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# An iterative algorithm for the generalized reflexive solutions of the general coupled matrix equations

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## Abstract

The general coupled matrix equations

$$\sum_{j=1}^q A_{ij} X_j B_{ij} = M_i, \quad i = 1, 2, \dots, p$$

(including the generalized coupled Sylvester matrix equations as special cases) have numerous applications in control and system theory. In this paper, an iterative algorithm is constructed to solve the general coupled matrix equations and their optimal approximation problem over generalized reflexive matrix solution  $(X_1, X_2, \dots, X_q)$ . When the general coupled matrix equations are consistent over generalized reflexive matrices, the generalized reflexive solution can be determined automatically by the iterative algorithm within finite iterative steps in the absence of round-off errors. The least Frobenius norm generalized reflexive solution of the general coupled matrix equations can be derived when an appropriate initial matrix group is chosen. Furthermore, the unique optimal approximation generalized reflexive solution  $(\tilde{X}_1, \tilde{X}_2, \dots, \tilde{X}_q)$  to a given matrix group  $(X_1^0, X_2^0, \dots, X_q^0)$  in Frobenius norm can be derived by finding the least-norm generalized reflexive solution  $(\tilde{X}_1^*, \tilde{X}_2^*, \dots, \tilde{X}_q^*)$  of the corresponding general coupled matrix equations  $\sum_{j=1}^q A_{ij} \tilde{X}_j B_{ij} = \tilde{M}_i, i = 1, 2, \dots, p$ , where  $\tilde{X}_j = X_j - X_j^0, \tilde{M}_i = M_i - \sum_{j=1}^q A_{ij} X_j^0 B_{ij}$ . A numerical example is given to illustrate the effectiveness of the proposed iterative algorithm.

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**Keywords:** iterative method; generalized reflexive solution; general coupled matrix equations; optimal approximate solution

## 1 Introduction

Let  $P \in \mathcal{R}^{m \times m}$  and  $Q \in \mathcal{R}^{n \times n}$  be two real generalized reflection matrices, i.e.,  $P^T = P, P^2 = I_m, Q^T = Q, Q^2 = I_n, I_n$  denotes the  $n$  order identity matrix. A matrix  $A \in \mathcal{R}^{m \times n}$  is called generalized reflexive matrix with respect to the matrix pair  $(P, Q)$  if  $PAQ = A$ . The set of all  $m$ -by- $n$  real generalized reflexive matrices with respect to a matrix pair  $(P, Q)$  is denoted by  $\mathcal{R}_r^{m \times n}(P, Q)$ . We denote by the superscript  $T$  the transpose of a matrix. In a matrix space  $\mathcal{R}^{m \times n}$ , we define an inner product as  $\langle A, B \rangle = \text{tr}(B^T A)$  for all  $A, B \in \mathcal{R}^{m \times n}$ ,  $\|A\|_F$  represents the Frobenius norm of  $A$ ,  $\mathcal{R}(A)$  represents the column space of  $A$ ,  $\text{vec}(\cdot)$  represents the vector operator, i.e.,  $\text{vec}(A) = (a_1^T, a_2^T, \dots, a_n^T)^T \in \mathcal{R}^{mn}$  for the matrix  $A =$

$(a_1, a_2, \dots, a_n) \in \mathcal{R}^{m \times n}$ ,  $a_i \in \mathcal{R}^m$ ,  $i = 1, 2, \dots, n$ ,  $A \otimes B$  stands for the Kronecker product of matrices  $A$  and  $B$ .

Least-squares-based iterative algorithms are very important in system identification, parameter estimation, and signal processing, including the recursive least squares (RLS) and iterative least squares (ILS) methods for solving the solutions of some matrix equations, for example, the Lyapunov matrix equation, Sylvester matrix equations, and coupled matrix equations as well. For example, novel gradient-based iterative (GI) method [1–5] and least-squares-based iterative methods [3, 4, 6] with highly computational efficiencies for solving (coupled) matrix equations are presented and have good stability performances based on the hierarchical identification principle, which regards the unknown matrix as the system parameter matrix to be identified. In this paper, we consider the following two problems.

**Problem I** Let  $P_j \in \mathcal{R}^{m_j \times m_j}$  and  $Q_j \in \mathcal{R}^{n_j \times n_j}$  be generalized reflection matrices. For given matrices  $A_{ij} \in \mathcal{R}^{r_i \times m_j}$ ,  $B_{ij} \in \mathcal{R}^{n_j \times s_i}$ , and  $M_i \in \mathcal{R}^{r_i \times s_i}$ , find a generalized reflexive matrix solution group  $(X_1, X_2, \dots, X_q)$  with  $X_j \in \mathcal{R}_r^{m_j \times n_j}(P_j, Q_j)$  such that

$$\sum_{j=1}^q A_{ij} X_j B_{ij} = M_i, \quad i = 1, 2, \dots, p. \tag{1}$$

**Problem II** When Problem I is consistent, let  $S_E$  denote the set of the generalized reflexive solution group of Problem I, *i.e.*,

$$S_E = \left\{ (X_1, X_2, \dots, X_q) \mid \sum_{j=1}^q A_{ij} X_j B_{ij} = M_i, i = 1, 2, \dots, p, X_j \in \mathcal{R}_r^{m_j \times n_j}(P_j, Q_j) \right\}.$$

For a given generalized reflexive matrix group

$$(X_1^0, X_2^0, \dots, X_q^0) \in \mathcal{R}_r^{m_1 \times n_1}(P_1, Q_1) \times \mathcal{R}_r^{m_2 \times n_2}(P_2, Q_2) \times \dots \times \mathcal{R}_r^{m_q \times n_q}(P_q, Q_q),$$

find  $(\widehat{X}_1, \widehat{X}_2, \dots, \widehat{X}_q) \in S_E$  such that

$$\sum_{j=1}^q \|\widehat{X}_j - X_j^0\|^2 = \min_{(X_1, X_2, \dots, X_q) \in S_E} \left\{ \sum_{j=1}^q \|X_j - X_j^0\|^2 \right\}. \tag{2}$$

General coupled matrix equation (1) (including the generalized coupled Sylvester matrix equations as special cases) may arise in many areas of control and system theory. Problem II occurs frequently in experiment design; see, for instance, [7].

