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The growth and value distribution of Laplace-Stieltjes transformations with infinite order in the right half-plane

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Abstract

By introducing the concept of X_U-order functions, we study the growth of analytic functions defined by Laplace-Stieltjes transformations which converge on the right half-plane. Some necessary and sufficient conditions on finite X_U-order of these functions have been obtained. We also investigate the value distribution of Laplace-Stieltjes transformations with finite X_U-order and obtain the existence of X_U-points and X_U-points dealing with multiple values of two Laplace-Stieltjes transformations which converge on the right half-plane. The main results of this paper are improvement and extension of some theorems given by Shang and Gao.

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1 Introduction and basic notes

Consider the Laplace-Stieltjes transforms

$$F(s) = \int_0^{+\infty} e^{-sx}\,d\alpha(x), \qquad s = \sigma + it, \tag{1}$$

where $\alpha(x)$ is a bounded variation on any interval $[0, Y]$ $(0 < Y < +\infty)$, and σ and t are two real variables. We choose a sequence $\{\lambda_n\}_{n=1}^{\infty}$

$$0 = \lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_n \uparrow +\infty, \tag{2}$$

which satisfies the following conditions:

$$\limsup_{n\to+\infty}(\lambda_{n+1} - \lambda_n) < +\infty, \qquad \limsup_{n\to+\infty}\frac{n}{\lambda_n} = D < \infty, \tag{3}$$

$$\limsup_{n\to+\infty}\frac{\log A_n^*}{\lambda_n} = 0, \tag{4}$$

where

$$A_n^* = \sup_{\lambda_n < x \le \lambda_{n+1}, -\infty < t < +\infty}\left|\int_{\lambda_n}^{x} e^{-ity}\,d\alpha(y)\right|.$$

Remark 1.1 The Dirichlet series is regarded as a special example of Laplace-Stieltjes transformations, and considerable attention has been paid to the growth and the value distribution of analytic functions defined by the Dirichlet series; see [1–3] for some recent results.

In 1963, Yu [4] proved the Valiron-Knopp-Bohr formula of the associated abscissas of bounded convergence, absolute convergence and uniform convergence of Laplace-Stieltjes.

Theorem A *Suppose that Laplace-Stieltjes transformations* (1) *satisfy the first formula of* (3) *and* $\limsup_{n\to+\infty}\frac{\log n}{\lambda_n}<+\infty$. *Then*

$$\limsup_{n\to+\infty}\frac{\log A_n^*}{\lambda_n}\le\sigma_u^F\le\limsup_{n\to+\infty}\frac{\log A_n^*}{\lambda_n}+\limsup_{n\to+\infty}\frac{\log n}{\lambda_n},$$

where σ_u^F *is called the abscissa of uniformly convergent* $F(s)$.

It follows from (3), (4) and Theorem A that $\sigma_u^F=0$, *i.e.*, $F(s)$ is analytic in the right half-plane. Put

$$\mu(\sigma,F)=\max_{n\in N}\left\{A_n^*e^{-\lambda_n\sigma}\right\}\quad(\sigma>0),\qquad M(\sigma,F)=\sup_{-\infty<t<+\infty}\left|F(\sigma+it)\right|,$$

$$M_u(\sigma,F)=\sup_{0<x<+\infty,-\infty<t<+\infty}\left|\int_0^x e^{-(\sigma+it)y}\,d\alpha(y)\right|\quad(\sigma>0).$$

Remark 1.2 The concepts of $M_u(\sigma,F)$, $\mu(\sigma,F)$ of analytic functions represented by Laplace-Stieltjes transformations convergent in the complex plane were first introduced by Yu.

Remark 1.3 From (4), for any $\sigma>0$, we have

$$\limsup_{n\to+\infty}\frac{\log A_n^*-\lambda_n\sigma}{\lambda_n}=-\sigma<0\quad\text{or}\quad\limsup_{n\to+\infty}\log A_n^*e^{-\lambda_n\sigma}=-\infty.$$

This implies that $\mu(\sigma,F)$ exists.

Many problems of analytic functions defined by Laplace-Stieltjes transformations have been studied and some important results have been obtained in [5–11]. In those papers, the authors mainly used the technique of a type function $U(x)$ to control the denominator in the definition of order. In 2012, Kong [12] investigated the growth of the Laplace-Stieltjes transforms convergent in the right half-plane by using a type function of the infinite order. In [13], the authors also investigated the growth and value distribution of infinite order analytic functions represented by Laplace-Stieltjes transformations convergent in the right half-plane. *They introduced a completely new technique based on the concept of* $X(x)$ *to control the growth order of the numerator* $\log M_u(\sigma,F)$ *or* $\log\mu(\sigma,F)$, *and obtained the main theorems as follows.*

Theorem B (see [13]) *If the Laplace-Stieltjes transformation* $F(s)$ *of infinite order has finite* X*-order, and sequence* (2) *satisfies* (3) *and* (4), *then we have*

$$\limsup_{\sigma\to 0^+}\frac{X(\log^+ M_u(\sigma,F))}{\log\frac{1}{\sigma}}=\rho^* \quad\Longleftrightarrow\quad \limsup_{\sigma\to 0^+}\frac{X(\log^+\mu(\sigma,F))}{\log\frac{1}{\sigma}}=\rho^*.$$

Theorem C (see [13]) *If the Laplace-Stieltjes transformation* $F(s)$ *of infinite order and sequence* (2) *satisfies* (3) *and* (4), *then we have*

$$\limsup_{\sigma\to 0^+}\frac{X(\log^+ M_u(\sigma,F))}{\log\frac{1}{\sigma}}=\rho^* \quad\Longleftrightarrow\quad \limsup_{n\to\infty}\frac{X(\lambda_n)}{\log^+\frac{\lambda_n}{\log^+ A_n^*}}=\rho^*,$$

where $0<\rho^*<\infty$.

