Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Some geometric properties of the metric space $V[\lambda, p]$

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Abstract

In this study, we consider the space $V[\lambda, p]$ with an invariant metric. Then, we examine some geometric properties of the linear metric space $V[\lambda, p]$ such as property (β), property (H) and k-NUC property.

MSC: 40A05; 46A45; 46B20

Keywords: linear metric space; Luxemburg norm; de la Vallée-Poussin mean; k-NUC property; property (H)

1 Introduction

Let X be a vector space over the scalar field of real numbers and d be an invariant metric on X. We denote $B_d(X)$ and $S_d(X)$ as follows:

$$B_d(X) = \{x \in X : d(x, \mathbf{0}) \leq r\} \quad \text{and}$$
$$S_d(X) = \{x \in X : d(x, \mathbf{0}) = r\}.$$

Let (X, d) be a linear metric space and $B_d(X)$ (resp., $S_d(X)$) be a closed unit ball (resp., the unit sphere) of X. A linear metric space (X, d) has property (β) if and only if for each $r > 0$ and $\varepsilon > 0$, there exists $\delta > 0$ such that for each element $x \in B_d(0, r)$ and each sequence (x_n) in $B_d(0, r)$ with $\operatorname{sep}(x_n) \geq \varepsilon$, there is an index k for which $d(\frac{x+x_k}{2}, \mathbf{0}) \leq 1 - \delta$, where $\operatorname{sep}(x_n) = \inf\{d(x_n, x_m) : n \neq m\} > \varepsilon$ [1]. If for each $x \in S_d(0, r)$ and $(x_n) \subset S_d(0, r)$, $x_n \xrightarrow{w} x$ implies $x_n \to x$, a linear metric space (X, d) is said to have property (H). Let $k \geq 2$ be an integer. A linear metric space (X, d) is said to be k-nearly uniform convex (k-NUC) if for every $\varepsilon > 0$ and $r > 0$, there exists $\delta > 0$ such that for any sequence $(x_n) \subset B_d(0, r)$ with $\operatorname{sep}(x_n) \geq \varepsilon$, there are $s_1, s_2, \ldots, s_k$ such that $d(\frac{x_{s_1}+x_{s_2}+\cdots+x_{s_k}}{k}, \mathbf{0}) \leq r - \delta$ [2]. These properties have been studied by Mongkolkeha and Pumam [3], Sanhan and Suantai [4], Cui *et al.* [5] and Cui and Hudzik [6].

Ahuja *et al.* [7] introduced the notions of strict convexity and U.C.I (uniform convexity) in linear metric spaces which are generalizations of the corresponding concepts in linear normed spaces. Later, Sastry and Naidu [8] introduced the notions of U.C.II and U.C.III in linear metric spaces and showed that these three forms are not always equivalent. Further, Junde *et al.* [9, 10] showed that if a linear metric space is complete and U.C.I, then it is reflexive.

In summability theory, de la Vallée-Poussin mean was first used to define the (V, λ)-summability by Leindler [11]. (V, λ)-summable sequences have been studied by many authors including Et *et al.* [12, 13], Savas [14–18], Savas and Malkowsky [19] and Şimsek *et al.* [20, 21]. Let $\Lambda = (\lambda_k)$ be a nondecreasing sequence of positive real numbers tending to infinity and let $\lambda_1 = 1$ and $\lambda_{k+1} \leq \lambda_k + 1$. The generalized de la Vallée-Poussin mean is defined by $t_n(x) = \frac{1}{\lambda_n}\sum_{k\in I_n} x_k$, where $I_n = [n - \lambda_n + 1, n]$ for $n = 1, 2, \ldots$. A sequence $x = (x_k)$ is said to be (V,λ)-summable to a number ℓ if $t_n(x) \to \ell$ as $n \to \infty$. If $\lambda_n = n$, then (V,λ)-summability is reduced to Cesàro summability.