Many theoretical and numerical results on Eq. (1) and some of its special cases have been obtained. Ding and Chen [1] presented the gradient-based iterative algorithms by applying the gradient search principle and the hierarchical identification principle for Eq. (1) with  $q = p$ . Wu *et al.* [8, 9] gave the finite iterative solutions to coupled Sylvester-conjugate matrix equations. Wu *et al.* [10] gave the finite iterative solutions to a class of complex matrix equations with conjugate and transpose of the unknowns. Jonsson and Kågström [11, 12] proposed recursive block algorithms for solving the coupled Sylvester matrix equations and the generalized Sylvester and Lyapunov matrix equations. By extending the idea of

conjugate gradient method, Dehghan and Hajarian [13] constructed an iterative algorithm to solve Eq. (1) with  $q = p$  over generalized bisymmetric matrices. Very recently, Huang *et al.* [14] presented a finite iterative algorithm for the one-sided and generalized coupled Sylvester matrix equations over generalized reflexive solutions. Yin *et al.* [15] presented a finite iterative algorithm for the two-sided and generalized coupled Sylvester matrix equations over reflexive solutions. Zhou *et al.* [16] gave the gradient based iterative solutions for coupled matrix equation (1) and a more general case over general solution. Li *et al.* [17] presented a numerical solution to linear matrix equation  $\sum_{j=1}^p A_j X B_j = C$  by finite steps iteration. For more results, we refer to [7, 18–36]. However, to our knowledge, the generalized reflexive solution to the more general coupled matrix equation (1) and the optimal approximation reflexive solution have not been derived. In this paper, we consider the generalized reflexive solution of Eq. (1) and the optimal approximation generalized reflexive solution.

This paper is organized as follows. In Section 2, we solve Problem I by constructing an iterative algorithm. The convergence of the proposed algorithm is proved. For an arbitrary initial matrix group, we can obtain a generalized reflexive solution group of Problem I within finite iteration steps in the absence of round-off errors. Furthermore, for a special initial matrix group, we can obtain the least Frobenius norm solutions of Problem I. Then in Section 3, we give the optimal approximate solution group of Problem II by finding the least Frobenius norm generalized reflexive solution group of the corresponding general coupled matrix equations. In Section 4, a numerical example is given to illustrate the effectiveness of our method. Finally, some conclusions are drawn in Section 5.

## 2 An iterative algorithm for solving Problem I

In this section, we first introduce an iterative algorithm to solve Problem I, then we prove its convergence. In addition, we give the least-norm generalized reflexive solutions of Problem I when an appropriate initial iterative matrix group is chosen.

### Algorithm 2.1

Step 1: Input matrices  $A_{ij} \in \mathcal{R}^{r_i \times m_j}$ ,  $B_{ij} \in \mathcal{R}^{n_j \times s_i}$ ,  $M_i \in \mathcal{R}^{r_i \times s_i}$ , and generalized reflection matrices  $P_j \in \mathcal{R}^{m_j \times m_j}$  and  $Q_j \in \mathcal{R}^{n_j \times n_j}$ ,  $i = 1, \dots, p$ ,  $j = 1, \dots, q$ ;

Step 2: Choose an arbitrary matrix group

$$(X_1(1), X_2(1), \dots, X_q(1)) \in \mathcal{R}_r^{m_1 \times n_1}(P_1, Q_1) \times \mathcal{R}_r^{m_2 \times n_2}(P_2, Q_2) \times \dots \times \mathcal{R}_r^{m_q \times n_q}(P_q, Q_q).$$

Compute

$$R(1) = \text{diag} \left( M_1 - \sum_{l=1}^q A_{1l} X_l(1) B_{1l}, M_2 - \sum_{l=1}^q A_{2l} X_l(1) B_{2l}, \dots, M_p - \sum_{l=1}^q A_{pl} X_l(1) B_{pl} \right),$$

$$S_j(1) = \frac{1}{2} \left[ \sum_{i=1}^p A_{ij}^T \left( M_i - \sum_{l=1}^q A_{il} X_l(1) B_{il} \right) B_{ij}^T + \sum_{i=1}^p P_i A_{ij}^T \left( M_i - \sum_{l=1}^q A_{il} X_l(1) B_{il} \right) B_{ij}^T Q_i \right],$$

$$k := 1;$$

Step 3: If  $R(k) = 0$ , then stop and  $(X_1(k), X_2(k), \dots, X_q(k))$  is the solution group of Eq. (1); else if  $R(k) \neq 0$ , but  $S_j(k) = 0$ ,  $j = 1, \dots, q$ , then stop and Eq. (1) is not consistent over a generalized reflexive matrix group; else  $k := k + 1$ ;

Step 4: Compute

$$\begin{aligned}
 X_j(k) &= X_j(k-1) + \frac{\|R(k-1)\|_F^2}{\sum_{l=1}^q \|S_l(k-1)\|_F^2} S_j(k-1), \quad j = 1, \dots, q, \\
 R(k) &= \text{diag} \left( M_1 - \sum_{l=1}^q A_{1l} X_l(k) B_{1l}, M_2 - \sum_{l=1}^q A_{2l} X_l(k) B_{2l}, \dots, M_p - \sum_{l=1}^q A_{pl} X_l(k) B_{pl} \right) \\
 &= R(k-1) - \frac{\|R(k-1)\|_F^2}{\sum_{l=1}^q \|S_l(k-1)\|_F^2} \\
 &\quad \times \text{diag} \left( \sum_{l=1}^q A_{1l} S_l(k-1) B_{1l}, \sum_{l=1}^q A_{2l} S_l(k-1) B_{2l}, \dots, \sum_{l=1}^q A_{pl} S_l(k-1) B_{pl} \right), \\
 S_j(k) &= \frac{1}{2} \left[ \sum_{i=1}^p A_{ij}^T \left( M_i - \sum_{l=1}^q A_{il} X_l(k) B_{il} \right) B_{ij}^T + \sum_{i=1}^p P_j A_{ij}^T \left( M_i - \sum_{l=1}^q A_{il} X_l(k) B_{il} \right) B_{ij}^T Q_j \right] \\
 &\quad + \frac{\|R(k)\|_F^2}{\|R(k-1)\|_F^2} S_j(k-1);
 \end{aligned}$$

Step 5: Go to Step 3.

