned}$$

Proof By Hölder's inequality (*cf.* [20]) and (10), it follows

$$\begin{aligned} &\left[\int_0^\infty \frac{f(x)\,dx}{\ln^\lambda e(n+\alpha)^x}\right]^p \\ &\quad = \left\{\int_0^\infty \frac{1}{\ln^\lambda e(n+\alpha)^x}\left[\frac{x^{(1-\frac{\lambda}{2})/q}}{\ln^{(1-\frac{\lambda}{2})/p}(n+\alpha)}\frac{f(x)}{(n+\alpha)^{\frac{1}{p}}}\right]\left[\frac{\ln^{(1-\frac{\lambda}{2})/p}(n+\alpha)}{x^{(1-\frac{\lambda}{2})/q}}(n+\alpha)^{\frac{1}{p}}\right]dx\right\}^p \\ &\quad \le \int_0^\infty \frac{\ln^{\frac{\lambda}{2}-1}(n+\alpha)}{\ln^\lambda e(n+\alpha)^x}\frac{x^{(1-\frac{\lambda}{2})(p-1)}f^p(x)\,dx}{n+\alpha}\left\{\int_0^\infty \frac{(n+\alpha)^{q-1}}{\ln^\lambda e(n+\alpha)^x}\frac{\ln^{(1-\frac{\lambda}{2})(q-1)}(n+\alpha)}{x^{1-\frac{\lambda}{2}}}\,dx\right\}^{p-1} \end{aligned}$$

$$= \left\{ \frac{\omega(n)(n+\alpha)^{q-1}}{\ln^{q(\frac{\lambda}{2}-1)+1}(n+\alpha)} \right\}^{p-1} \int_0^\infty \frac{\ln^{\frac{\lambda}{2}-1}(n+\alpha)}{\ln^\lambda e(n+\alpha)^x} \frac{x^{(1-\frac{\lambda}{2})(p-1)} f^p(x)\,dx}{n+\alpha}$$

$$= \left[B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right) \right]^{p-1} \frac{n+\alpha}{\ln^{\frac{p\lambda}{2}-1}(n+\alpha)} \int_0^\infty \frac{\ln^{\frac{\lambda}{2}-1}(n+\alpha)}{\ln^\lambda e(n+\alpha)^x} \frac{x^{(1-\frac{\lambda}{2})(p-1)} f^p(x)\,dx}{n+\alpha}.$$

Then by the Lebesgue term-by-term integration theorem (*cf.* [21]), we have

$$J \le \left[B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right) \right]^{\frac{1}{q}} \left\{ \sum_{n=1}^\infty \int_0^\infty \frac{\ln^{\frac{\lambda}{2}-1}(n+\alpha)}{\ln^\lambda e(n+\alpha)^x} \frac{x^{(1-\frac{\lambda}{2})(p-1)} f^p(x)\,dx}{n+\alpha} \right\}^{\frac{1}{p}}$$

$$= \left[B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right) \right]^{\frac{1}{q}} \left\{ \int_0^\infty \sum_{n=1}^\infty \frac{\ln^{\frac{\lambda}{2}-1}(n+\alpha)}{\ln^\lambda e((n+\alpha))^x} \frac{x^{(1-\frac{\lambda}{2})(p-1)} f^p(x)\,dx}{n+\alpha} \right\}^{\frac{1}{p}}$$

$$= \left[B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right) \right]^{\frac{1}{q}} \left\{ \int_0^\infty \varpi(x) x^{p(1-\frac{\lambda}{2})-1} f^p(x)\,dx \right\}^{\frac{1}{p}},$$

and (11) follows. Still by Hölder's inequality, we have

$$\left[\sum_{n=1}^\infty \frac{a_n}{\ln^\lambda e(n+\alpha)^x} \right]^q$$

$$= \left\{ \sum_{n=1}^\infty \frac{1}{\ln^\lambda e(n+\alpha)^x} \left[\frac{x^{(1-\frac{\lambda}{2})/q}}{\ln^{(1-\frac{\lambda}{2})/p}(n+\alpha)} \frac{1}{(n+\alpha)^{\frac{1}{p}}} \right] \left[\frac{\ln^{(1-\frac{\lambda}{2})/p}(n+\alpha)}{x^{(1-\frac{\lambda}{2})/q}} (n+\alpha)^{\frac{1}{p}} a_n \right] \right\}^q$$