Let w be the space of all real sequences. Let $p = (p_k)$ be a bounded sequence of positive real numbers. Şimşek *et al.* [20] defined the space $V[\lambda,p]$ as follows:

$$V[\lambda,p] = \left\{x = (x_k) \in \omega : \sum_{k=1}^{\infty}\left(\frac{1}{\lambda_k}\sum_{j\in I_k}|x_j|\right)^{p_k} < \infty\right\}.$$

If $\lambda_k = k$, then $V[\lambda,p] = \operatorname{ces}(p)$ [22]. If $\lambda_k = k$ and $p_k = p$ for all $k \in \mathrm{N}$, then $V[\lambda,p] = \operatorname{ces}_p$ [23]. Paranorm on $V[\lambda,p]$ is given by

$$h(x) = \left(\sum_{k=1}^{\infty}\left(\frac{1}{\lambda_k}\sum_{j\in I_k}|x_j|\right)^{p_k}\right)^{\frac{1}{M}},$$

where $M = \max\{1, H\}$ and $H = \sup p_k$. If $p_k = p$ for all $k \in \mathrm{N}$, the notation $V_p(\lambda)$ is used in place of $V[\lambda,p]$ and the norm on $V_p(\lambda)$ is as follows:

$$\|x\|_{V_p(\lambda)} = \left(\sum_{k=1}^{\infty}\left(\frac{1}{\lambda_k}\sum_{j\in I_k}|x_j|\right)^{p}\right)^{\frac{1}{p}}.$$

$\rho : V_\rho[\lambda,p] \to [0,\infty]$, $\rho(x) = (\sum_{k=1}^{\infty}(\frac{1}{\lambda_k}\sum_{j\in I_k}|x_j|)^{p_k})$ is a modular on $V_\rho[\lambda,p]$ and the Luxemburg norm on $V_\rho[\lambda,p]$ is defined by $\|x\|_L = \inf\{\sigma > 0 : \rho(\frac{x}{\sigma}) \leq 1\}$ for all $x \in V_\rho[\lambda,p]$. The Amemiya norm on the space $V_\rho[\lambda,p]$ can be similarly introduced as follows:

$$\|x\|_A = \inf_{\sigma>0}\frac{1}{\sigma}\big(1 + \rho(\sigma x)\big) \quad \text{for all } x \in V_\rho[\lambda,p].$$

2 Main results

In this part of the paper, our main purpose is to define a metric on $V[\lambda,p]$ and show that $V[\lambda,p]$ possesses property (β), property (H) and k-NUC property. Let $p = (p_k)$ be a bounded sequence of real numbers with $p_k > 1$ for all $k \in \mathrm{N}$. The mapping $d(x,y) = (\sum_{k=1}^{\infty}(\frac{1}{\lambda_k}\sum_{j\in I_k}|x(j) - y(j)|)^{p_k})^{1/H}$ is a metric on the space $V[\lambda,p]$, where $M = \max(1, H = \sup p_k)$ and $m = \inf p_k$ since the function $|t|^p$ is convex for $p > 1$. First, we will show that the space $V[\lambda,p]$ has property (β) under the above metric. To do this, we need the following two lemmas. To prove these lemmas, we use the technique given in Sanhan and Mongkolkeha [1].

Lemma 2.1 *Let $y, z \in (V[\lambda,p], d)$. If $\beta \in (0,1)$, then*

$$\big(d(y + z, \mathbf{0})\big)^M \leq \big(d(y,\mathbf{0})\big)^M + 2^M\beta\big(d(y,\mathbf{0})\big)^M + \frac{2^M}{\beta^{M-1}}\big(d(z,\mathbf{0})\big)^M.$$