Obviously, it can be seen that  $X_j(k), S_j(k) \in R_r^{m_j \times n_j}(P_j, Q_j)$  for all  $j = 1, \dots, q$  and  $k = 1, 2, \dots$

**Lemma 2.1** For the sequences  $\{R(k)\}, \{S_j(k)\}$  ( $j = 1, 2, \dots, q$ ) generated by Algorithm 2.1, and  $m \geq 2$ , we have

$$\text{tr}((R(s))^T R(t)) = 0, \quad \sum_{j=1}^q \text{tr}((S_j(s))^T S_j(t)) = 0, \quad s, t = 1, 2, \dots, m, s \neq t. \tag{3}$$

The proof of Lemma 2.1 is similar to that of Lemma 2.2 in [21] and is omitted.

**Lemma 2.2** Suppose  $(X_1^*, X_2^*, \dots, X_q^*)$  is an arbitrary generalized reflexive solution group of Problem I, then for any initial generalized reflexive matrix group  $(X_1(1), X_2(1), \dots, X_q(1))$ , we have

$$\sum_{j=1}^q \text{tr}((X_j^* - X_j(k))^T S_j(k)) = \|R(k)\|_F^2, \quad k = 1, 2, \dots, \tag{4}$$

where the sequences  $\{X_j(k)\}, \{S_j(k)\}$  and  $\{R(k)\}$  are generated by Algorithm 2.1.

*Proof* We prove the conclusion by induction for the positive integer  $k$ .

For  $k = 1$ , we have that

$$\begin{aligned}
 &\sum_{j=1}^q \text{tr}((X_j^* - X_j(1))^T S_j(1)) \\
 &= \sum_{j=1}^q \text{tr} \left( (X_j^* - X_j(1))^T \left[ \frac{1}{2} \left( \sum_{i=1}^p A_{ij}^T \left( M_i - \sum_{l=1}^q A_{il} X_l(1) B_{il} \right) B_{ij}^T \right. \right. \right.
 \end{aligned}$$

$$\begin{aligned}
 & + \sum_{i=1}^p P_j A_{ij}^T \left( M_i - \sum_{l=1}^q A_{il} X_l(1) B_{il} \right) B_{ij}^T Q_j \Bigg) \Bigg] \\
 & = \sum_{j=1}^q \operatorname{tr} \left( (X_j^* - X_j(1))^T \left[ \sum_{i=1}^p A_{ij}^T \left( M_i - \sum_{l=1}^q A_{il} X_l(1) B_{il} \right) B_{ij}^T \right] \right) \\
 & = \sum_{i=1}^p \operatorname{tr} \left( \left( M_i - \sum_{l=1}^q A_{il} X_l(1) B_{il} \right)^T \sum_{j=1}^q A_{ij} (X_j^* - X_j(1)) B_{ij} \right) \\
 & = \operatorname{tr} \left( \operatorname{diag} \left( \left( M_1 - \sum_{l=1}^q A_{1l} X_l(1) B_{1l} \right)^T, \left( M_2 - \sum_{l=1}^q A_{2l} X_l(1) B_{2l} \right)^T, \dots, \right. \right. \\
 & \quad \left. \left. \left( M_p - \sum_{l=1}^q A_{pl} X_l(1) B_{pl} \right)^T \right) \right. \\
 & \quad \left. \times \operatorname{diag} \left( \sum_{j=1}^q A_{1j} (X_j^* - X_j(1)) B_{1j}, \sum_{j=1}^q A_{2j} (X_j^* - X_j(1)) B_{2j}, \dots, \sum_{j=1}^q A_{pj} (X_j^* - X_j(1)) B_{pj} \right) \right) \\
 & = \operatorname{tr} \left( \operatorname{diag} \left( \left( M_1 - \sum_{l=1}^q A_{1l} X_l(1) B_{1l} \right)^T, \left( M_2 - \sum_{l=1}^q A_{2l} X_l(1) B_{2l} \right)^T, \dots, \right. \right. \\
 & \quad \left. \left. \left( M_p - \sum_{l=1}^q A_{pl} X_l(1) B_{pl} \right)^T \right) \right. \\
 & \quad \left. \times \operatorname{diag} \left( M_1 - \sum_{j=1}^q A_{1j} X_j(1) B_{1j}, M_2 - \sum_{j=1}^q A_{2j} X_j(1) B_{2j}, \dots, M_p - \sum_{j=1}^q A_{pj} X_j(1) B_{pj} \right) \right) \\
 & = \|R(1)\|^2.
 \end{aligned}$$

Assume that (4) holds for  $k = m$ . When  $k = m + 1$ , by Algorithm 2.1, we have that

$$\begin{aligned}
 & \sum_{j=1}^q \operatorname{tr} \left( (X_j^* - X_j(m+1))^T S_j(m+1) \right) \\
 & = \sum_{j=1}^q \operatorname{tr} \left( (X_j^* - X_j(m+1))^T \left[ \frac{1}{2} \left( \sum_{i=1}^p A_{ij}^T \left( M_i - \sum_{l=1}^q A_{il} X_l(m+1) B_{il} \right) B_{ij}^T \right. \right. \right. \\
 & \quad \left. \left. + \sum_{i=1}^p P_j A_{ij}^T \left( M_i - \sum_{l=1}^q A_{il} X_l(m+1) B_{il} \right) B_{ij}^T Q_j \right) + \frac{\|R(m+1)\|_F^2}{\|R(m)\|_F^2} S_j(m) \right] \right) \\
 & = \sum_{j=1}^q \operatorname{tr} \left( (X_j^* - X_j(m+1))^T \left[ \sum_{i=1}^p A_{ij}^T \left( M_i - \sum_{l=1}^q A_{il} X_l(m+1) B_{il} \right) B_{ij}^T \right] \right) \\
 & \quad + \frac{\|R(m+1)\|_F^2}{\|R(m)\|_F^2} \sum_{j=1}^q \operatorname{tr} \left( (X_j^* - X_j(m+1))^T S_j(m) \right) \\
 & = \sum_{i=1}^p \operatorname{tr} \left( \left( M_i - \sum_{l=1}^q A_{il} X_l(m+1) B_{il} \right)^T \sum_{j=1}^q A_{ij} (X_j^* - X_j(m+1)) B_{ij} \right) \\
 & \quad + \frac{\|R(m+1)\|_F^2}{\|R(m)\|_F^2} \sum_{j=1}^q \operatorname{tr} \left( (X_j^* - X_j(m))^T S_j(m) \right)
 \end{aligned}$$

