Masumoto *Journal of Inequalities and Applications* 2013, **2013**:282
http://www.journalofinequalitiesandapplications.com/content/2013/1/282

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

On critical extremal length for the existence of holomorphic mappings of once-holed tori

Makoto Masumoto*

Dedicated to Professor Hari M Srivastava

*Correspondence:
masumoto@yamaguchi-u.ac.jp
Department of Mathematics,
Yamaguchi University, Yamaguchi,
753-8512, Japan

Abstract

Let $\mathfrak{T}_a[Y_0]$ be the set of marked once-holed tori which allows a holomorphic mapping into a given Riemann surface Y_0 with marked handle. We compare it with the subset $\mathfrak{T}_\infty[Y_0]$ of marked once-holed tori X such that there is a holomorphic mapping $f : X \to Y_0$ for which the cardinal numbers of $f^{-1}(p)$, $p \in Y_0$, are bounded. We show that while $\mathfrak{T}_\infty[Y_0]$ is a proper subset of $\mathfrak{T}_a[Y_0]$ apart from a few exceptions, their critical extremal lengths are identical.

MSC: Primary 30F99; secondary 32H02

Keywords: Riemann surface; once-holed torus; holomorphic mapping; extremal length

1 Introduction

Let R be a Riemann surface of positive genus. By a *mark of handle* of R we mean an ordered pair $\chi = \{a, b\}$ of simple loops a and b on R whose geometric intersection number $a \times b$ is equal to one. The pair $Y = (R, \chi)$ is called a *Riemann surface with marked handle*.

Let $Y' = (R', \chi')$ be another Riemann surface with marked handle, where $\chi' = \{a', b'\}$. If $f : R \to R'$ is holomorphic and maps a and b onto loops freely homotopic to a' and b' on R', respectively, then we say that f is a holomorphic mapping of Y into Y' and use the notation $f : Y \to Y'$. If, in addition, $f : R \to R'$ is conformal, that is, if $f : R \to R'$ is holomorphic and injective, then $f : Y \to Y'$ is called conformal.

A noncompact Riemann surface of genus one with exactly one boundary component is called a *once-holed torus*. A *marked once-holed torus* means a once-holed torus with marked handle. Let $\mathfrak{T}$ denote the set of marked once-holed tori, where two marked once-holed tori are identified with each other if there is a conformal mapping of one onto the other. It is a three-dimensional real analytic manifold with boundary (see [1, §7]).

For later use, we introduce some notations. Let $\mathbb{H}$ stand for the upper half-plane: $\mathbb{H} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$. For $\tau \in \mathbb{H}$, let G_τ denote the additive group generated by 1 and τ. Then $T_\tau := \mathbb{C}/G_\tau$ is a torus, that is, a compact Riemann surface of genus one. The two oriented segments $[0,1]$ and $[0,\tau]$ are projected onto simple loops a_τ and b_τ forming a mark χ_τ of handle of T_τ. Set $X_\tau = (T_\tau, \chi_\tau)$. For $l \in [0,1)$, we define $T_\tau^{(l)} = T_\tau \setminus \pi_\tau([0,l])$, where $\pi_\tau : \mathbb{C} \to T_\tau$ is the natural projection. Then $T_\tau^{(l)}$ is a once-holed torus. We choose a mark $\chi_\tau^{(l)}$ of handle of $T_\tau^{(l)}$ so that the inclusion mapping of $T_\tau^{(l)}$ into T_τ is a conformal mapping of $X_\tau^{(l)} := (T_\tau^{(l)}, \chi_\tau^{(l)})$ into X_τ. The correspondence $(\tau, l) \mapsto X_\tau^{(l)}$ defines a bijection of $\mathbb{H} \times [0,1)$

onto $\mathfrak{T}$ (see, for example, [2]). In other words, every marked once-holed torus is realized as a horizontal slit domain of a torus with marked handle, or a marked torus, uniquely up to conformal automorphisms of the marked torus.