$$\le \left\{ \sum_{n=1}^\infty \frac{\ln^{\frac{\lambda}{2}-1}(n+\alpha)}{\ln^\lambda e(n+\alpha)^x} \frac{x^{(1-\frac{\lambda}{2})(p-1)}}{(n+\alpha)} \right\}^{q-1} \sum_{n=1}^\infty \frac{(n+\alpha)^{q-1}}{\ln^\lambda e(n+\alpha)^x} \frac{\ln^{(1-\frac{\lambda}{2})(q-1)}(n+\alpha)}{x^{1-\frac{\lambda}{2}}} a_n^q$$

$$= \frac{[\varpi(x)]^{q-1}}{x^{\frac{q\lambda}{2}-1}} \sum_{n=1}^\infty \frac{(n+\alpha)^{q-1}}{\ln^\lambda e(n+\alpha)^x} x^{\frac{\lambda}{2}-1} \ln^{(1-\frac{\lambda}{2})(q-1)}(n+\alpha) a_n^q.$$

Then by the Lebesgue term-by-term integration theorem, we have

$$L_1 \le \left\{ \int_0^\infty \sum_{n=1}^\infty \frac{(n+\alpha)^{q-1}}{\ln^\lambda e(n+\alpha)^x} x^{\frac{\lambda}{2}-1} \ln^{(1-\frac{\lambda}{2})(q-1)}(n+\alpha) a_n^q\,dx \right\}^{\frac{1}{q}}$$

$$= \left\{ \sum_{n=1}^\infty \left[\ln^{\frac{\lambda}{2}}(n+\alpha) \int_0^\infty \frac{x^{\frac{\lambda}{2}-1}\,dx}{\ln^\lambda e(n+\alpha)^x} \right] (n+\alpha)^{q-1} \ln^{q(1-\frac{\lambda}{2})-1}(n+\alpha) a_n^q \right\}^{\frac{1}{q}}$$

$$= \left\{ \sum_{n=1}^\infty \omega(n)(n+\alpha)^{q-1} \ln^{q(1-\frac{\lambda}{2})-1}(n+\alpha) a_n^q \right\}^{\frac{1}{q}},$$

and then in view of (10), inequality (12) follows. □

3 Main results

We introduce two functions

$$\Phi(x) := x^{p(1-\frac{\lambda}{2})-1} \quad (x>0) \quad \text{and} \quad \Psi(n) := (n+\alpha)^{q-1} \ln^{q(1-\frac{\lambda}{2})-1}(n+\alpha) \quad (n \in \mathbf{N}),$$

wherefrom, $[\Phi(x)]^{1-q} = x^{\frac{q\lambda}{2}-1}$, and $[\Psi(n)]^{1-p} = \frac{\ln^{\frac{p\lambda}{2}-1}(n+\alpha)}{n+\alpha}$.

Theorem 1 *If* $0 < \lambda \le 2, \alpha \ge \frac{1}{2}, p > 1, \frac{1}{p} + \frac{1}{q} = 1, f(x), a_n \ge 0, f \in L_{p,\Phi}(R_+), a = \{a_n\}_{n=1}^{\infty} \in l_{q,\Psi}$, $\|f\|_{p,\Phi} > 0$ *and* $\|a\|_{q,\Psi} > 0$, *then we have the following equivalent inequalities*:

$$I := \sum_{n=1}^{\infty} \int_0^{\infty} \frac{a_n f(x)\,dx}{\ln^{\lambda} e(n+\alpha)^x} = \int_0^{\infty} \sum_{n=1}^{\infty} \frac{a_n f(x)\,dx}{\ln^{\lambda} e(n+\alpha)^x} < B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \|f\|_{p,\Phi} \|a\|_{q,\Psi}, \tag{13}$$

$$J = \left\{ \sum_{n=1}^{\infty} \bigl[\Psi(n)\bigr]^{1-p} \left[\int_0^{\infty} \frac{f(x)}{\ln^{\lambda} e(n+\alpha)^x}\,dx \right]^p \right\}^{\frac{1}{p}} < B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \|f\|_{p,\Phi}, \tag{14}$$

$$L := \left\{ \int_0^{\infty} \bigl[\Phi(x)\bigr]^{1-q} \left[\sum_{n=1}^{\infty} \frac{a_n}{\ln^{\lambda} e(n+\alpha)^x} \right]^q dx \right\}^{\frac{1}{q}} < B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \|a\|_{q,\Psi}, \tag{15}$$

where the constant $B(\frac{\lambda}{2}, \frac{\lambda}{2})$ *is the best possible in the above inequalities.*