Proof Let $y, z \in (V[\lambda, p], d)$ and $0 < \beta < 1$. Then

$$\begin{aligned}
\big(d(y+z,\mathbf{0})\big)^M &= \sum_{k=1}^{\infty}\bigg(\frac{1}{\lambda_k}\sum_{j\in I_k}\big|y(j)+z(j)\big|\bigg)^{p_k} \\
&\le \sum_{k=1}^{\infty}\bigg((1-\beta)\frac{1}{\lambda_k}\sum_{j\in I_k}\big|y(j)\big| + \beta\frac{1}{\lambda_k}\sum_{j\in I_k}\bigg|y(j)+\frac{z(j)}{\beta}\bigg|\bigg)^{p_k} \\
&\le (1-\beta)\sum_{k=1}^{\infty}\bigg(\frac{1}{\lambda_k}\sum_{j\in I_k}\big|y(j)\big|\bigg)^{p_k} + \beta\sum_{k=1}^{\infty}\bigg(\frac{1}{\lambda_k}\sum_{j\in I_k}\bigg|y(j)+\frac{z(j)}{\beta}\bigg|\bigg)^{p_k} \\
&\le \sum_{k=1}^{\infty}\bigg(\frac{1}{\lambda_k}\sum_{j\in I_k}\big|y(j)\big|\bigg)^{p_k} + 2^M\beta\sum_{k=1}^{\infty}\bigg(\frac{1}{\lambda_k}\sum_{j\in I_k}\big|y(j)\big|\bigg)^{p_k} \\
&\quad + 2^M\sum_{k=1}^{\infty}\bigg(\frac{1}{\lambda_k}\sum_{j\in I_k}\bigg|\frac{z(j)}{\beta}\bigg|\bigg)^{p_k} \\
&\le \sum_{k=1}^{\infty}\bigg(\frac{1}{\lambda_k}\sum_{j\in I_k}\big|y(j)\big|\bigg)^{p_k} + 2^M\beta\sum_{k=1}^{\infty}\bigg(\frac{1}{\lambda_k}\sum_{j\in I_k}\big|y(j)\big|\bigg)^{p_k} \\
&\quad + \frac{2^M}{\beta^{M-1}}\sum_{k=1}^{\infty}\bigg(\frac{1}{\lambda_k}\sum_{j\in I_k}\big|z(j)\big|\bigg)^{p_k} \\
&= \big(d(y,\mathbf{0})\big)^M + 2^M\beta\big(d(y,\mathbf{0})\big)^M + \frac{2^M}{\beta^{M-1}}\big(d(z,\mathbf{0})\big)^M. \qquad \square
\end{aligned}$$

Lemma 2.2 *Let $y, z \in (V[\lambda, p], d)$. Then for any $\varepsilon > 0$ and $L > 0$, there exists $\delta > 0$ such that*

$$\big|\big(d(y+z,\mathbf{0})\big)^M - \big(d(y,\mathbf{0})\big)^M\big| < \varepsilon,$$

where $(d(y,\mathbf{0}))^M \le L$ and $(d(z,\mathbf{0}))^M \le \delta$.

Proof Let $\varepsilon > 0$ and $L > 0$. For $\beta = \frac{\varepsilon}{2^{M+1}(L+\varepsilon)}$, we take $\delta = \frac{\varepsilon\beta^{M-1}}{2^{M+1}}$. From Lemma 2.1, we have

$$\begin{aligned}
\big(d(y+z,\mathbf{0})\big)^M &\le \big(d(y,\mathbf{0})\big)^M + 2^M\beta\big(d(y,\mathbf{0})\big)^M + \frac{2^M}{\beta^{M-1}}\big(d(z,\mathbf{0})\big)^M \\
&\le \big(d(y,\mathbf{0})\big)^M + 2^M\beta L + \frac{2^M}{\beta^{M-1}}\delta \\
&\le \big(d(y,\mathbf{0})\big)^M + 2^M\frac{\varepsilon}{2^{M+1}}\frac{L}{L+\varepsilon} + \frac{2^M}{\beta^{M-1}}\frac{\varepsilon\beta^{M-1}}{2^{M+1}} \\
&\le \big(d(y,\mathbf{0})\big)^M + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&\le \big(d(y,\mathbf{0})\big)^M + \varepsilon \qquad (2.1)
\end{aligned}$$

and

$$\begin{aligned}
\big(d(y,\mathbf{0})\big)^M &\le \big(d(y+z,\mathbf{0})\big)^M + 2^M\beta\big(d(y+z,\mathbf{0})\big)^M + \frac{2^M}{\beta^{M-1}}\big(d(-z,\mathbf{0})\big)^M \\
&\le \big(d(y+z,\mathbf{0})\big)^M + 2^M\beta\Big(\big(d(y,\mathbf{0})\big)^M + \varepsilon\Big) + \frac{2^M}{\beta^{M-1}}\delta
\end{aligned}$$

$$\begin{aligned}
&\leq \big(d(y+z,\mathbf{0})\big)^M + 2^M\beta(L+\varepsilon) + \frac{2^M}{\beta^{M-1}}\frac{\varepsilon\beta^{M-1}}{2^{M+1}} \\
&= \big(d(y+z,\mathbf{0})\big)^M + 2^M\frac{\varepsilon}{2^{M+1}(L+\varepsilon)}(L+\varepsilon) + \frac{\varepsilon}{2} \\
&= \big(d(y+z,\mathbf{0})\big)^M + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \big(d(y+z,\mathbf{0})\big)^M + \varepsilon. \qquad (2.2)
\end{aligned}$$

