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On generalized difference lacunary statistical convergence in a paranormed space

Selma Altundağ*

*Correspondence: scaylan@sakarya.edu.tr
Mathematics Department, Faculty of Arts and Science, Sakarya University, Sakarya, 54187, Turkey

Abstract
In this article, we introduce the concept of Δ^m-lacunary statistical convergence and Δ^m-lacunary strong convergence in a paranormed space. Also, we establish some connections between these concepts.

1 Introduction

In order to extend convergence of sequences, the notion of statistical convergence was introduced by Fast [1] and Steinhaus [2] and several generalizations and applications of this concept have been investigated by various authors [3, 4]. This notion was studied in normed spaces by Kolk [5], in locally convex Hausdorff topological spaces by Maddox [6], in topological Hausdorff groups by Çakallı [7] and in probabilistic normed space by Karakuş [8]. Recently, Alotaibi and Alroqi [9] extended this notion in paranormed spaces.

In this article, we study the concept of statistical convergence from difference sequence spaces which are defined over paranormed space.

2 Preliminaries and definitions

Let K be a subset of the set of natural numbers $\mathbb{N}$. Then the asymptotic density of K denoted by $\delta(K) = \lim_n \frac{1}{n}|\{k \leq n : k \in K\}|$, where the vertical bars denote the cardinality of the enclosed set in [10].

A number sequence $x = (x_k)$ is said to be statistically convergent to the number L if for each $\varepsilon > 0$, the set $K(\varepsilon) = \{k \leq n : |x_k - L| \geq \varepsilon\}$ has asymptotic density zero, *i.e.*,

$$\lim_n \frac{1}{n}\left|\left\{k \leq n : |x_k - L| \geq \varepsilon\right\}\right| = 0.$$

In this case we write $st\text{-}\lim x = L$. This concept was studied by [11, 12].

By a lacunary $\theta = (k_r)$; $r = 0, 1, 2, \ldots$, where $k_0 = 0$, we shall mean an increasing sequence of non-negative integers with $k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and $h_r = k_r - k_{r-1}$. The ratio $\frac{k_r}{k_{r-1}}$ will be denoted by q_r.

The notion of difference sequence space $X(\Delta)$ was introduced by Kızmaz [13] as follows:

$$X(\Delta) = \left\{x = (x_k) : (\Delta x_k) \in X\right\}$$

for $X = l_\infty, c, c_0$, where $\Delta x_k = x_k - x_{k+1}$ for all $k \in \mathbb{N}$.

The notion of difference sequence spaces was further generalized by Et and Çolak [14] as follows:

$$X(\Delta^m) = \{x = (x_k) \in w : (\Delta^m x_k) \in X\}$$

for $X = l_\infty$, c and c_0, where $m \in \mathbb{N}$, $\Delta^m x_k = \Delta^{m-1} x_k - \Delta^{m-1} x_{k+1}$, $\Delta^0 x_k = x_k$.

The sequence x is said to be Δ^m-statistically convergent to the number L provided that for each $\varepsilon > 0$,

$$\lim_n \frac{1}{n} \left| \left\{ k \le n : \left| \Delta^m x_k - L \right| \ge \varepsilon \right\} \right| = 0.$$

The set of all Δ^m-statistically convergent sequences was denoted by $S(\Delta^m)$ in [15]. Furthermore, this notion was studied in [16, 17].

A paranorm is a function $g : X \longrightarrow \mathbb{R}$ defined on a linear space X such that for all $x, y, z \in X$,

(i) $g(x) = 0$ if $x = \theta$;
(ii) $g(-x) = g(x)$;
(iii) $g(x + y) \le g(x) + g(y)$;
(iv) If (α_n) is a sequence of scalars with $\alpha_n \longrightarrow \alpha_0$ $(n \longrightarrow \infty)$ and $x_n, a \in X$ with $x_n \longrightarrow a$ $(n \longrightarrow \infty)$ in the sense that $g(x_n - a) \longrightarrow 0$ $(n \longrightarrow \infty)$, then $\alpha_n x_n \longrightarrow \alpha_0 a$ $(n \longrightarrow \infty)$, in the sense that $g(\alpha_n x_n - \alpha_0 a) \longrightarrow 0$ $(n \longrightarrow \infty)$.

A paranorm g for which $g(x) = 0$ implies $x = \theta$ is called a total paranorm on X and the pair (X, g) is called a total paranormed space.