Let Y_0 be a Riemann surface with marked handle. We are interested in the set $\mathfrak{T}_a[Y_0]$ of marked once-holed tori X for which there is a holomorphic mapping of X into Y_0. It possesses an interesting quantitative property. In [3] (see also [4]) we have established that there is a nonnegative number $\lambda_a[Y_0]$ such that

(i) if $\operatorname{Im}\tau \geqq 1/\lambda_a[Y_0]$, then $X_\tau^{(l)} \notin \mathfrak{T}_a[Y_0]$ for any l, while

(ii) if $\operatorname{Im}\tau < 1/\lambda_a[Y_0]$, then $X_\tau^{(l)} \in \mathfrak{T}_a[Y_0]$ for some l,

where $1/0 = +\infty$. If Y_0 is a marked torus, then $\mathfrak{T}_a[Y_0] = \mathfrak{T}$ and hence $\lambda_a[Y_0] = 0$. Otherwise, $\lambda_a[Y_0] > 0$ by [5, Theorem 1 and Proposition 1].

In this article we compare $\mathfrak{T}_a[Y_0]$ with the set $\mathfrak{T}_\infty[Y_0]$ of marked once-holed tori X such that there is a holomorphic mapping $f: X \to Y_0$ for which the supremum $d(f)$ of the cardinal numbers of $f^{-1}(p)$, $p \in R_0$, is finite. As is shown in [3], it possesses a property similar to that of $\mathfrak{T}_a[Y_0]$: There is a nonnegative number $\lambda_\infty[Y_0]$ such that

(i) if $\operatorname{Im}\tau \geqq 1/\lambda_\infty[Y_0]$, then $X_\tau^{(l)} \notin \mathfrak{T}_\infty[Y_0]$ for any l, while

(ii) if $\operatorname{Im}\tau < 1/\lambda_\infty[Y_0]$, then $X_\tau^{(l)} \in \mathfrak{T}_\infty[Y_0]$ for some l.

Since

$$\mathfrak{T}_\infty[Y_0] \subset \mathfrak{T}_a[Y_0],$$

we have

$$\lambda_\infty[Y_0] \geqq \lambda_a[Y_0]. \tag{1}$$

We first establish the following theorem.

Theorem 1 *If Y_0 is not a marked torus or a marked once-holed torus, then $\mathfrak{T}_\infty[Y_0]$ is a proper subset of $\mathfrak{T}_a[Y_0]$.*

Nevertheless, the sign of equality actually holds in (1).

Theorem 2 *For any marked Riemann surface Y_0, the equality $\lambda_\infty[Y_0] = \lambda_a[Y_0]$ holds.*

The proofs of Theorems 1 and 2 will be given in the next section.

2 Proofs

We begin with the proof of Theorem 1. Let $Y_0 = (R_0, \chi_0)$, where $\chi_0 = \{a_0, b_0\}$, be a Riemann surface with marked handle which is not a marked torus or a marked once-holed torus. We consider the loops a_0 and b_0 as elements of the fundamental group $\pi_1(R_0)$ of R_0. Let $\tilde{R}_0$ be the covering Riemann surface of R_0 corresponding to the subgroup $\langle a_0, b_0\rangle$ of $\pi_1(R_0)$ generated by a_0 and b_0. Since R_0 is not a torus, $\tilde{R}_0$ is a once-holed torus. We choose a mark $\tilde{\chi}_0 = \{\tilde{a}_0, \tilde{b}_0\}$ of handle of $\tilde{R}_0$ so that the natural projection $\pi_0: \tilde{R}_0 \to R_0$ is a holomorphic mapping of the marked once-holed torus $\tilde{Y}_0 := (\tilde{R}_0, \tilde{\chi}_0)$ onto Y_0. Then $\tilde{Y}_0$ is an element of $\mathfrak{T}_a[Y_0]$.

Let f be an arbitrary holomorphic mapping of $\tilde{Y}_0$ into Y_0. Since it maps $\tilde{a}_0$ and $\tilde{b}_0$ onto loops freely homotopic to a_0 and b_0, respectively, it is lifted to a holomorphic mapping $\tilde{f}$ of

$\tilde{Y}_0$ into itself satisfying $\pi_0 \circ \tilde{f} = f$. By Huber [6, Satz II] (see also Marden-Richards-Rodin [7, Theorem 5]), we infer that $\tilde{f}$ is a conformal automorphism of $\tilde{Y}_0$. Since R_0 is not a torus or a once-holed torus, we conclude that $d(f) = d(\pi_0) = \infty$ and hence $\tilde{Y}_0 \notin \mathfrak{T}_\infty[Y_0]$. This completes the proof of Theorem 1.