Proof By the Lebesgue term-by-term integration theorem, there are two expressions for I in (13). In view of (11), for $\varpi(x) < B(\frac{\lambda}{2}, \frac{\lambda}{2})$, we have (14). By Hölder's inequality, we have

$$I = \sum_{n=1}^{\infty} \left[\Psi^{\frac{-1}{q}}(n) \int_0^{\infty} \frac{1}{\ln^{\lambda} e(n+\alpha)^x} f(x)\,dx \right] \bigl[\Psi^{\frac{1}{q}}(n) a_n \bigr] \le J \|a\|_{q,\Psi}. \tag{16}$$

Then by (14), we have (13). On the other hand, assuming that (13) is valid, setting

$$a_n := \bigl[\Psi(n)\bigr]^{1-p} \left[\int_0^{\infty} \frac{1}{\ln^{\lambda} e(n+\alpha)^x} f(x)\,dx \right]^{p-1}, \quad n \in \mathbf{N},$$

then $J^{p-1} = \|a\|_{q,\Psi}$. By (11), we find $J < \infty$. If $J = 0$, then (14) is trivially valid; if $J > 0$, then by (13), we have

$$\|a\|_{q,\Psi}^{q} = J^p = I < B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \|f\|_{p,\Phi} \|a\|_{q,\Psi}, \quad \textit{i.e.}$$

$$\|a\|_{q,\Psi}^{q-1} = J < B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \|f\|_{p,\Phi},$$

that is, (14) is equivalent to (13). In view of (12), for $[\varpi(x)]^{1-q} > [B(\frac{\lambda}{2}, \frac{\lambda}{2})]^{1-q}$, we have (15). By Hölder's inequality, we find

$$I = \int_0^{\infty} \bigl[\Phi^{\frac{1}{p}}(x) f(x) \bigr] \left[\Phi^{\frac{-1}{p}}(x) \sum_{n=1}^{\infty} \frac{1}{\ln^{\lambda} e(n+\alpha)^x} a_n \right] dx \le \|f\|_{p,\Phi} L. \tag{17}$$

Then by (15), we have (13). On the other hand, assuming that (13) is valid, setting

$$f(x) := \bigl[\Phi(x)\bigr]^{1-q} \left[\sum_{n=1}^{\infty} \frac{1}{\ln^{\lambda} e(n+\alpha)^x} a_n \right]^{q-1}, \quad x \in (0, \infty),$$

then $L^{q-1} = \|f\|_{p,\Phi}$. By (12), we find $L < \infty$. If $L = 0$, then (15) is trivially valid; if $L > 0$, then by (13), we have

$$\|f\|_{p,\Phi}^{p} = L^q = I < B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right)\|f\|_{p,\Phi}\|a\|_{q,\Psi}, \quad i.e.,$$
$$\|f\|_{p,\Phi}^{p-1} = L < B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right)\|a\|_{q,\Psi},$$

that is, (15) is equivalent to (13). Hence, inequalities (13), (14) and (15) are equivalent.

For $0 < \varepsilon < \frac{p\lambda}{2}$, setting $\widetilde{f}(x) = x^{\frac{\lambda}{2}+\frac{\varepsilon}{p}-1}$, $x \in (0,1)$; $\widetilde{f}(x) = 0$, $x \in [1,\infty)$, and $\widetilde{a}_n = \frac{1}{n+\alpha} \times \ln^{\frac{\lambda}{2}-\frac{\varepsilon}{q}-1}(n+\alpha)$, $n \in \mathbf{N}$, if there exists a positive number k ($\le B(\frac{\lambda}{2}, \frac{\lambda}{2})$) such that (13) is valid as we replace $B(\frac{\lambda}{2}, \frac{\lambda}{2})$ with k, then, in particular, it follows