$$\begin{aligned}
 & - \frac{\|R(m+1)\|_F^2}{\sum_{j=1}^q \|S_j(m)\|_F^2} \sum_{j=1}^q \text{tr}((S_j(m))^T S_j(m)) \\
 & = \text{tr} \left( \text{diag} \left( \left( M_1 - \sum_{l=1}^q A_{1l} X_l(m+1) B_{1l} \right)^T, \left( M_2 - \sum_{l=1}^q A_{2l} X_l(m+1) B_{2l} \right)^T, \dots, \right. \right. \\
 & \quad \left. \left. \left( M_p - \sum_{l=1}^q A_{pl} X_l(m+1) B_{pl} \right)^T \right) \right) \\
 & \quad \times \text{diag} \left( \sum_{j=1}^q A_{1j} (X_j^* - X_j(m+1)) B_{1j}, \sum_{j=1}^q A_{2j} (X_j^* - X_j(m+1)) B_{2j}, \dots, \right. \\
 & \quad \left. \sum_{j=1}^q A_{pj} (X_j^* - X_j(m+1)) B_{pj} \right) \\
 & \quad + \frac{\|R(m+1)\|_F^2}{\|R(m)\|_F^2} \|R(m)\|_F^2 - \frac{\|R(m+1)\|_F^2}{\sum_{j=1}^q \|S_j(m)\|_F^2} \sum_{j=1}^q \|S_j(m)\|_F^2 \\
 & = \text{tr} \left( \text{diag} \left( \left( M_1 - \sum_{l=1}^q A_{1l} X_l(m+1) B_{1l} \right)^T, \left( M_2 - \sum_{l=1}^q A_{2l} X_l(m+1) B_{2l} \right)^T, \dots, \right. \right. \\
 & \quad \left. \left. \left( M_p - \sum_{l=1}^q A_{pl} X_l(m+1) B_{pl} \right)^T \right) \right) \\
 & \quad \times \text{diag} \left( M_1 - \sum_{j=1}^q A_{1j} X_j(m+1) B_{1j}, M_2 - \sum_{j=1}^q A_{2j} X_j(m+1) B_{2j}, \dots, \right. \\
 & \quad \left. M_p - \sum_{j=1}^q A_{pj} X_j(m+1) B_{pj} \right) + \|R(m+1)\|_F^2 - \|R(m)\|_F^2 \\
 & = \|R(m+1)\|_F^2.
 \end{aligned}$$

Therefore, (4) holds for  $k = m + 1$ . Thus (4) holds by the principle of induction. This completes the proof.  $\square$

**Remark 2.1** If there exists a positive number  $k$  such that  $S_j(k) = 0, j = 1, 2, \dots, q$ , but  $R(k) \neq 0$ , then by Lemma 2.2, we have that Eq. (1) is not consistent over generalized reflexive matrices.

**Theorem 2.1** Suppose that Problem I is consistent, then for an arbitrary initial matrix group  $(X_1, X_2, \dots, X_q)$  with  $X_j \in \mathcal{R}_r^{m_j \times n_j}(P_j, Q_j)$ , a generalized reflexive solution group of Problem I can be obtained with finite iteration steps in the absence of round-off errors.

*Proof* If  $R(k) \neq 0, k = 1, 2, \dots, m = \sum_{i=1}^p r_i s_i$ , by Lemma 2.2 and Remark 2.1, we have  $S_j(k) \neq 0$  for all  $j = 1, 2, \dots, q$  and  $k = 1, 2, \dots, m$ , then we can compute  $R(m+1)$  and  $(X_1(m+1), X_2(m+1), \dots, X_q(m+1))$  by Algorithm 2.1.

By Lemma 2.1, we have

$$\text{tr}((R(m+1))^T R(k)) = 0, \quad k = 1, 2, \dots, m$$

and

$$\text{tr}((R(k))^T R(l)) = 0, \quad k, l = 1, 2, \dots, m, k \neq l.$$

It can be seen that the set of  $R(1), R(2), \dots, R(m)$  is an orthogonal basis of the matrix subspace

$$S = \{L | L = \text{diag}(L_1, L_2, \dots, L_p), L_i \in \mathcal{R}^{r_i \times s_i}, i = 1, 2, \dots, p\},$$

which implies that  $R(m + 1) = 0$ , i.e.,  $(X_1(m + 1), X_2(m + 1), \dots, X_q(m + 1))$  with  $X_j(m + 1) \in \mathcal{R}_r^{m_j \times n_j}(P_j, Q_j)$  is a generalized reflexive solution group of Problem I. This completes the proof.  $\square$

To show the least Frobenius norm generalized reflexive solution of Problem I, we first introduce the following result.

**Lemma 2.3** (see [23, Lemma 2.4]) *Suppose that the consistent system of linear equation  $Ax = b$  has a solution  $x^* \in R(A^T)$ , then  $x^*$  is a unique least Frobenius norm solution of the system of linear equations.*

By Lemma 2.3, the following result can be obtained.