For the proof of Theorem 2, we make a remark. Let $X = (T, \chi)$, where $\chi = \{a, b\}$, be a marked once-holed torus. Then the extremal length $\lambda(X)$ of the free homotopy class of a is called the *basic extremal length* of X. Note that $\lambda(X_\tau^{(l)}) = 1/\operatorname{Im}\tau$ (see [8, Proposition 1]).

Now, take an arbitrary $\tau \in \mathbb{H}$ with $\operatorname{Im}\tau < 1/\lambda_a[Y_0]$. Then, for some $l \in [0,1)$, there is a holomorphic mapping f of $X_\tau^{(l)}$ into Y_0. Recall that $T_\tau^{(l)}$ is the horizontal slit domain $T_\tau \setminus \pi_\tau([0,l])$ of the torus T_τ. Choose a canonical exhaustion $\{S_n\}$ of $T_\tau^{(l)}$ so that each S_n is a once-holed torus including the loops in $\chi_\tau^{(l)}$. Since the inclusion mapping $S_n \to T_\tau^{(l)}$ is a conformal mapping of the marked once-holed torus $W_n := (S_n, \chi_\tau^{(l)})$ into $X_\tau^{(l)}$, the restriction f_n of f to S_n is a holomorphic mapping of W_n into Y_0. As S_n is relatively compact in $T_n^{(l)}$, we know that $d(f_n) < \infty$. Consequently, W_n belongs to $\mathfrak{T}_\infty[Y_0]$.

To estimate the basic extremal length of W_n, take an arbitrary positive number ε less than $\operatorname{Im}\tau/2$. Let H_ε be the horizontal strip $\{z \in \mathbb{C} \mid \varepsilon < \operatorname{Im} z < \operatorname{Im}\tau - \varepsilon\}$. Since $\{S_n\}$ is increasing with $\bigcup_n S_n = T_\tau^{(l)}$, for all sufficiently large n, the subdomain S_n includes the ring domain $\pi_\tau(H_\varepsilon)$. It follows that

$$\operatorname{Im}\tau - 2\varepsilon < \frac{1}{\lambda(W_n)} \leqq \frac{1}{\lambda_\infty[Y_0]},$$

which implies that

$$\operatorname{Im}\tau \leqq \frac{1}{\lambda_\infty[Y_0]}.$$

As τ was an arbitrary point of $\mathbb{H}$ satisfying $\operatorname{Im}\tau < 1/\lambda_a[Y_0]$, we deduce that

$$\frac{1}{\lambda_a[Y_0]} \leqq \frac{1}{\lambda_\infty[Y_0]},$$

or

$$\lambda_\infty[Y_0] \leqq \lambda_a[Y_0].$$

Theorem 2 has been proved.

3 Topological relations between $\mathfrak{T}_a[Y_0]$ and $\mathfrak{T}_\infty[Y_0]$

The arguments in the proof of Theorem 2 easily lead us to the following theorem.

Theorem 3 *The closure of* $\mathfrak{T}_\infty[Y_0]$ *is identical with* $\mathfrak{T}_a[Y_0]$.

Proof We begin with recalling a global coordinate system on the space $\mathfrak{T}$ of marked once-holed tori. Let $X = (T, \chi)$ be a marked once-holed torus, where $\chi = \{a, b\}$. Observe that $\dot{\chi} := \{b, a^{-1}\}$ is a mark of handle of T. Also, if c is a simple loop homotopic to ab^{-1}, then $\ddot{\chi} := \{c, a\}$ is another mark of handle of T. Set $\dot{X} = (T, \dot{\chi})$ and $\ddot{X} = (T, \ddot{\chi})$. Then the basic extremal lengths of X, $\dot{X}$ and $\ddot{X}$ define a global coordinate system on $\mathfrak{T}$. In fact, we introduce a real analytic structure into $\mathfrak{T}$ so that the mapping $\Lambda : X \mapsto (\lambda(X), \lambda(\dot{X}), \lambda(\ddot{X}))$ is a real analytic diffeomorphism of $\mathfrak{T}$ into $\mathbb{R}^3$ (see [1]).