$$\begin{aligned}
\widetilde{I} &:= \sum_{n=1}^{\infty}\int_0^{\infty}\frac{1}{\ln^{\lambda} e(n+\alpha)^x}\widetilde{a}_n\widetilde{f}(x)\,dx < k\|\widetilde{f}\|_{p,\Phi}\|\widetilde{a}\|_{q,\Psi} \\
&= k\left\{\int_0^1 \frac{dx}{x^{-\varepsilon+1}}\right\}^{\frac{1}{p}}\left\{\frac{1}{(1+\alpha)\ln^{\varepsilon+1}(1+\alpha)} + \sum_{n=2}^{\infty}\frac{1}{(n+\alpha)\ln^{\varepsilon+1}(n+\alpha)}\right\}^{\frac{1}{q}} \\
&< k\left(\frac{1}{\varepsilon}\right)^{\frac{1}{p}}\left\{\frac{1}{(1+\alpha)\ln^{\varepsilon+1}(1+\alpha)} + \int_1^{\infty}\frac{1}{(x+\alpha)\ln^{\varepsilon+1}(x+\alpha)}\,dx\right\}^{\frac{1}{q}} \\
&= \frac{k}{\varepsilon}\left\{\frac{\varepsilon}{(1+\alpha)\ln^{\varepsilon+1}(1+\alpha)} + \frac{1}{\ln^{\varepsilon}(1+\alpha)}\right\}^{\frac{1}{q}},
\end{aligned} \tag{18}$$

$$\begin{aligned}
\widetilde{I} &= \sum_{n=1}^{\infty}\frac{1}{n+\alpha}\ln^{\frac{\lambda}{2}-\frac{\varepsilon}{q}-1}(n+\alpha)\int_0^1\frac{1}{\ln^{\lambda} e(n+\alpha)^x}x^{\frac{\lambda}{2}+\frac{\varepsilon}{p}-1}\,dx \\
&\overset{t=x\ln(n+\alpha)}{=} \sum_{n=1}^{\infty}\frac{1}{(n+\alpha)\ln^{\varepsilon+1}(n+\alpha)}\int_0^{\ln(n+\alpha)}\frac{1}{(t+1)^{\lambda}}t^{\frac{\lambda}{2}+\frac{\varepsilon}{p}-1}\,dt \\
&= B\left(\frac{\lambda}{2}+\frac{\varepsilon}{p}, \frac{\lambda}{2}-\frac{\varepsilon}{p}\right)\sum_{n=1}^{\infty}\frac{1}{(n+\alpha)\ln^{\varepsilon+1}(n+\alpha)} - A(\varepsilon) \\
&> B\left(\frac{\lambda}{2}+\frac{\varepsilon}{p}, \frac{\lambda}{2}-\frac{\varepsilon}{p}\right)\int_1^{\infty}\frac{1}{(y+\alpha)\ln^{\varepsilon+1}(y+\alpha)}\,dy - A(\varepsilon) \\
&= \frac{1}{\varepsilon\ln^{\varepsilon}(1+\alpha)}B\left(\frac{\lambda}{2}+\frac{\varepsilon}{p}, \frac{\lambda}{2}-\frac{\varepsilon}{p}\right) - A(\varepsilon),
\end{aligned}$$

$$A(\varepsilon) := \sum_{n=1}^{\infty}\frac{1}{(n+\alpha)\ln^{\varepsilon+1}(n+\alpha)}\int_{\ln(n+\alpha)}^{\infty}\frac{1}{(t+1)^{\lambda}}t^{\frac{\lambda}{2}+\frac{\varepsilon}{p}-1}\,dt. \tag{19}$$

We find

$$\begin{aligned}
0 < A(\varepsilon) &\le \sum_{n=1}^{\infty}\frac{1}{(n+\alpha)\ln^{\varepsilon+1}(n+\alpha)}\int_{\ln(n+\alpha)}^{\infty}\frac{1}{t^{\lambda}}t^{\frac{\lambda}{2}+\frac{\varepsilon}{p}-1}\,dt \\
&= \frac{1}{\frac{\lambda}{2}-\frac{\varepsilon}{p}}\sum_{n=1}^{\infty}\frac{1}{(n+\alpha)\ln^{\frac{\lambda}{2}+\frac{\varepsilon}{q}+1}(n+\alpha)} < \infty,
\end{aligned}$$