**Theorem 2.2** *Suppose that Problem I is consistent. If we choose the initial iterative matrices  $X_j(1) = \sum_{i=1}^p A_{ij}^T K_i B_{ij}^T + \sum_{i=1}^p P_j A_{ij}^T K_i B_{ij}^T Q_j$ ,  $j = 1, 2, \dots, q$ , where  $K_i \in \mathcal{R}^{r_i \times s_i}$ ,  $i = 1, 2, \dots, p$  are arbitrary matrices, especially,  $X_j(1) = 0 \in \mathcal{R}^{m_j \times n_j}(P_j, Q_j)$ , then the solution group  $(X_1^*, X_2^*, \dots, X_q^*)$  generated by Algorithm 2.1 is the unique least Frobenius norm generalized reflexive solution group of Problem I.*

*Proof* We know the solvability of Eq. (1) over generalized reflexive matrices is equivalent to the following matrix equations:

$$\begin{cases} \sum_{j=1}^q A_{ij} X_j B_{ij} = M_i & (i = 1, 2, \dots, p), \\ \sum_{j=1}^q A_{ij} P_j X_j Q_j B_{ij} = M_i & (i = 1, 2, \dots, p). \end{cases} \quad (5)$$

Then the system of matrix equations (5) is equivalent to

$$\begin{pmatrix} B_{11}^T \otimes A_{11} & \cdots & B_{1q}^T \otimes A_{1q} \\ \vdots & \cdots & \vdots \\ B_{p1}^T \otimes A_{p1} & \cdots & B_{pq}^T \otimes A_{pq} \\ B_{11}^T Q_1 \otimes A_{11} P_1 & \cdots & B_{1q}^T Q_q \otimes A_{1q} P_q \\ \vdots & \cdots & \vdots \\ B_{p1}^T Q_1 \otimes A_{p1} P_1 & \cdots & B_{pq}^T Q_q \otimes A_{pq} P_q \end{pmatrix} \begin{pmatrix} \text{vec}(X_1) \\ \vdots \\ \text{vec}(X_q) \end{pmatrix} = \begin{pmatrix} \text{vec}(M_1) \\ \vdots \\ \text{vec}(M_p) \\ \text{vec}(M_1) \\ \vdots \\ \text{vec}(M_p) \end{pmatrix}. \quad (6)$$

Let  $X_j(1) = \sum_{i=1}^p A_{ij}^T K_i B_{ij}^T + \sum_{i=1}^p P_j A_{ij}^T K_i B_{ij}^T Q_j$ ,  $j = 1, 2, \dots, q$ , where  $K_i \in \mathcal{R}^{r_i \times s_i}$  are arbitrary matrices, then

$$\begin{aligned} & \begin{pmatrix} \text{vec}(X_1(1)) \\ \vdots \\ \text{vec}(X_q(1)) \end{pmatrix} \\ &= \begin{pmatrix} \text{vec}(\sum_{i=1}^p A_{i1}^T K_i B_{i1}^T + \sum_{i=1}^p P_1 A_{i1}^T K_i B_{i1}^T Q_1) \\ \vdots \\ \text{vec}(\sum_{i=1}^p A_{iq}^T K_i B_{iq}^T + \sum_{i=1}^p P_q A_{iq}^T K_i B_{iq}^T Q_q) \end{pmatrix} \\ &= \begin{pmatrix} B_{11} \otimes A_{11}^T & \cdots & B_{p1} \otimes A_{p1}^T & Q_1 B_{11} \otimes P_1 A_{11}^T & \cdots & Q_1 B_{p1} \otimes P_1 A_{p1}^T \\ \vdots & \cdots & \vdots & \cdots & \cdots & \vdots \\ B_{1q} \otimes A_{1q}^T & \cdots & B_{pq} \otimes A_{pq}^T & Q_q B_{1q} \otimes P_q A_{1q}^T & \cdots & Q_q B_{pq} \otimes P_q A_{pq}^T \end{pmatrix} \begin{pmatrix} \text{vec}(K_1) \\ \vdots \\ \text{vec}(K_p) \\ \text{vec}(K_1) \\ \vdots \\ \text{vec}(K_p) \end{pmatrix} \\ &= \begin{pmatrix} B_{11}^T \otimes A_{11} & \cdots & B_{1q}^T \otimes A_{1q} \\ \vdots & \vdots & \vdots \\ B_{p1}^T \otimes A_{p1} & \cdots & B_{pq}^T \otimes A_{pq} \\ B_{11}^T Q_1 \otimes A_{11} P_1 & \cdots & B_{1q}^T Q_q \otimes A_{1q} P_q \\ \vdots & \vdots & \vdots \\ B_{p1}^T Q_1 \otimes A_{p1} P_1 & \cdots & B_{pq}^T Q_q \otimes A_{pq} P_q \end{pmatrix}^T \begin{pmatrix} \text{vec}(K_1) \\ \vdots \\ \text{vec}(K_p) \\ \text{vec}(K_1) \\ \vdots \\ \text{vec}(K_p) \end{pmatrix} \\ &\in R \left( \begin{pmatrix} B_{11}^T \otimes A_{11} & \cdots & B_{1q}^T \otimes A_{1q} \\ \vdots & \vdots & \vdots \\ B_{p1}^T \otimes A_{p1} & \cdots & B_{pq}^T \otimes A_{pq} \\ B_{11}^T Q_1 \otimes A_{11} P_1 & \cdots & B_{1q}^T Q_q \otimes A_{1q} P_q \\ \vdots & \vdots & \vdots \\ B_{p1}^T Q_1 \otimes A_{p1} P_1 & \cdots & B_{pq}^T Q_q \otimes A_{pq} P_q \end{pmatrix}^T \right). \end{aligned}$$

Furthermore, we can see that all generalized reflexive matrix solution groups  $(X_1(k), X_2(k), \dots, X_q(k))$  generated by Algorithm 2.1 satisfy

$$\begin{pmatrix} \text{vec}(X_1(1)) \\ \vdots \\ \text{vec}(X_q(1)) \end{pmatrix} \in R \left( \begin{pmatrix} B_{11}^T \otimes A_{11} & \cdots & B_{1q}^T \otimes A_{1q} \\ \vdots & \vdots & \vdots \\ B_{p1}^T \otimes A_{p1} & \cdots & B_{pq}^T \otimes A_{pq} \\ B_{11}^T Q_1 \otimes A_{11} P_1 & \cdots & B_{1q}^T Q_q \otimes A_{1q} P_q \\ \vdots & \vdots & \vdots \\ B_{p1}^T Q_1 \otimes A_{p1} P_1 & \cdots & B_{pq}^T Q_q \otimes A_{pq} P_q \end{pmatrix}^T \right).$$

By Lemma 2.3 we know that  $(X_1^*, X_2^*, \dots, X_q^*)$  is the least Frobenius norm generalized reflexive solution group of the system of linear equations (6). Since a vector operator is isomorphic,  $(X_1^*, X_2^*, \dots, X_q^*)$  is the unique least Frobenius norm generalized reflexive solution group of the system of matrix equations (5). Thus  $(X_1^*, X_2^*, \dots, X_q^*)$  is the unique least Frobenius norm generalized reflexive solution group of Problem I.  $\square$

### 3 The solution of Problem II

In this section, we show that the optimal approximate generalized reflexive solution group of Problem II to a given generalized reflexive matrix group can be derived by finding the least Frobenius norm generalized reflexive solution group of the corresponding general coupled matrix equations.