Note that each seminorm (norm) p on X is a paranorm (total) but converse need not be true.

The concept of paranorm is a generalization of absolute value [18].

A modulus function f is a function from $[0, \infty)$ to $[0, \infty)$ such that

(i) $f(x) = 0$ *if and only if* $x = 0$;
(ii) $f(x + y) \le f(x) + f(y)$ *for all* $x, y \ge 0$;
(iii) f *increasing*;
(iv) f *is continuous from at the right zero.*

Since $|f(x) - f(y)| \le f(|x - y|)$, it follows from condition (iv) that f is continuous on $[0, \infty)$. Furthermore, we have $f(nx) \le nf(x)$ for all $n \in \mathbb{N}$ from condition (ii) and so

$$f(x) = f\left(nx\frac{1}{n}\right) \le nf\left(\frac{x}{n}\right).$$

Hence, for all $n \in \mathbb{N}$,

$$\frac{1}{n} f(x) \le f\left(\frac{x}{n}\right).$$

A modulus may be bounded or unbounded. For example, $f(x) = x^p$ for $0 < p \le 1$ is unbounded, but $f(x) = \frac{x}{1+x}$ is bounded. Ruckle [19] used the idea of a modulus function f to construct a class of FK spaces

$$L(f) = \left\{ x = (x_k) : \sum f(|x_k|) < \infty \right\}.$$

In [9], the notion of statistical convergence was defined in a paranormed space.

Definition 2.1 A sequence $x = (x_k)$ is said to be statistically convergent to the number L in (X, g) if for each $\varepsilon > 0$,

$$\lim_n \frac{1}{n} \bigl|\bigl\{k \leq n : g(x_k - L) \geq \varepsilon\bigr\}\bigr| = 0.$$

In this case, we write $g(st)$-$\lim x = L$. We denote the set of all $g(st)$-convergent sequences by S_g [9].

Definition 2.2 A sequence $x = (x_k)$ is said to be strongly p-Cesaro summable $(0 < p < \infty)$ to the limit L in (X, g) if

$$\lim_n \frac{1}{n} \sum_{j=1}^{n} \bigl(g(x_j - L)\bigr)^p = 0,$$

and we write it as $x_k \longrightarrow L([C_1, g]_p)$. In this case, L is called the $[C_1, g]_p$-lim it of x [9].

In this article, we shall study the concept of Δ^m-lacunary statistical convergence, Δ^m-lacunary strong convergence and Δ^m-lacunary strong convergence with respect to a modulus function in a paranormed space.

3 Generalized difference statistical convergence in a paranormed space

Definition 3.1 A sequence $x = (x_k)$ is said to be Δ^m-statistically convergent to the number L in (X, g) if for each $\varepsilon > 0$,

$$\lim_n \frac{1}{n} \bigl|\bigl\{k \leq n : g\bigl(\Delta^m x_k - L\bigr) \geq \varepsilon\bigr\}\bigr| = 0.$$

In this case, we write $S_g(\Delta^m)$-$\lim x = L$. We denote the set of all Δ^m-statistically convergent sequences in (X, g) by $S_g(\Delta^m)$.

Definition 3.2 Let θ be a lacunary sequence. A sequence $x = (x_k)$ is said to be Δ^m-lacunary statistically convergent to the number L in (X, g) if for each $\varepsilon > 0$,

$$\lim_r \frac{1}{h_r} \bigl|\bigl\{k \in I_r : g\bigl(\Delta^m x_k - L\bigr) \geq \varepsilon\bigr\}\bigr| = 0.$$

In this case, we write $S_g^\theta(\Delta^m)$-$\lim x = L$. We denote the set of all Δ^m-lacunary statistically convergent sequences in (X, g) by $S_g^\theta(\Delta^m)$.

Definition 3.3 A sequence $x = (x_k)$ is said to be strongly Δ^m-Cesaro summable to the limit L in (X, g) if

$$\lim_n \frac{1}{n} \sum_{k=1}^{n} \bigl(g\bigl(\Delta^m x_k - L\bigr)\bigr) = 0,$$

and we write it as $x_k \longrightarrow L(|\sigma_1|_g(\Delta^m))$. In this case L is called the $|\sigma_1|_g(\Delta^m)$-lim of x.