Remark 1.4 In Theorems B and C, the definitions of X-order and the function $X(x)$ are introduced in Section 2.

Thus, a question arises naturally: What will happen when $\rho^*=\infty$ *in Theorems* B *and* C*?*

In this paper, we investigate the above question by using the type functions $U(x)$ to enlarge the growth of the denominator $\log\frac{1}{\sigma}$, where $U(x)=x^{\rho(x)}$ satisfies the following conditions:

(i) $\rho(x)$ is monotone and $\lim_{x\to\infty}\rho(x)=\infty$;
(ii) $\lim_{x\to\infty}\frac{\log U(x')}{\log U(x)}=1$, where $x'=x(1+\frac{1}{\log U(x)})$.

Theorem 1.1 *If the Laplace-Stieltjes transformation* $F(s)$ *of infinite order has infinite* X*-order, and sequence* (2) *satisfies* (3) *and* (4), *then we have*

$$\limsup_{\sigma\to 0^+}\frac{X(\log^+ M_u(\sigma,F))}{\log U(\frac{1}{\sigma})}=T \quad\Longleftrightarrow\quad \limsup_{\sigma\to 0^+}\frac{X(\log^+\mu(\sigma,F))}{\log U(\frac{1}{\sigma})}=T,$$

where $0<T<\infty$.

Remark 1.5 If the Laplace-Stieltjes transformation $F(s)$ of infinite order has infinite X-order and satisfies

$$\limsup_{\sigma\to 0^+}\frac{X(\log M_u(\sigma,F))}{\log U(\frac{1}{\sigma})}=T, \tag{5}$$

then T is called the X_U-order of the Laplace-Stieltjes transform $F(s)$.

Remark 1.6 From Lemma 2.1 and Lemma 2.2 in Section 2, we can prove the conclusion of Theorem 1.1 easily.

Theorem 1.2 *If the Laplace-Stieltjes transformation* $F(s)$ *has infinite* X*-order and sequence* (2) *satisfies* (3) *and* (4), *then we have*

$$\limsup_{\sigma\to 0^+}\frac{X(\log^+ M_u(\sigma,F))}{\log U(\frac{1}{\sigma})}=T \quad\Longleftrightarrow\quad \limsup_{n\to\infty}\frac{X(\lambda_n)}{\log^+ U(\frac{\lambda_n}{\log^+ A_n^*})}=T.$$

From Theorem 1.2, we further investigate the value distribution of analytic functions with infinite X-order represented by Laplace-Stieltjes transformations convergent in the right half-plane and obtain the following theorems.

Theorem 1.3 *Suppose that sequence* (2) *satisfies* (3) *and* (4) *and the Laplace-Stieltjes transformation* $F(s)$ *has infinite order. Let* $\alpha(x) = \alpha_1(x) + i\alpha_2(x)$, *where* $\alpha_1(x)$ *is an increasing function, and for any positive number* $K > 0$ *and* $|\delta|$, $\alpha_2(x)$ *satisfies* (5) *and*

$$\bigl|\alpha_2(x+\delta) - \alpha_2(x)\bigr| \le K\bigl|\alpha_1(x+\delta) - \alpha_1(x)\bigr|, \quad 0 \le x, x+\delta < +\infty.$$

Then $s = 0$ *is the* X_U*-point of* $F(s)$ *with finite* X_U*-order* $\varrho \ge T$, *that is, for any* $\eta > 0$, *the inequality*

$$\limsup_{\sigma\to 0^+} \frac{X(\overline{n}(\sigma, 0, \eta, F = a))}{\log U(\frac{1}{\sigma})} = \varrho \ge T$$

holds for any $a \in \mathbb{C}$ *with one exception, where* $\overline{n}(\sigma, 0, \eta, F = a)$ *is the counting function of distinct zero of the function* $F(s) - a$ *in the strip* $\{s : \Re(s) > \sigma, |\Im(s)| < \eta\}$.

Theorem 1.4 *Suppose that sequence* (2) *satisfies* (3) *and* (4), *and the Laplace-Stieltjes transformation* $F(s)$ *is of infinite order. Let* $\alpha(x) = \int_0^x r(y)e^{it_0 y}\,dy$, *where* $r(y)$ *is a continuous function on* $y \in [0, +\infty)$, $r(y) \ge 0$, t_0 *is a positive real number, and if* $F(s)$ *satisfies* (5), *then* $s = it_0$ *is the* X_U*-point of* $F(s)$ *with finite* X_U*-order* $\varrho \ge T$, *that is, for any* $\eta > 0$, *the inequality*

$$\limsup_{\sigma\to 0^+} \frac{X(\overline{n}(\sigma, it_0, \eta, F = a))}{\log U(\frac{1}{\sigma})} = \varrho \ge T$$

holds for any $a \in \mathbb{C}$ *with one exception, where* $\overline{n}(\sigma, it_0, \eta, F = a)$ *is the counting function of distinct zeros of the function* $F(s) - a$ *in the strip* $\{s : \Re(s) > \sigma, |\Im(s) - t_0| < \eta\}$.