When Problem I is consistent, the set of generalized reflexive solution group of Problem I denoted by  $S_E$  is not empty. For a given matrix pair  $(X_1^0, X_2^0, \dots, X_q^0)$  with  $X_j^0 \in \mathcal{R}_r^{m_j \times n_j}(P_j, Q_j)$ ,  $j = 1, 2, \dots, q$ , we have

$$\sum_{j=1}^q A_{ij} X_j B_{ij} = M_i$$

$$\iff \sum_{j=1}^q A_{ij} (X_j - X_j^0) B_{ij} = M_i - \sum_{j=1}^q A_{ij} X_j^0 B_{ij}, \quad i = 1, 2, \dots, p. \tag{7}$$

Set  $\tilde{X}_j = X_j - X_j^0$ ,  $\tilde{M}_i = M_i - \sum_{j=1}^q A_{ij} X_j^0 B_{ij}$ , then solving Problem II is equivalent to finding the least Frobenius norm generalized reflexive solution group  $(\tilde{X}_1^*, \tilde{X}_2^*, \dots, \tilde{X}_q^*)$  of the corresponding general coupled matrix equations

$$\sum_{j=1}^q A_{ij} \tilde{X}_j B_{ij} = \tilde{M}_i, \quad i = 1, 2, \dots, p. \tag{8}$$

Let initial iteration matrices be

$$\tilde{X}_j(1) = \sum_{i=1}^p A_{ij}^T K_i B_{ij}^T + \sum_{i=1}^p P_j A_{ij}^T K_i B_{ij}^T Q_j, \quad j = 1, 2, \dots, q,$$

where  $K_i \in \mathcal{R}^{r_i \times s_i}$ ,  $i = 1, 2, \dots, p$  are arbitrary matrices, especially,  $\tilde{X}_j(1) = 0 \in \mathcal{R}^{m_j \times n_j}(P_j, Q_j)$ ,  $j = 1, 2, \dots, q$ . By using Algorithm 2.1, we can get the least Frobenius norm generalized reflexive solution group  $(\tilde{X}_1^*, \tilde{X}_2^*, \dots, \tilde{X}_q^*)$  of (8). Thus the generalized reflexive solution group of Problem II can be represented as

$$(\hat{X}_1, \hat{X}_2, \dots, \hat{X}_q) = (\tilde{X}_1^* + X_1^0, \tilde{X}_2^* + X_2^0, \dots, \tilde{X}_q^* + X_q^0).$$

### 4 A numerical example

In this section, we show a numerical example to illustrate the proposed iterative method. All the tests are performed by MATLAB 7.8.

**Example 4.1** Consider the generalized reflexive solution of the general coupled matrix equations

$$\begin{cases} A_{11} X_1 B_{11} + A_{12} X_2 B_{12} = M_1, \\ A_{21} X_1 B_{21} + A_{22} X_2 B_{22} = M_2, \end{cases} \tag{9}$$

where

$$A_{11} = \begin{pmatrix} 1 & 3 & -5 & 7 & -9 \\ 2 & 0 & 4 & 6 & -1 \\ 0 & -2 & 9 & 6 & -8 \\ 3 & 6 & 2 & 2 & -3 \\ -5 & 5 & -22 & -1 & -11 \\ 8 & 4 & -6 & -9 & -9 \end{pmatrix}, \quad B_{11} = \begin{pmatrix} 4 & 8 & -5 & 4 \\ -1 & 5 & -2 & 3 \\ 3 & 9 & 2 & -6 \\ -2 & 7 & -8 & 1 \end{pmatrix},$$

$$A_{12} = \begin{pmatrix} 6 & -5 & 7 & -9 \\ 2 & 4 & 6 & -11 \\ 9 & -12 & 3 & -8 \\ 13 & 6 & 4 & -15 \\ -5 & 15 & -13 & -11 \\ 2 & 9 & -6 & -9 \end{pmatrix}, \quad B_{12} = \begin{pmatrix} 7 & 1 & 8 & -6 \\ -4 & 5 & -2 & 3 \\ 3 & -12 & 0 & 8 \\ 1 & 6 & 9 & 4 \\ -5 & 8 & -2 & 9 \end{pmatrix},$$

$$A_{21} = \begin{pmatrix} 14 & 5 & -1 & 7 & 1 \\ -2 & 3 & -2 & 5 & 4 \\ 13 & 4 & 2 & -3 & 6 \\ -8 & 1 & -5 & 4 & 8 \end{pmatrix}, \quad B_{21} = \begin{pmatrix} 1 & 3 & -5 & 8 & 2 \\ -11 & 5 & -6 & 2 & 5 \\ 13 & 2 & 7 & -9 & 7 \\ -9 & 6 & -5 & 12 & 1 \end{pmatrix},$$

$$A_{22} = \begin{pmatrix} 1 & 2 & -5 & 8 \\ -5 & 5 & -7 & 3 \\ 2 & 4 & 9 & -6 \\ -3 & 7 & -12 & 11 \end{pmatrix}, \quad B_{22} = \begin{pmatrix} 2 & 4 & 8 & -5 & 4 \\ 7 & -1 & 5 & -2 & 3 \\ 6 & 3 & 9 & 2 & -6 \\ 5 & -2 & 7 & -8 & 1 \\ 1 & 4 & -3 & -2 & 6 \end{pmatrix},$$

$$M_1 = \begin{pmatrix} 941 & -615 & -299 & -2,420 \\ 701 & 153 & -1,088 & -1,350 \\ 746 & -1,846 & -1,931 & -1,352 \\ 1,298 & 1,133 & 305 & -2,594 \\ 1,953 & -4,224 & 2,945 & -3,630 \\ 698 & -1,980 & 1,488 & -2,192 \end{pmatrix},$$

$$M_2 = \begin{pmatrix} 1,242 & 1,986 & 1,365 & -1,150 & 2,727 \\ 1,569 & 477 & 1,215 & -439 & 384 \\ -2,845 & 1,447 & -1,795 & 818 & 1,722 \\ 3,924 & 687 & 3,806 & -2,488 & 1,229 \end{pmatrix}.$$

Let

$$P_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad Q_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

$$P_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

be generalized reflection matrices.