Definition 3.4 A sequence $x = (x_k)$ is said to be strongly Δ^m-lacunary strongly summable to the limit L in (X,g) if

$$\lim_r \frac{1}{h_r} \sum_{k\in I_r} \bigl(g\bigl(\Delta^m x_k - L\bigr)\bigr) = 0,$$

and we write it as $x_k \longrightarrow L(N_g^\theta(\Delta^m))$. In this case L is called the $N_g^\theta(\Delta^m)$-lim of x.

Theorem 3.1 *Let θ be a lacunary sequence and (X,g) be a paranormed space. Then*

(i) *If $x_k \longrightarrow L(N_g^\theta(\Delta^m))$, then $x_k \longrightarrow L(S_g^\theta(\Delta^m))$ and the inclusion is strict*;

(ii) *If x is a Δ^m-bounded sequence and $x_k \longrightarrow L(S_g^\theta(\Delta^m))$, then $x_k \longrightarrow L(N_g^\theta(\Delta^m))$*;

(iii) $l_g^\infty(\Delta^m) \cap S_g^\theta(\Delta^m) = l_g^\infty(\Delta^m) \cap N_g^\theta(\Delta^m)$.

Proof (i) If $\varepsilon > 0$ and $x_k \to L(N_g^\theta(\Delta^m))$, we can write

$$\sum_{k\in I_r} g\bigl(\Delta^m x_k - L\bigr) \ge \sum_{\substack{k\in I_r \\ g(\Delta^m x_k - L)\ge\varepsilon}} g\bigl(\Delta^m x_k - L\bigr) \ge \varepsilon \bigl|\bigl\{k \in I_r : g\bigl(\Delta^m x_k - L\bigr) \ge \varepsilon\bigr\}\bigr|,$$

which yields the result.

In order to prove that the inclusion $N_g^\theta(\Delta^m) \subset S_g^\theta(\Delta^m)$ is proper, let θ be given and $X = N_0^\theta(\Delta, \frac{1}{h_r}) = \{x = (x_k) : |\frac{1}{h_r}\sum_{k\in I_r} \Delta x_k|^{\frac{1}{h_r}} \to 0, r \to \infty\}$ with the paranorm $g(x) = |x_1| + \sup_r |\frac{1}{h_r}\sum_{k\in I_r} \Delta x_k|^{\frac{1}{h_r}}$. Define $x = (x_k)$ to be $2h_r 1^{h_r}$ at the first term in I_r for every $r \ge 1$, $x_k = h_r(1^{h_r} - 2^{h_r} - \cdots - (k-1)^{h_r})$ between the second term and $([\sqrt{h_r}] + 1)$th term in I_r, $x_k = h_r(1^{h_r} - 2^{h_r} - \cdots - ([\sqrt{h_r}])^{h_r})$ at the $([\sqrt{h_r}] + 2)$th term in I_r and $x_k = 0$ otherwise.

We see that

$$\Delta x_k = \begin{cases} h_r 1^{h_r}, h_r 2^{h_r}, \ldots, h_r[\sqrt{h_r}]^{h_r}, & \text{at the first } [\sqrt{h_r}] \text{ integers in } I_r, \\ 0, & \text{otherwise.} \end{cases}$$

and

$$g(\Delta x_k) = \begin{cases} 1, 2, \ldots, [\sqrt{h_r}], & \text{at the first } [\sqrt{h_r}] \text{ integers in } I_r, \\ 0, & \text{otherwise.} \end{cases}$$

Note that x is not Δ-bounded in (X,g). We have, for every $\varepsilon > 0$,

$$\frac{1}{h_r}\bigl|\bigl\{k \in I_r : g(\Delta x_k) \ge \varepsilon)\bigr\}\bigr| = \frac{[\sqrt{h_r}]}{h_r} \to 0$$

as $r \to \infty$, *i.e.*, $x_k \to 0(S_g^\theta(\Delta))$. On the other hand,

$$\frac{1}{h_r}\sum_{k\in I_r} g(\Delta x_k) = \frac{1}{h_r}\frac{[\sqrt{h_r}]([\sqrt{h_r}] + 1)}{2} \to \frac{1}{2} \ne 0;$$

hence $x_k \nrightarrow 0(N_g^\theta(\Delta))$.