Theorem 1.5 *Under the assumptions of Theorem* 1.4, l (≥ 1) *is a positive integer. Then* $s = 0$ *is the* X_U*-point dealing with multiple values of* $F(s)$ *with finite* X_U*-order* $\varrho \ge T$, *that is, for any* $\eta > 0$, *the inequality*

$$\limsup_{\sigma\to 0^+} \frac{X(\overline{n}^{l)}(\sigma, 0, \eta, F = a))}{\log U(\frac{1}{\sigma})} = \varrho \ge T$$

holds for any $a \in \mathbb{C}$ *with at most* q $(> 1 + [\frac{1}{l}])$ *exceptions, where* $\overline{n}^{l)}(\sigma, 0, \eta, F = a)$ *is the counting function of distinct zeros of the function* $F(s) - a$ *in the strip* $\{s : \Re(s) > \sigma, |\Im(s)| < \eta\}$, *whose multiplicities are not greater than* l.

Theorem 1.6 *Under the assumptions of Theorem* 1.4, l (≥ 1) *is a positive integer. Then* $s = it_0$ *is the* X_U*-point dealing with multiple values of* $F(s)$ *with finite* X_U*-order* $\varrho \ge T$, *that is, for any* $\eta > 0$, *the inequality*

$$\limsup_{\sigma\to 0^+} \frac{X(\overline{n}^{l)}(\sigma, it_0, \eta, F = a))}{\log U(\frac{1}{\sigma})} = \varrho \ge T$$

holds for any $a \in \mathbb{C}$ *with at most* $q\ (> 1 + [\frac{1}{l}])$ *possible exceptions, where* $\overline{n}^{l)}(\sigma, it_0, \eta, F = a)$ *is the counting function of distinct zeros of the function* $F(s) - a$ *in the strip* $\{s : \Re(s) > \sigma, |\Im(s) - t_0| < \eta\}$, *whose multiplicities are not greater than* l.

The structure of this paper is as follows. In Section 2, we introduce the concepts of X-order and X_U-order. Section 3 is devoted to proving Theorem 1.2. Section 4 is devoted to proving Theorems 1.3-1.6.

2 The definitions of X-order and X_U-order

We first introduce the concept of X-order of such functions as follows.

Definition 2.1 [14] If the Laplace-Stieltjes transform $F(s)$ satisfies $\sigma_u^F = 0$ (sequence (2) satisfies (3) and (4)) and

$$\limsup_{\sigma \to 0^+} \frac{\log^+ \log^+ M_u(\sigma, F)}{\log \frac{1}{\sigma}} = \infty,$$

then $F(s)$ is called a Laplace-Stieltjes transform of infinite order.

By studying a lot of papers, we found that to control the growth of the molecule $M_u(\sigma, F)$ or $\mu(\sigma, F)$ in the definition of order, many mathematicians proposed the type functions $U(x)$ to enlarge the growth of the denominator $\log \frac{1}{\sigma}$ or $-\sigma$ (see [4, 6, 7, 10, 11]). In this paper, we investigate the growth of the Laplace-Stieltjes transform of infinite order by using a class of functions to reduce the growth of $M_u(\sigma, F)$ or $\mu(\sigma, F)$ which is different from the previous form. Thus, we should give the definition of the new function as follows.

Let $\mathfrak{F}$ be the class of all functions $X(x)$ satisfying the following conditions:

(i) $X(x)$ is defined on $[a, +\infty)$, $a > 0$, is positive, strictly increasing, differential and tends to $+\infty$ as $x \to +\infty$;

(ii) $xX'(x) = o(1)$ as $x \to +\infty$.

Definition 2.2 If the Laplace-Stieltjes transformation $F(s)$ of infinite order satisfies

$$\limsup_{\sigma \to 0^+} \frac{X(\log M_u(\sigma, F))}{\log \frac{1}{\sigma}} = \rho^*,$$

where $X(x) \in \mathfrak{F}$, then ρ^* is called the X-order of the Laplace-Stieltjes transform $F(s)$.

Remark 2.1 In particular, if we take $X(x) = \log_p x$, $p \geq 2$, $p \in N_+$, where $\log_1 x = \log x$ and $\log_p x = \log(\log_{p-1} x)$, X-order is p-order of the Laplace-Stieltjes transformations with infinite order.

Remark 2.2 In addition, X-order is more precise than p-order to some extent. In fact, for $p\ (\geq 2)$ being a positive integer, we can find a function $X(x) \in \mathfrak{F}$ and a positive real function $M(x)$ satisfying

$$\limsup_{x \to \infty} \frac{X(\log M(x))}{\log x} = A \quad (0 < A < \infty)$$

and

$$\limsup_{x\to\infty}\frac{\log_p(\log M(x))}{\log x}=\infty \quad\text{and}\quad \limsup_{x\to\infty}\frac{\log_{p+1}(\log M(x))}{\log x}=0.$$

For example, let $M(x)=\exp_{p+1}\{(t\log x)^{1/d}\}$, $X(x)=(\log_p x)^d$, where t is a finite positive real constant and $0<d<1$. We can get that $\rho_p(M)=\infty$, $\rho_{p+1}(M)=0$ and $\rho_X(M)=t$, where $\rho_p(f)$ denotes the p-order of f and $\rho_X(f)$ the X-order of f.

Remark 2.3 If $\rho^*=\infty$ in Definition 2.1, then $F(s)$ is called a Laplace-Stieltjes transform of infinite X-order.