From (2.1) and (2.2), we obtain that $|(d(y+z,\mathbf{0}))^M - (d(y,\mathbf{0}))^M| < \varepsilon$. □

Theorem 2.3 *The space* $(V[\lambda,p],d)$ *has property* (β).

Proof Let $\varepsilon > 0$ and $(x_n) \subset B(V[\lambda,p],d)$ such that $\operatorname{sep}(x_n) \geq \varepsilon$ and $x \in B(V[\lambda,p],d)$. We take $y^N = (0,0,\ldots,0,\sum_{k=1}^N y(k), y(N+1), y(N+2),\ldots)$. By using the diagonal method, we can find a subsequence (x_{n_r}) of (x_n) for each $N \in \mathrm{N}$ such that $(x_{n_r}(k))$ converges for each $k \in \mathrm{N}$ with $1 \leq k \leq N$, since $(x_n(k))_{k=1}^\infty$ is bounded for each $k \in \mathrm{N}$. Therefore, there is $t_N \in \mathrm{N}$ for each $N \in \mathrm{N}$ such that $\operatorname{sep}((x_n^N)_{r>t_N}) \geq \varepsilon$. So, there is a sequence of positive integers $(t_N)_{N=1}^\infty$ with $t_1 < t_2 < t_3 \cdots$ such that $d(x_{t_N}^N,\mathbf{0}) \geq \frac{\varepsilon}{2}$ for all $N \in \mathrm{N}$. Then there exists $\kappa > 0$ such that for all $N \in \mathrm{N}$,

$$\sum_{k=N}^\infty \bigg(\frac{1}{\lambda_k}\sum_{j\in I_k}|x_{t_N}|\bigg)^{p_k} \geq \kappa. \qquad (2.3)$$

By Lemma 2.2, there exists δ_0 such that

$$\big|\big(d(y+z,\mathbf{0})\big)^M - \big(d(y,\mathbf{0})\big)^M\big| < \frac{\kappa}{2^m}, \qquad (2.4)$$

where $(d(y,\mathbf{0}))^M < j^M$ and $(d(z,\mathbf{0}))^M \leq \delta_0$. There exists $N_1 \in \mathrm{N}$ such that $(d(x^{N_1},\mathbf{0}))^M \leq \delta_0$ if $x \in B(V[\lambda,p])$ and $(d(x,\mathbf{0}))^M \leq \delta_0$. Let us take $y = x_{t_{N_1}}^{N_1}$ and $z = x^{N_1}$. Hence, we have

$$\sum_{k=N_1}^\infty \bigg(\frac{1}{\lambda_k}\sum_{j\in I_k}\bigg|\frac{x(j)+x_{t_{N_1}}(j)}{2}\bigg|\bigg)^{p_k} \leq \sum_{k=N_1}^\infty \bigg(\frac{1}{\lambda_k}\sum_{j\in I_k}\bigg|\frac{x_{t_{N_1}}(j)}{2}\bigg|\bigg)^{p_k} + \frac{\kappa}{2^m}. \qquad (2.5)$$

From (2.3), (2.4), (2.5) and by using the convexity of the function $f(t) = |t|^{p_k}$ for all $k \in \mathrm{N}$, we obtain that