and then $A(\varepsilon) = O(1)$ ($\varepsilon \to 0^+$). Hence by (18) and (19), it follows

$$\begin{aligned}&\frac{1}{\ln^{\varepsilon}(1+\alpha)}B\left(\frac{\lambda}{2}+\frac{\varepsilon}{p},\frac{\lambda}{2}-\frac{\varepsilon}{p}\right)-\varepsilon O(1)\\&\quad< k\left\{\frac{\varepsilon}{(1+\alpha)\ln^{\varepsilon+1}(1+\alpha)}+\frac{1}{\ln^{\varepsilon}(1+\alpha)}\right\}^{\frac{1}{q}},\end{aligned} \tag{20}$$

and $B(\frac{\lambda}{2},\frac{\lambda}{2}) \le k$ ($\varepsilon \to 0^+$). Hence $k = B(\frac{\lambda}{2},\frac{\lambda}{2})$ is the best value of (13).

By equivalence, the constant factor $B(\frac{\lambda}{2},\frac{\lambda}{2})$ in (14) and (15) is the best possible. Otherwise, we can imply a contradiction by (16) and (17) that the constant factor in (13) is not the best possible. □

Remark 1 (i) Define the first type half-discrete Hilbert-type operator $T_1 : L_{p,\Phi}(R_+) \to l_{p,\Psi^{1-p}}$ as follows: For $f \in L_{p,\Phi}(R_+)$, we define $T_1 f \in l_{p,\Psi^{1-p}}$, satisfying

$$T_1 f(n) = \int_0^\infty \frac{1}{\ln^{\lambda} e(n+\alpha)^x} f(x)\,dx, \quad n \in \mathbf{N}.$$

Then by (14) it follows $\|T_1 f\|_{p,\Psi^{1-p}} \le B(\frac{\lambda}{2},\frac{\lambda}{2})\|f\|_{p,\Phi}$, and then T_1 is a bounded operator with $\|T_1\| \le B(\frac{\lambda}{2},\frac{\lambda}{2})$. Since by Theorem 1 the constant factor in (14) is the best possible, we have $\|T_1\| = B(\frac{\lambda}{2},\frac{\lambda}{2})$.

(ii) Define the second type half-discrete Hilbert-type operator $T_2 : l_{q,\Psi} \to L_{q,\Phi^{1-q}}(R_+)$ as follows: For $a \in l_{q,\Psi}$, we define $T_2 a \in L_{q,\Phi^{1-q}}(R_+)$, satisfying

$$T_2 a(x) = \sum_{n=1}^{\infty} \frac{1}{\ln^{\lambda} e(n+\alpha)^x} a_n, \quad x \in (0,\infty).$$

Then by (15) it follows $\|T_2 a\|_{q,\Phi^{1-q}} \le B(\frac{\lambda}{2},\frac{\lambda}{2})\|a\|_{q,\Psi}$, and then T_2 is a bounded operator with $\|T_2\| \le B(\frac{\lambda}{2},\frac{\lambda}{2})$. Since by Theorem 1 the constant factor in (15) is the best possible, we have $\|T_2\| = B(\frac{\lambda}{2},\frac{\lambda}{2})$.

Remark 2 For $p = q = 2$, $\lambda = 1$, $\lambda_1 = \lambda_2 = \frac{1}{2}$, $\alpha = \frac{1}{2}$ in (13), (14) and (15), we have (7) and the following equivalent inequalities:

$$\left\{\sum_{n=1}^{\infty}\frac{1}{n+\frac{1}{2}}\left[\int_0^\infty \frac{f(x)}{\ln e(n+\frac{1}{2})^x}\,dx\right]^2\right\}^{\frac{1}{2}} < \pi\|f\|, \tag{21}$$

$$\left\{\int_0^\infty\left[\sum_{n=1}^{\infty}\frac{a_n}{\ln e(n+\frac{1}{2})^x}\right]^2 dx\right\}^{\frac{1}{2}} < \pi\left\{\sum_{n=1}^{\infty}\left(n+\frac{1}{2}\right)a_n^2\right\}^{\frac{1}{2}}. \tag{22}$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
ZH wrote and reformed the article. BY conceived of the study, and participated in its design and coordination. All authors read and approved the final manuscript.

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