Lemma 2.1 *Let $X(x)\in\mathfrak{F}$ and let $\beta(x)$ be the function satisfying*

$$\limsup_{x\to\infty}\frac{\log^+\beta(x)}{\log x}=\varrho \quad (0\le\varrho<\infty).$$

If $M(x)$ satisfies $\limsup_{x\to\infty}\frac{X(\log M(x))}{\log x}=\nu\ (>0)$, then we have

$$\limsup_{x\to\infty}\frac{X(\beta(x)\log M(x))}{\log x}=\nu.$$

Proof We consider two cases as follows.

Case 1. If $\beta(x)$ is not a constant. From the assumptions of Lemma 2.1, we can get that $\beta(x)\to\infty$ as $x\to\infty$. Thus, for sufficiently large x, we have $\beta(x)>1$. From $X(x)\in\mathfrak{F}$, we have $\lim_{x\to\infty}\log M(x)=\infty$. Then from the Cauchy mean value theorem, there exists $\xi\,(\log M(x)<\xi<\beta(x)\log M(x))$ satisfying

$$\frac{X(\beta(x)\log M(x))-X(\log M(x))}{\log(\beta(x)\log M(x))-\log\log M(x)}=\frac{X'(\xi)}{(\log\xi)'}=\xi X'(\xi),$$

that is,

$$X\bigl(\beta(x)\log M(x)\bigr)=X\bigl(\log M(x)\bigr)+\log\beta(x)\xi X'(\xi). \tag{6}$$

Since $xX'(x)=o(1)$ as $x\to+\infty$ and $\limsup_{x\to\infty}\frac{\log\beta(x)}{\log x}=\varrho\ (0\le\varrho<\infty)$, by (6), we can get the conclusion of Lemma 2.1 easily.

Case 2. If $\beta(x)$ is a constant. By using the same argument as in Case 1, we can prove the conclusion of Lemma 2.1 easily.

Thus, the conclusion of this lemma is true. □

The following lemma is very crucial in the study of the growth of analytic functions represented by Laplace-Stieltjes transforms convergent in the right half-plane which show the relation between $M_u(\sigma,F)$ and $\mu(\sigma,F)$ of such functions.

Lemma 2.2 [8, 11] *If the abscissa $\sigma_u^F=0$ of the uniformly convergent Laplace-Stieltjes transformation and the sequence* (2) *satisfies* (3), *then for any given $\varepsilon\in(0,1)$ and for σ (>0) sufficiently reaching* 0, *we have*

$$\frac{1}{3}\mu(\sigma,F)\le M_u(\sigma,F)\le K(\varepsilon)\mu\bigl((1-\varepsilon)\sigma,F\bigr)\frac{1}{\sigma},$$

where $K(\varepsilon)$ is a constant depending on ε, (3) *and*

$$\log^+ x = \begin{cases} \log x, & x \ge 1, \\ 0, & x < 1. \end{cases}$$

3 The proof of Theorem 1.2

We prove the conclusions of Theorem 1.2 by using the properties of two functions $X(x)$ and $U(x)$. This method is different from the previous method of [13] to some extent.

We first prove '$\Longleftarrow$' of Theorem 1.2. Suppose that

$$\limsup_{n\to\infty} \frac{X(\lambda_n)}{\log U(\frac{\lambda_n}{\log A_n^*})} = T. \tag{7}$$

Then, for any positive real number $\tau > 0$, for sufficiently large n, we have

$$\lambda_n < W\left((T+\tau)\log U\left(\frac{\lambda_n}{\log^+ A_n^*}\right)\right),$$

where $W(x)$ is the inverse function of $X(x)$. Let $V(x)$ and $U(x)$ be two reciprocally inverse functions, then we have

$$V\left(\exp\left\{\frac{1}{T+\tau}X(\lambda_n)\right\}\right) < \frac{\lambda_n}{\log^+ A_n^*}, \qquad \log^+ A_n^* \le \lambda_n\left(V\left(\exp\left\{\frac{1}{T+\tau}X(\lambda_n)\right\}\right)\right)^{-1}.$$

Thus, we have

$$\log^+\left(A_n^* e^{-\lambda_n\sigma}\right) \le \lambda_n\left(\left(V\left(\exp\left\{\frac{1}{T+\tau}X(\lambda_n)\right\}\right)\right)^{-1} - \sigma\right). \tag{8}$$

For any fixed and sufficiently small $\sigma > 0$, set

$$G = W\left((T+\tau)\log U\left(\frac{1}{\sigma} + \frac{1}{\sigma\log U(\frac{1}{\sigma})}\right)\right),$$

that is,

$$\frac{1}{\sigma} + \frac{1}{\sigma\log U(\frac{1}{\sigma})} = V\left(\exp\left\{\frac{1}{T+\tau}X(G)\right\}\right). \tag{9}$$

If $\lambda_n \le G$, for sufficiently large n, let $V(\exp\{\frac{1}{T+\tau}X(\lambda_n)\}) \ge 1$, from $\sigma > 0$, (8), (9) and the definition of $U(x)$, we have

$$\begin{aligned} \log^+ A_n^* e^{-\lambda_n\sigma} &\le G\left(\left(V\left(\exp\left\{\frac{1}{T+\tau}X(\lambda_n)\right\}\right)\right)^{-1} - \sigma\right) \\ &\le G = W\left((T+\tau)\log U\left(\frac{1}{\sigma} + \frac{1}{\sigma\log U(\frac{1}{\sigma})}\right)\right) \\ &\le W\left((T+\tau)\log\left[\left(1+o(1)\right)U\left(\frac{1}{\sigma}\right)\right]\right). \end{aligned} \tag{10}$$