$$\begin{aligned}
\bigg(d\bigg(\frac{y+z}{2},\mathbf{0}\bigg)\bigg)^M &= \sum_{k=1}^\infty \bigg(\frac{1}{\lambda_k}\sum_{j\in I_k}\bigg|\frac{x(j)+x_{t_{N_1}}(j)}{2}\bigg|\bigg)^{p_k} \\
&= \sum_{k=1}^{N_1-1} \bigg(\frac{1}{\lambda_k}\sum_{j\in I_k}\bigg|\frac{x(j)+x_{t_{N_1}}(j)}{2}\bigg|\bigg)^{p_k} + \sum_{k=N_1}^\infty \bigg(\frac{1}{\lambda_k}\sum_{j\in I_k}\bigg|\frac{x(j)+x_{t_{N_1}}(j)}{2}\bigg|\bigg)^{p_k} \\
&\leq \sum_{k=1}^{N_1-1} \bigg(\frac{1}{\lambda_k}\sum_{j\in I_k}\bigg|\frac{x(j)+x_{t_{N_1}}(j)}{2}\bigg|\bigg)^{p_k} + \sum_{k=N_1}^\infty \bigg(\frac{1}{\lambda_k}\sum_{j\in I_k}\bigg|\frac{x_{t_{N_1}}(k)}{2}\bigg|\bigg)^{p_k} + \frac{\kappa}{2^m} \\
&\leq \frac{1}{2}\sum_{k=1}^{N_1-1} \bigg(\frac{1}{\lambda_k}\sum_{j\in I_k}\big|x(j)\big|\bigg)^{p_k} + \frac{1}{2}\sum_{k=1}^{N_1-1} \bigg(\frac{1}{\lambda_k}\sum_{j\in I_k}\big|x_{t_{N_1}}(j)\big|\bigg)^{p_k}
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{1}{2^m} \sum_{k=N_1}^{\infty} \left(\frac{1}{\lambda_k} \sum_{j \in I_k} \left| x_{t_{N_1}}(j) \right| \right)^{p_k} + \frac{\kappa}{2^m} \\
&\leq \frac{1}{2} \sum_{k=1}^{N_1 - 1} \left(\frac{1}{\lambda_k} \sum_{j \in I_k} \left| x(j) \right| \right)^{p_k} + \frac{1}{2} \sum_{k=1}^{\infty} \left(\frac{1}{\lambda_k} \sum_{j \in I_k} \left| x_{t_{N_1}}(j) \right| \right)^{p_k} \\
&\quad - \frac{2^m - 2}{2^{m+1}} \sum_{k=N_1}^{\infty} \left(\frac{1}{\lambda_k} \sum_{j \in I_k} \left| x_{t_{N_1}}(j) \right| \right)^{p_k} + \frac{\kappa}{2^m} \\
&< \frac{j^M}{2} + \frac{j^M}{2} - \frac{2^m - 2}{2^{m+1}} \kappa + \frac{\kappa}{2^m} \\
&= j^M - \frac{\kappa}{2}.
\end{aligned}
$$

Therefore, we have $d(\frac{y+z}{2}, \mathbf{0}) < (j^M - \frac{\kappa}{2})^{1/M} < j - \delta$ whenever $\delta \in (0, j - (j^M - \frac{\kappa}{2})^{1/M})$. Consequently, the space $(V[\lambda, p], d)$ possesses property (β). □

Now, we will show that the space $(V[\lambda, p], d)$ has k-NUC property.

Theorem 2.4 *The space $V[\lambda, p]$ is k-NUC for any integer $k \geq 2$.*

Proof Let $\varepsilon > 0$ and $(x_n) \subset B_d(V[\lambda, p])$ with $\operatorname{sep}(x_n) \geq \varepsilon$. For each $m \in \mathrm{N}$, let

$$
x_n^m = \left(0, 0, \ldots, x_n(m), x_n(m+1), \ldots \right). \tag{2.6}
$$

Since the sequence $(x_n(i))_{i=1}^{\infty}$ is bounded for each $i \in \mathrm{N}$, by using the diagonal method, we can find a subsequence (x_{n_l}) of (x_n) such that $(x_{n_l}(k))$ converges for each $k \in \mathrm{N}$. Therefore, there is an increasing sequence t_m with $\operatorname{sep}((x_{n_l}^m)_{l > t_m}) \geq \varepsilon$. Hence, there exists a sequence of positive integers $(r_m)_{m=1}^{\infty}$ with $r_1 < r_2 < r_3 < \cdots$ such that $d(x_{r_m}^m, \mathbf{0}) \geq \frac{\varepsilon}{2}$ for all $m \in \mathrm{N}$. Then there is $\zeta > 0$ such that

$$
\sum_{k=m}^{\infty} \left(\frac{1}{\lambda_k} \sum_{j \in I_k} |x_{r_m}| \right)^{p_k} \geq \zeta. \tag{2.7}
$$