Now, let $X = (T, \chi)$ be an arbitrary element of $\mathfrak{T}_a[Y_0]$. Take a canonical exhaustion $\{S_n\}$ of T for which each S_n is a once-holed torus including the loops in χ, and set $W_n = (S_n, \chi)$. Since $X = X_\tau^{(l)}$ for some $\tau \in \mathbb{H}$ and $l \in [0, 1)$, the proof of Theorem 2 shows that the basic extremal length $\lambda(W_n)$ tends to $\lambda(X)$ as $n \to \infty$. By changing marks of handles, we infer that $\{\lambda(\dot{W}_n)\}$ and $\{\lambda(\ddot{W}_n)\}$ converge to $\lambda(\dot{X})$ and $\lambda(\ddot{X})$, respectively, and hence that $\Lambda(W_n) \to \Lambda(X)$ as $n \to \infty$. Since each W_n belongs to $\mathfrak{T}_\infty[Y_0]$, the marked once-holed torus X belongs to the closure $\overline{\mathfrak{T}_\infty[Y_0]}$ of $\mathfrak{T}_\infty[Y_0]$. We thus obtain $\mathfrak{T}_a[Y_0] \subset \overline{\mathfrak{T}_\infty[Y_0]}$. Because $\mathfrak{T}_a[Y_0]$ is closed and includes $\mathfrak{T}_\infty[Y_0]$, we conclude that $\overline{\mathfrak{T}_\infty[Y_0]} = \mathfrak{T}_a[Y_0]$. □

Since $\mathfrak{T}_\infty[Y_0]$ and $\mathfrak{T}_a[Y_0]$ are (noncompact) domains with Lipschitz boundary by [3], we see that Theorem 3 is an improvement of Theorem 2. Also, we obtain the following corollary.

Corollary 1 *The interiors of* $\mathfrak{T}_\infty[Y_0]$ *and* $\mathfrak{T}_a[Y_0]$ *coincide with each other.*

If Y_0 is a marked torus, then $\mathfrak{T}_a[Y_0]$ is identical with $\mathfrak{T}$ (see [1]). Hence so is $\mathfrak{T}_\infty[Y_0]$ by Corollary 1.

Competing interests
The author declares that he has no competing interests.

Acknowledgements
This research is supported in part by JSPS KAKENHI Grant Number 22540196. The author is grateful to the referees for their invaluable comments.

Received: 15 December 2012 Accepted: 22 May 2013 Published: 5 June 2013

References

1. Masumoto, M: Conformal mappings of a once-holed torus. J. Anal. Math. **66**, 117-136 (1995)
2. Shiba, M: The moduli of compact continuations of an open Riemann surface of genus one. Trans. Am. Math. Soc. **301**, 299-311 (1987)
3. Masumoto, M: Holomorphic mappings of once-holed tori. Preprint
4. Masumoto, M: Conformal and holomorphic mappings of once-holed tori. Global J. Math. Sci. **1**, 24-30 (2012)
5. Masumoto, M: Holomorphic mappings and basic extremal lengths of once-holed tori. Nonlinear Anal. **71**, e1178-e1181 (2009)
6. Huber, H: Über Analytische Abbildungen Riemannscher Flächen in sich. Comment. Math. Helv. **27**, 1-73 (1953)
7. Marden, A, Richards, I, Rodin, B: Analytic self-mappings of Riemann surfaces. J. Anal. Math. **18**, 197-225 (1967)
8. Masumoto, M: Estimates of the Euclidean span for an open Riemann surface of genus one. Hiroshima Math. J. **14**, 573-582 (1992)

doi:10.1186/1029-242X-2013-282
Cite this article as: Masumoto: On critical extremal length for the existence of holomorphic mappings of once-holed tori. *Journal of Inequalities and Applications* 2013 **2013**:282.