If $\lambda_n > G$, from (8) and (9), we have

$$\begin{aligned}\log^+ A_n^* e^{-\lambda_n \sigma} &\le \lambda_n \left(\left(V\left(\exp\left\{ \frac{1}{T+\tau} X(G) \right\} \right) \right)^{-1} - \sigma \right) \\ &\le \lambda_n \left(\left(\frac{1}{\sigma} + \frac{1}{\sigma \log U(\frac{1}{\sigma})} \right)^{-1} - \sigma \right) < 0. \end{aligned} \tag{11}$$

For sufficiently large n, from (10) and (11), we have

$$\log \mu(\sigma, F) \le W\left((T+\tau) \log\left[\left(1 + o(1)\right) U\left(\frac{1}{\sigma} \right) \right] \right).$$

Since τ is arbitrary, by Theorem 1.1 and Lemma 2.1, we can get

$$\limsup_{\sigma \to 0^+} \frac{X(\log M_u(\sigma, F))}{\log U(\frac{1}{\sigma})} \le T.$$

Suppose that

$$\limsup_{\sigma \to 0^+} \frac{X(\log M_u(\sigma, F))}{\log U(\frac{1}{\sigma})} < T.$$

Thus, there exists any real number ε $(0 < \varepsilon < \frac{T}{2})$. For any positive integer n and any sufficiently small $\sigma > 0$, from Lemma 2.2, we have

$$\log^+ A_n^* e^{-\lambda_n \sigma} \le \log M_u(\sigma, F) \le W\left((T - 2\varepsilon) \log U\left(\frac{1}{\sigma} \right) \right). \tag{12}$$

From (7), there exists a subsequence $\{\lambda_{n(p)}\}$; for sufficiently large p, we have

$$X(\lambda_{n(p)}) > (T - \varepsilon) \log U\left(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}^*} \right). \tag{13}$$

Take a sequence $\{\sigma_p\}$ satisfying

$$W\left((T - 2\varepsilon) \log U\left(\frac{1}{\sigma_p} \right) \right) = \frac{\log^+ A_{n(p)}^*}{1 + \log U(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}^*})}. \tag{14}$$

From (12) and (14), we get

$$\log^+ A_{n(p)}^* - \lambda_{n(p)} \sigma_p \le W\left((T - 2\varepsilon) \log U\left(\frac{1}{\sigma_p} \right) \right) = \frac{\log^+ A_{n(p)}^*}{1 + \log U(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}^*})},$$

that is,

$$\frac{1}{\sigma_p} \le \frac{\lambda_{n(p)}}{\log^+ A_{n(p)}^*} \left(1 + \frac{1}{\log U(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}^*})} \right).$$

Thus, we have

$$U\left(\frac{1}{\sigma_p}\right) \le U\left(\frac{\lambda_{n(p)}}{\log^+ A^*_{n(p)}}\left(1+\frac{1}{\log U(\frac{\lambda_{n(p)}}{\log^+ A^*_{n(p)}})}\right)\right) \le U^{1+o(1)}\left(\frac{\lambda_{n(p)}}{\log^+ A^*_{n(p)}}\right). \tag{15}$$

From (14) and (15), we have

$$\begin{aligned}\lambda_{n(p)} &= \frac{\lambda_{n(p)}}{\log^+ A^*_{n(p)}} W\left((T-2\varepsilon)\log U\left(\frac{1}{\sigma_p}\right)\right)\left(1+\log U\left(\frac{\lambda_{n(p)}}{\log^+ A^*_{n(p)}}\right)\right)\\ &= \frac{\lambda_{n(p)}}{\log^+ A^*_{n(p)}} W\left((T-2\varepsilon)\big(1+o(1)\big)\log U\left(\frac{\lambda_{n(p)}}{\log^+ A^*_{n(p)}}\right)\right)\left(1+\log U\left(\frac{\lambda_{n(p)}}{\log^+ A^*_{n(p)}}\right)\right).\end{aligned}$$

Thus, from the Cauchy mean value theorem, there exists a real number ξ between $\frac{\lambda_{n(p)}}{\log^+ A^*_{n(p)}}(1+\log U(\frac{\lambda_{n(p)}}{\log^+ A^*_{n(p)}})W(T-2\varepsilon)(1+o(1))\log U(\frac{\lambda_{n(p)}}{\log^+ A^*_{n(p)}})$ and $W(T-2\varepsilon)(1+o(1))\times \log U(\frac{\lambda_{n(p)}}{\log^+ A^*_{n(p)}})$ such that

$$\begin{aligned}X(\lambda_{n(p)}) &= X\left(\frac{\lambda_{n(p)}}{\log^+ A^*_{n(p)}}\left(1+\log U\left(\frac{\lambda_{n(p)}}{\log^+ A^*_{n(p)}}\right)\right)\right.\\ &\qquad \left.\times\, W\left((T-2\varepsilon)\big(1+o(1)\big)\log U\left(\frac{\lambda_{n(p)}}{\log^+ A^*_{n(p)}}\right)\right)\right)\\ &= X\left(W\left((T-2\varepsilon)\big(1+o(1)\big)\log U\left(\frac{\lambda_{n(p)}}{\log^+ A^*_{n(p)}}\right)\right)\right)\\ &\qquad + \log\left(\frac{\lambda_{n(p)}}{\log^+ A^*_{n(p)}}\left(1+\log U\left(\frac{\lambda_{n(p)}}{\log^+ A^*_{n(p)}}\right)\right)\right)\xi X'(\xi).\end{aligned}$$

Since

$$\lim_{p\to\infty}\frac{\log(\frac{\lambda_{n(p)}}{\log^+ A^*_{n(p)}}(1+\log U(\frac{\lambda_{n(p)}}{\log^+ A^*_{n(p)}})))}{\log U(\frac{\lambda_{n(p)}}{\log^+ A^*_{n(p)}})}=0,$$

then for sufficiently large p, we have

$$X(\lambda_{n(p)}) = (T-2\varepsilon)\big(1+o(1)\big)\log U\left(\frac{\lambda_{n(p)}}{\log^+ A^*_{n(p)}}\right) + K_2\xi X'(\xi)\log U\left(\frac{\lambda_{n(p)}}{\log^+ A^*_{n(p)}}\right), \tag{16}$$

where K_2 is a constant.