We will find the generalized reflexive solution of Eq. (9) by using Algorithm 2.1. It can be verified that Eq. (9) is consistent over generalized reflexive matrices and the solutions are

$$X_1^* = \begin{pmatrix} -2 & 9 & 2 & 5 \\ 3 & 1 & 11 & -1 \\ 7 & 3 & -7 & 3 \\ 11 & 1 & 3 & -1 \\ -2 & 5 & 2 & 9 \end{pmatrix}, \quad X_2^* = \begin{pmatrix} 14 & 16 & -1 & 3 & 4 \\ 9 & 7 & 0 & 9 & 7 \\ -3 & -8 & -8 & 3 & 8 \\ 3 & 4 & 1 & 14 & 16 \end{pmatrix}.$$

Because of the influence of the error of calculation, the residual  $R(k)$  is usually unequal to zero in the process of iteration, where  $k = 1, 2, \dots$ . For any chosen positive number  $\varepsilon$ , however small it is, e.g.,  $\varepsilon = 1.0000e-010$ , whenever  $\|R(k)\| < \varepsilon$ , stop the iteration,  $X_1(k)$  and  $X_2(k)$  are regarded to be generalized reflexive solutions of Eq. (9). Let the initially iterative matrices be as follows:

$$X_1(1) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad X_2(1) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

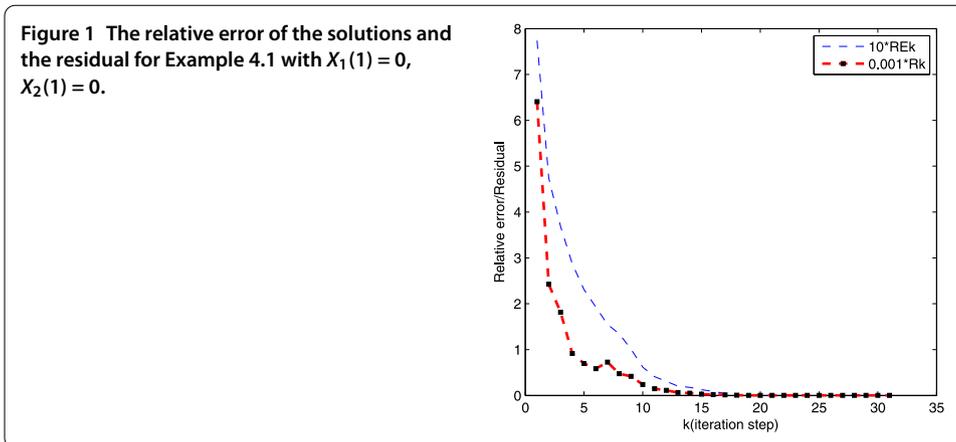
By Algorithm 2.1, we have

$$X_1^* = X_1(30) = \begin{pmatrix} -2.0000 & 9.0000 & 2.0000 & 5.0000 \\ 3.0000 & 1.0000 & 11.0000 & -1.0000 \\ 7.0000 & 3.0000 & -7.0000 & 3.0000 \\ 11.0000 & 1.0000 & 3.0000 & -1.0000 \\ -2.0000 & 5.0000 & 2.0000 & 9.0000 \end{pmatrix},$$

$$X_2^* = X_2(30) = \begin{pmatrix} 14.0000 & 16.0000 & -1.0000 & 3.0000 & 4.0000 \\ 9.0000 & 7.0000 & 0 & 9.0000 & 7.0000 \\ -3.0000 & -8.0000 & -8.0000 & 3.0000 & 8.0000 \\ 3.0000 & 4.0000 & 1.0000 & 14.0000 & 16.0000 \end{pmatrix},$$

$$\|R(30)\| = 6.4815e-012 < \varepsilon.$$

Thus we obtain the reflexive solution of Eq. (9). The relative error and the residual of the solution are shown in Figure 1, where the relative error  $REk = \frac{\|X_1(k) - X_1^*\| + \|X_2(k) - X_2^*\|}{\|X_1^*\| + \|X_2^*\|}$  and the residual  $Rk = \|R(k)\|$ .



Let  $S_E$  denote the set of all generalized reflexive solution group of Eq. (9). For two given generalized reflexive matrices

$$X_1^0 = \begin{pmatrix} 3 & -1 & 2 & 2 \\ 3 & -2 & 0 & 0 \\ 1 & -3 & -1 & -3 \\ 0 & 0 & 3 & 2 \\ -2 & 2 & -3 & -1 \end{pmatrix}, \quad X_2^0 = \begin{pmatrix} 2 & 4 & -2 & 2 & 0 \\ 1 & 3 & 0 & 1 & 3 \\ 5 & -2 & 2 & -5 & 2 \\ 2 & 0 & 2 & 2 & 4 \end{pmatrix},$$

we will find the optimal approximate generalized reflexive solution group to the given matrix group  $(X_1^0, X_2^0)$  in  $S_E$  in Frobenius norm, *i.e.*, find  $(\widehat{X}_1, \widehat{X}_2) \in S_E$  such that