Let $\alpha > 0$ such that $1 < \alpha < \lim_{k \to \infty} \inf p_k$. Let $\varepsilon_1 = \frac{n^{\alpha-1} - 1}{(n-1)n^{\alpha}} \frac{\zeta}{2}$ for $k \geq 2$. From Lemma 2.2, there is a $\delta > 0$ such that

$$
\left| \left(d(y + z, \mathbf{0}) \right)^M - \left(d(y, \mathbf{0}) \right)^M \right| < \varepsilon_1, \tag{2.8}
$$

where $(d(y, \mathbf{0}))^M < r^M$ and $(d(z, \mathbf{0}))^M \leq \delta$. Then there exist positive integers m_i ($i = 1, 2, \ldots, n-1$) with $m_1 < m_2 < \cdots < m_{n-1}$ such that $d(x_i^{m_i}, \mathbf{0}) \leq \delta$. Now, define $m_n = m_{n-1} + 1$. Then we have $d(x_{r_{m_n}}^{m_n}, \mathbf{0}) \geq \zeta$ for all $m \in \mathrm{N}$. For $1 \leq i \leq n-1$, let $s_i = i$ and $s_n = r_{m_n}$. By using (2.6), (2.7), (2.8) and the convexity of the function $f_i(u) = |u|^{p_i}$ ($i \in \mathrm{N}$), we obtain