From (13) and (16), we can get a contradiction. Thus, we can get

$$\limsup_{\sigma\to 0^+}\frac{X(\log M_u(\sigma,F))}{\log U(\frac{1}{\sigma})} = T.$$

Hence, the sufficiency of Theorem 1.2 is completed.

We can prove the necessity of Theorem 1.2 by using a similar argument as in the proof of sufficiency of Theorem 1.2.

Thus, the proof of Theorem 1.2 is completed.

4 Proofs of Theorems 1.3-1.6

In this section, we give the definition of X_U-order of Laplace-Stieltjes transformations in the level half-strip as follows.

Definition 4.1 Let $F(s)$ be an analytic function with infinite X-order represented by Laplace-Stieltjes transformations convergent in the right half-plane. Set $S(t_0, l) = \{\sigma + it : \sigma > 0, |t - t_0| \leq l\}$, where t_0 is a real number and l is a positive number. Let $X \in \mathfrak{F}$ and

$$\tau_S^{X_U} = \limsup_{\sigma\to 0^+} \frac{X(\log^+ M_S(\sigma, F))}{\log U(\frac{1}{\sigma})},$$

where $M_S(\sigma, F) = \sup_{|t-t_0|\leq l} |F(\sigma + it)|$. Then $\tau_S^{X_U}$ is called the X_U-order of $F(S)$ in the level half-strip $S(t_0, l)$.

To prove Theorems 1.3-1.6, we need some lemmas as follows.

Lemma 4.1 *If the Laplace-Stieltjes transformation $F(s)$ is of infinite X-order, sequence* (2) *satisfies* (3) *and* (4), *and $\alpha(x) = \alpha_1(x) + i\alpha_2(x)$, where $\alpha_1(x)$ is an increasing function, and for any positive number $K > 0$ and $|\delta|$, $\alpha_2(x)$ satisfies*

$$\left|\alpha_2(x+\delta) - \alpha_2(x)\right| \leq K\left|\alpha_1(x+\delta) - \alpha_1(x)\right|, \quad 0 \leq x, x+\delta < +\infty,$$

then for any $\varepsilon > 0$, we have

$$T = \limsup_{\sigma\to 0^+} \frac{X(\log^+ M_u(\sigma, F))}{\log U(\frac{1}{\sigma})} = \limsup_{\sigma\to 0^+} \frac{X(\log^+ M_{S_\varepsilon}(\sigma, F))}{\log U(\frac{1}{\sigma})} = \tau_{S_\varepsilon}^{X_U}.$$

Proof We will prove this lemma by using a similar argument to that in [11]. From the assumptions of Lemma 4.1, for any $0 < x \leq \infty$,

$$\begin{aligned} M_{S_\varepsilon}(\sigma, F) &\geq \left|\int_0^\infty e^{-\sigma y}\, d\alpha(y)\right| \geq \int_0^\infty e^{-\sigma y}\, d\alpha_1(y) \\ &\geq \frac{1}{K+1}\int_0^\infty e^{-\sigma y}\left|d\alpha_1(y)\right| \geq \frac{1}{K+1}\int_0^x e^{-\sigma y}\left|d\alpha_1(y)\right| \\ &\geq \frac{1}{K+1}\left|\int_0^\infty e^{-(\sigma+it)y}\, d\alpha(y)\right|. \end{aligned}$$

Then

$$M(\sigma, F) \geq M_{S_\varepsilon}(\sigma, F) \geq \frac{1}{K+1} M_u(\sigma, F) \geq \frac{1}{K+1} M(\sigma, F).$$

Since $F(s)$ is an analytic function with infinite X-order, from the above inequality and the definition of X_U-order, we can get the conclusion of Lemma 4.1. □

Lemma 4.2 [11, Lemma 2.4] *Let*

$$z = \frac{1 - \sinh s}{1 + \sinh s}, \quad s \in B = \left\{s : \Re(s) > 0, |\Im(s)| < \frac{\pi}{2}\right\}.$$

Then

(i) *this mapping maps the horizontal half-strip B to the unit disc $\{z : |z| < 1\}$, and its inverse mapping is*

$$s = \Psi(z) = \sinh^{-1}\frac{1-z}{1+z};$$

(ii) $\min_{0\leq\theta\leq 2\pi}\Re[\Psi(re^{i\theta})] \geq \Psi(r)$ $(0 < r < 1)$;

(iii) $\max_{0\leq\theta\leq\frac{\pi}{4}}\Re[\Psi(re^{i\theta})] \leq \Psi(r^2)$ $(0 < r < 1)$;

(iv) $\Psi(\{z : |z| < r\}) \subseteq \{s : \Re(s) > \Psi(r), |\Im(s)| < \frac{\pi}{2}\}$ $(0 < r < 1)$.