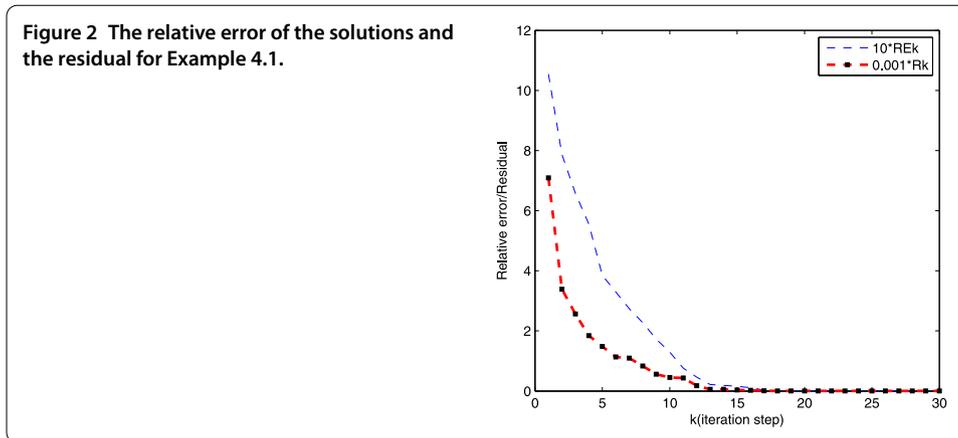
$$\|\widehat{X}_1 - X_1^0\| + \|\widehat{X}_2 - X_2^0\| = \min_{(X_1, X_2) \in S_E} \|X_1 - X_1^0\| + \|X_2 - X_2^0\|.$$

Let  $\widetilde{X}_1 = X_1 - X_1^0, \widetilde{X}_2 = X_2 - X_2^0, \widetilde{M}_1 = M_1 - A_{11}X_1^0B_{11} - A_{12}X_2^0B_{12}, \widetilde{M}_2 = M_2 - A_{21}X_1^0B_{21} - A_{22}X_2^0B_{22}$ . By the method mentioned in Section 3, we can obtain the least-norm generalized reflexive solution group  $(\widetilde{X}_1^*, \widetilde{X}_2^*)$  of the matrix equations  $A_{11}\widetilde{X}_1B_{11} + A_{12}\widetilde{X}_2B_{12} = \widetilde{M}_1, A_{21}\widetilde{X}_1B_{21} + A_{22}\widetilde{X}_2B_{22} = \widetilde{M}_2$  by choosing the initially iterative matrices  $\widetilde{X}_1(1) = 0$  and  $\widetilde{X}_2(1) = 0$ . Then by Algorithm 2.1, we have that

$$\widetilde{X}_1^* = \widetilde{X}_1(29) = \begin{pmatrix} -5.0000 & 10.0000 & 0.0000 & 3.0000 \\ -0.0000 & 3.0000 & 11.0000 & -1.0000 \\ 6.0000 & 6.0000 & -6.0000 & 6.0000 \\ 11.0000 & 1.0000 & -0.0000 & -3.0000 \\ -0.0000 & 3.0000 & 5.0000 & 10.0000 \end{pmatrix},$$

$$\widetilde{X}_2^* = \widetilde{X}_2(29) = \begin{pmatrix} 12.0000 & 12.0000 & 1.0000 & 1.0000 & 4.0000 \\ 8.0000 & 4.0000 & 0 & 8.0000 & 4.0000 \\ -8.0000 & -6.0000 & -10.0000 & 8.0000 & 6.0000 \\ 1.0000 & 4.0000 & -1.0000 & 12.0000 & 12.0000 \end{pmatrix},$$

$$\|R(29)\| = 1.4095e-011 < \varepsilon = 1.0000e-010,$$



and the optimal approximate generalized reflexive solution to the matrix group  $(X_1^0, X_2^0)$  in Frobenius norm is

$$\widehat{X}_1 = \widetilde{X}_1^* + X_1^0 = \begin{pmatrix} -2.0000 & 9.0000 & 2.0000 & 5.0000 \\ 3.0000 & 1.0000 & 11.0000 & -1.0000 \\ 7.0000 & 3.0000 & -7.0000 & 3.0000 \\ 11.0000 & 1.0000 & 3.0000 & -1.0000 \\ -2.0000 & 5.0000 & 2.0000 & 9.0000 \end{pmatrix},$$

$$\widehat{X}_2 = \widetilde{X}_2^* + X_2^0 = \begin{pmatrix} 14.0000 & 16.0000 & -1.0000 & 3.0000 & 4.0000 \\ 9.0000 & 7.0000 & 0 & 9.0000 & 7.0000 \\ -3.0000 & -8.0000 & -8.0000 & 3.0000 & 8.0000 \\ 3.0000 & 4.0000 & 1.0000 & 14.0000 & 16.0000 \end{pmatrix}.$$

The relative error and the residual of the solution are shown in Figure 2, where the relative error  $REk = \frac{\|\widetilde{X}_1(k) + X_1^0 - X_1^*\| + \|\widetilde{X}_2(k) + X_2^0 - X_2^*\|}{\|X_1^*\| + \|X_2^*\|}$  and the residual  $Rk = \|R(k)\|$ .

From Figures 1 and 2, we can see that the iterative solutions generated by Algorithm 2.1 can quickly converge to the exact generalized reflexive solution of Eq. (9). This illustrates that our iterative algorithm is quite effective.

### 5 Conclusions

In this paper, an iterative algorithm is presented to solve the general coupled matrix equations  $\sum_{j=1}^q A_{ij}X_jB_{ij} = M_i$  ( $i = 1, 2, \dots, p$ ) and their optimal approximation problem over generalized reflexive matrices. When the general coupled matrix equations are consistent over generalized reflexive matrices, for any initially generalized reflexive matrix group, the generalized reflexive solution group can be obtained by the iterative algorithm within finite iterative steps in the absence of round-off errors. When a special kind of initial iteration matrix group is chosen, the unique least-norm generalized reflexive solution of the general coupled matrix equations can be derived. Furthermore, the optimal approximate generalized reflexive solution of the general coupled matrix equations to a given generalized reflexive matrix group can be derived by finding the least-norm generalized reflexive solution of new corresponding general coupled matrix equations. Finally, a numerical example is given to show that our iterative algorithm is quite effective.

#### Competing interests

The authors declare that they have no competing interests.

#### Authors' contributions

All authors have the same contributions on this paper. All authors read and approved the final manuscript.

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