$$
\begin{aligned}
&\left(d\left(\frac{x_{s_1} + x_{s_2} + \cdots + x_{s_n}}{n}, \mathbf{0} \right) \right)^M \\
&\quad = \sum_{k=1}^{\infty} \left(\frac{1}{\lambda_k} \sum_{j \in I_k} \left| \frac{x_{s_1}(j) + \cdots + x_{s_n}(j)}{n} \right| \right)^{p_k}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{k=1}^{m_1} \left(\frac{1}{\lambda_k} \sum_{j\in I_k} \left| \frac{x_{s_1}(j) + \cdots + x_{s_n}(j)}{n} \right| \right)^{p_k} + \sum_{k=m_1+1}^{\infty} \left(\frac{1}{\lambda_k} \sum_{j\in I_k} \left| \frac{x_{s_1}(j) + \cdots + x_{s_n}(j)}{n} \right| \right)^{p_k} \\
&\leq \sum_{k=1}^{m_1} \left(\frac{1}{\lambda_k} \sum_{j\in I_k} \left| \frac{x_{s_1}(j) + \cdots + x_{s_n}(j)}{n} \right| \right)^{p_k} + \sum_{k=m_1+1}^{\infty} \left(\frac{1}{\lambda_k} \sum_{j\in I_k} \left| \frac{x_{s_1}(j) + \cdots + x_{s_n}(j)}{n} \right| \right)^{p_k} + \varepsilon_1 \\
&\leq \sum_{k=1}^{m_1} \frac{1}{n} \sum_{i=1}^{n} \left(\frac{1}{\lambda_k} \sum_{j\in I_k} \left| x_{s_i}(j) \right| \right)^{p_k} + \sum_{k=m_1+1}^{m_2} \left(\frac{1}{\lambda_k} \sum_{j\in I_k} \left| \frac{x_{s_2}(j) + x_{s_3}(j) + \cdots + x_{s_n}(j)}{n} \right| \right)^{p_k} \\
&\quad + \sum_{k=m_2+1}^{\infty} \left(\frac{1}{\lambda_k} \sum_{j\in I_k} \left| \frac{x_{s_3}(j) + x_{s_4}(j) + \cdots + x_{s_n}(j)}{n} \right| \right)^{p_k} + 2\varepsilon_1 \\
&\leq \sum_{k=1}^{m_1} \frac{1}{n} \sum_{i=1}^{n} \left(\frac{1}{\lambda_k} \sum_{j\in I_k} \left| x_{s_i}(j) \right| \right)^{p_k} + \sum_{k=m_1+1}^{m_2} \frac{1}{n} \sum_{i=2}^{n} \left(\frac{1}{\lambda_k} \sum_{j\in I_k} \left| x_{s_i}(j) \right| \right)^{p_k} \\
&\quad + \sum_{k=m_2+1}^{m_3} \frac{1}{n} \sum_{i=3}^{n} \left(\frac{1}{\lambda_k} \sum_{j\in I_k} \left| x_{s_i}(j) \right| \right)^{p_k} + \cdots + \sum_{k=m_{n-1}+1}^{m_n} \frac{1}{n} \sum_{i=n-1}^{n} \left(\frac{1}{\lambda_k} \sum_{j\in I_k} \left| x_{s_i}(j) \right| \right)^{p_k} \\
&\quad + \sum_{k=m_n+1}^{\infty} \left(\frac{1}{\lambda_k} \sum_{j\in I_k} \left| \frac{x_{s_n}(j)}{n} \right| \right)^{p_k} + (n-1)\varepsilon_1 \\
&\leq \left(\frac{(d(x_{s_1},\theta))^M + (d(x_{s_2},\theta))^M + \cdots + (d(x_{s_n},\theta))^M}{n} \right) + \frac{1}{n} \sum_{k=1}^{m_n} \left(\frac{1}{\lambda_k} \sum_{j\in I_k} \left| x_{s_n}(j) \right| \right)^{p_k} \\
&\quad + \sum_{k=m_n+1}^{\infty} \left(\frac{1}{\lambda_k} \sum_{j\in I_k} \left| \frac{x_{s_n}(j)}{n} \right| \right)^{p_k} + (n-1)\varepsilon_1 \\
&\leq \frac{n-1}{n} r^M + \frac{1}{n} \sum_{k=1}^{m_n} \left(\frac{1}{\lambda_k} \sum_{j\in I_k} \left| x_{s_n}(j) \right| \right)^{p_k} + \frac{1}{n^\alpha} \sum_{k=m_n+1}^{\infty} \left(\frac{1}{\lambda_k} \sum_{j\in I_k} \left| \frac{x_{s_n}(j)}{n} \right| \right)^{p_k} + (n-1)\varepsilon_1 \\
&\leq r^M - \frac{r^M}{n} + \frac{1}{n} \left(r^M - \sum_{k=m_n+1}^{\infty} \left(\frac{1}{\lambda_k} \sum_{j\in I_k} \left| x_{s_n}(j) \right| \right)^{p_k} \right) \\
&\quad + \frac{1}{n^\alpha} \sum_{k=m_n+1}^{\infty} \left(\frac{1}{\lambda_k} \sum_{j\in I_k} \left| \frac{x_{s_n}(j)}{n} \right| \right)^{p_k} + (n-1)\varepsilon_1 \\
&\leq r^M + (n-1)\varepsilon_1 - \left(\frac{n^{\alpha-1}-1}{n^\alpha} \right) \zeta \\
&\leq r^M + (n-1) \frac{n^{\alpha-1}-1}{n^\alpha(n-1)} \left(\frac{\zeta}{2} \right) - \left(\frac{n^{\alpha-1}-1}{n^\alpha} \right) \zeta \\
&= r^M - \left(\frac{n^{\alpha-1}-1}{n^\alpha} \right) \left(\frac{\zeta}{2} \right).
\end{aligned}
$$

Thus, we have $d(\frac{x_{s_1}(j)+x_{s_2}(j)+\cdots+x_{s_n}(j)}{n}, \mathbf{0}) < (r^M - (\frac{n^{\alpha-1}-1}{n^\alpha})\frac{\zeta}{2})^{1/M} < r - \delta$ for $\delta \in (0, r - (r^M - (\frac{n^{\alpha-1}-1}{n^\alpha})\frac{\zeta}{2})^{1/M})$. Hence, $(V[\lambda,p],d)$ is k-NUC. □

Since k-NUC implies NUC and NUC implies property (H), by using the previous theorem, we can give the following result.

Corollary 2.5 *The space $(V[\lambda,p],d)$ has property (H).*

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
MC, MK and ME have contributed to all parts of the article. All authors read and approved the final manuscript.

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Received: 18 September 2012 Accepted: 26 December 2012 Published: 22 January 2013

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doi:10.1186/1029-242X-2013-28
Cite this article as: Çinar et al.: **Some geometric properties of the metric space** $V[\lambda, p]$. *Journal of Inequalities and Applications* 2013 **2013**:28.