Lemma 4.3 (see [14]) *Let f be an admissible function in the unit disc $\mathbb{D}$, let q be a positive integer, and let $a_1, \ldots, a_q$ be pairwise distinct complex numbers. Then, for $r \to 1^-$, $r \notin E$,*

$$(q-2)T(r,f) \leq \sum_{j=1}^{q}\overline{N}\left(r, \frac{1}{f-a_j}\right) + S(r,f),$$

where $E \subset (0,1)$ is a possibly occurring exceptional set with $\int_E \frac{dr}{1-r} < \infty$, and the term $\overline{N}(r, \frac{1}{f-a_j})$ is replaced by $\overline{N}(r,f)$ when some $a_j = \infty$. We use $S(r,f)$ to denote

$$S(r,f) = O\left\{\log\frac{1}{1-r}\right\} + O\left\{\log^+ T(r,f)\right\}$$

as $r \to 1^-$ possibly outside the set E such that $\int_E \frac{dr}{1-r} < \infty$. If the order of f is finite, the remainder $S(r,f)$ is an $O(\log\frac{1}{1-r})$ without any exceptional set.

Remark 4.1 Under the assumptions of Lemma 4.3, for a positive integer l, we can get the following inequality easily:

$$\left(q-2-\frac{2}{l}\right)T(r,f) \leq \sum_{j=1}^{q}\overline{N}^{l)}\left(r, \frac{1}{f-a_j}\right) + S(r,f),$$

where $\overline{N}^{l)}(r, \frac{1}{f-a_j})$ is the counting function of poles of the function $\frac{1}{f-a_j}$ with multiplicities $\leq l$ in $\{z : |z| \leq r\}$, each point counted only once.

Lemma 4.4 [15, p.282, (1.8)] *Let h be an analytic in the disc $|z| = r < 1$, then*

$$T(r,h) \leq \log M(r,h) \leq \frac{1+r}{1-r}T(r,h),$$

where $M(r,h)$ is the maximum modulus of h in the disc $|z| = r < 1$.

4.1 The proof of Theorem 1.3

Since sequence (2) satisfies (3) and (4), the Laplace-Stieltjes transformation $F(s)$ of infinite X-order, and $\limsup_{n\to\infty}\frac{X(\lambda_n)}{\log U(\frac{\lambda_n}{\log^+ A_n^*})} = T$ $(0 < T < \infty)$, from Theorem 1.2, we have

$$\limsup_{\sigma\to 0^+}\frac{X(\log^+ M_u(\sigma, F))}{\log U(\frac{1}{\sigma})} = T \tag{17}$$

and from Lemma 4.1 and (17), for any $\varepsilon > 0$, we have

$$\limsup_{\sigma\to 0^+} \frac{X(\log^+ M_{S_\varepsilon}(\sigma, F))}{\log U(\frac{1}{\sigma})} = T.$$

Thus, it follows

$$\limsup_{\sigma\to 0^+} \frac{X(\log M(\sigma, F, S_\varepsilon))}{\log U(\frac{1}{\sigma})} = T, \tag{18}$$

where $S_\varepsilon = \{s : \Re(s) > 0, |\Im(s)| \leq \varepsilon\}$ and $M(\sigma, F, S_\varepsilon) = \sup\{|F(s)| : \Re(s) \geq \sigma, s \in S_\varepsilon\}$.

Set $g(z) = F(\frac{2\varepsilon}{\pi}\Psi(z))$, where $\Psi(z)$ is stated as in Lemma 4.2. Then from Lemma 4.2, we get that $g(z)$ is analytic in the unit disc $|z| < 1$ and satisfies

$$M\left(\frac{2\varepsilon}{\pi}\Psi\left(r^2\right), F, S_{\varepsilon_1}\right) \leq M(r, g) \leq M\left(\frac{2\varepsilon}{\pi}\Psi(r), F, S_\varepsilon\right), \tag{19}$$

where $0 < \varepsilon_1 < \varepsilon$. Therefore, from (18), (19) and Lemma 2.1, we have

$$\limsup_{r\to 1^-} \frac{X(\log^+ M(r, g))}{\log \frac{1}{1-r}} = \infty, \qquad \limsup_{r\to 1^-} \frac{X(\log^+ M(r, g))}{\log U(\frac{1}{1-r})} = T. \tag{20}$$

From (20), Lemma 4.4 and Lemma 2.1, we can get that g is an admissible function in $|z| < 1$ and

$$\limsup_{r\to 1^-} \frac{X(T(r, g))}{\log U(\frac{1}{1-r})} = T. \tag{21}$$

Then from Lemma 4.3 and (21), we can get that at most there exists one exception a satisfying

$$\limsup_{r\to 1^-} \frac{X(\overline{N}(r, g = a))}{\log U(\frac{1}{1-r})} = \varrho \geq T. \tag{22}$$

Since

$$\overline{n}(r, g = a) \log \frac{1 + r}{2r} \leq \int_r^{\frac{1+r}{2}} \frac{\overline{n}(t, g = a)}{t}\, dt \leq \overline{N}\left(\frac{1 + r}{2}, g = a\right) \log \frac{1 + r}{2r}$$

and

$$\overline{N}(r, g = a) - \overline{N}(r_0, g = a) \leq \int_{r_0}^{r} \frac{\overline{n}(t, g = a)}{t}\, dt \leq \overline{n}(r, g = a) \log \frac{r}{r_0}, \quad r_0 < r < 1,$$

we have

$$\overline{n}(r, g = a) \leq \overline{N}\left(\frac{1 + r}{2}, g = a\right) \leq \overline{n}(r, g = a) \log \frac{r}{r_0} + O(1), \quad r_0 < r < 1. \tag{23}$$

Thus, from (22)-(23) and Lemma 2.1, we have

$$\limsup_{r\to 1^-} \frac{X(\overline{n}(r, g = a))}{\log U(\frac{1}{1-r})} = \varrho \geq T. \tag{24}$$

Hence, for any $\eta > 0$ and (24), the inequality

$$\limsup_{\sigma\to 0^+}\frac{X(\overline{n}(\sigma,0,\eta,F=a))}{\log U(\frac{1}{\sigma})}=\varrho\geq T$$

holds for any $a\in\mathbb{C}$ with one exception, where $\overline{n}(\sigma,0,\eta,F=a)$ is the counting function of zeros of the function $F(s)-a$ in the strip $\{s:\Re(s)>\sigma,|\Im(s)|<\eta\}$.

Thus, we complete the proof of Theorem 1.3.

4.2 The proof of Theorem 1.4

Since $r(y)$ is a continuous function on $y\in[0,+\infty)$, then we can get that $\alpha(x)=\int_0^x r(y)e^{it_0y}\,dy$ is a function of bounded variation on $x\in[0,Y]$ $(0<Y<\infty)$. Set

$$\widehat{S}_\varepsilon=\{s:\Re(s)>0,|\Im(s)-t_0|\leq\varepsilon\},\qquad M_{\widehat{S}_\varepsilon}(\sigma,F)=\sup_{|t-t_0|\leq\varepsilon}|F(\sigma+it)|.$$

From the assumptions of Theorem 1.6, for any real number x $(0<x\leq\infty)$, we have

$$\begin{aligned}M_{\widehat{S}_\varepsilon}(\sigma,F)&=\sup_{|t-t_0|\leq\varepsilon}|F(\sigma+it)|=\sup_{|t-t_0|\leq\varepsilon}\left|\int_0^\infty e^{-sy}r(y)e^{it_0y}\,dy\right|\\&\geq\int_0^\infty e^{-\sigma y}r(y)\,dy\geq\int_0^x e^{-\sigma y}r(y)\,dy\\&\geq\left|\int_0^x e^{-sy}r(y)e^{it_0y}\,dy\right|=\left|\int_0^\infty e^{-sy}\,d\alpha(y)\right|,\end{aligned}$$

that is,

$$M(\sigma,F)\leq M_u(\sigma,F)\leq M_{\widehat{S}_\varepsilon}(\sigma,F)\leq M(\sigma,F).\tag{25}$$

From the assumption of Theorem 1.4, by (25) and Theorem 1.1, we can get

$$\begin{aligned}\limsup_{\sigma\to 0^+}\frac{X(\log^+M_u(\sigma,F))}{\log U(\frac{1}{\sigma})}&=\limsup_{\sigma\to 0^+}\frac{X(\log^+M_{\widehat{S}_\varepsilon}(\sigma,F))}{\log U(\frac{1}{\sigma})}\\&=\limsup_{\sigma\to 0^+}\frac{X(\log^+M(\sigma,F))}{\log U(\frac{1}{\sigma})}=T.\end{aligned}\tag{26}$$

From (26), for any $\varepsilon>0$, we have

$$\limsup_{\sigma\to 0^+}\frac{X(\log^+M(\sigma,F,\widehat{S}_\varepsilon))}{\log U(\frac{1}{\sigma})}=T,$$

where $M(\sigma,F,\widehat{S}_\varepsilon)=\sup\{|F(s)|:\Re(s)\geq\sigma,s\in\widehat{S}_\varepsilon\}$.

Set $g(z)=F(\frac{2\varepsilon}{\pi}\Psi(z)+it_0)$, where $\Psi(z)$ is stated as in Lemma 4.2. Then from Lemma 4.2, we can get that $g(z)$ is analytic in the unit disc $|z|<1$ and satisfies

$$M\left(\frac{2\varepsilon}{\pi}\Psi(r^2),F,S_{\varepsilon_2}\right)\leq M(r,g)\leq M\left(\frac{2\varepsilon}{\pi}\Psi(r),F,S_\varepsilon\right),$$

where $0 < \varepsilon_2 < \varepsilon$. Therefore, from the above inequality and Lemma 4.2, we have

$$\limsup_{r\to 1^-} \frac{X(\log^+ M(r,g))}{\log \frac{1}{1-r}} = \infty, \qquad \limsup_{r\to 1^-} \frac{X(\log^+ M(r,g))}{\log U(\frac{1}{1-r})} = T.$$

Then, similar to the proof of Theorem 1.3, we can prove that for any $\eta > 0$, the inequality

$$\limsup_{\sigma\to 0^+} \frac{X(\overline{n}(\sigma, it_0, \eta, F = a))}{\log \frac{1}{\sigma}} = \varrho \geq \rho^*$$

holds for any $a \in \mathbb{C}$ with one exception, where $\overline{n}(\sigma, it_0, \eta, F = a)$ is stated as in Theorem 1.6.

Thus, we complete the proof of Theorem 1.4.

4.3 Proofs of Theorems 1.5 and 1.6

From Remark 4.1, by using the same argument as in Theorems 1.3 and 1.4, we can prove the conclusions of Theorems 1.5 and 1.6 easily.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

HYX and ZXX completed the main part of this article, ZXX corrected the main theorems. All authors read and approved the final manuscript.

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