(ii) Suppose that $x_k \to L(S_g^\theta(\Delta^m))$ and say $g(\Delta^m x_k - L) \leq M$ for all k. Given $\varepsilon > 0$, we get

$$\begin{aligned}\frac{1}{h_r}\sum_{k\in I_r} g(\Delta^m x_k - L) &= \frac{1}{h_r}\sum_{\substack{k\in I_r\\ g(\Delta^m x_k - L)\geq\varepsilon}} g(\Delta^m x_k - L) + \frac{1}{h_r}\sum_{\substack{k\in I_r\\ g(\Delta^m x_k - L)<\varepsilon}} g(\Delta^m x_k - L)\\ &\leq \frac{M}{h_r}\bigl|\bigl\{k\in I_r : g(\Delta^m x_k - L)\geq\varepsilon\bigr\}\bigr| + \varepsilon,\end{aligned}$$

from which the result follows.

(iii) This is an immediate consequence of (i) and (ii). □

Corollary 3.1 *If $x_k \to L(|\sigma_1|_g(\Delta^m))$, then $x_k \to L(S_g(\Delta^m))$. If $x \in l_g^\infty(\Delta^m)$ and if $x_k \to L(S_g(\Delta^m))$, then $x_k \to L(|\sigma_1|_g(\Delta^m))$.*

Theorem 3.2 *Let θ be a lacunary sequence and (X,g) be a paranormed space, then $S_g^\theta(\Delta^m) = S_g(\Delta^m)$ if and only if $1 < \liminf_r q_r \leq \limsup_r q_r < \infty$.*

Proof Suppose that $\liminf q_r > 1$, then there exists a $\delta > 0$ such that $q_r \geq 1+\delta$ for sufficiently large r, which implies that

$$\frac{h_r}{k_r} \geq \frac{\delta}{1+\delta}.$$

If $x_k \to L(S_g(\Delta^m))$, then for every $\varepsilon > 0$ and sufficiently large r, we have

$$\begin{aligned}\frac{1}{k_r}\bigl|\bigl\{k\leq k_r : g(\Delta^m x_k - L)\geq\varepsilon\bigr\}\bigr| &\geq \frac{1}{k_r}\bigl|\bigl\{k\in I_r : g(\Delta^m x_k - L)\geq\varepsilon\bigr\}\bigr|\\ &\geq \frac{\delta}{1+\delta}\frac{1}{h_r}\bigl|\bigl\{k\in I_r : g(\Delta^m x_k - L)\geq\varepsilon\bigr\}\bigr|,\end{aligned}$$

which proves the $S_g(\Delta^m) \subset S_g^\theta(\Delta^m)$.

Conversely, suppose that $\liminf q_r = 1$. Since θ is lacunary, we can select a subsequence (k_{r_j}) of θ satisfying $\frac{k_{r_j}}{k_{r_j-1}} < 1 + \frac{1}{j}$ and $\frac{k_{r_j-1}}{k_{r_{(j-1)}}} > j$, where $r_j \geq r_{j-1} + 2$ and $X = N_0^\theta(\Delta, \frac{1}{h_r}) = \{x = (x_k) : |\frac{1}{h_r}\sum_{k\in I_r}\Delta x_k|^{\frac{1}{h_r}} \to 0, r\to\infty\}$ with the paranorm $g(x) = |x_1| + \sup_r|\frac{1}{h_r}\sum_{k\in I_r}\Delta x_k|^{\frac{1}{h_r}}$.

Now define a sequence by

$$\Delta x_k = \begin{cases} h_r + k, & k\in I_{r_{(j)}}, j = 1,2,\ldots,\\ 0, & \text{otherwise.}\end{cases}$$

We can see that

$$g(\Delta x_k) = \begin{cases} 1, & k\in I_{r_{(j)}}, j = 1,2,\ldots,\\ 0, & \text{otherwise.}\end{cases}$$

and hence x is Δ-bounded in (X,g).

We can see that $x \notin N_g^\theta(\Delta^m)$, but $x \in \sigma_g^1(\Delta^m)$. Theorem 3.1(ii) implies that $x \notin S_g^\theta(\Delta^m)$, but it follows from Corollary 3.1 that $x \in S_g(\Delta^m)$. Hence $S_g(\Delta^m) \not\subset S_g^\theta(\Delta^m)$ and $S_g(\Delta^m) \subset S_g^\theta(\Delta^m)$ implies that $\liminf q_r > 1$.

To show for any lacunary sequence θ, $S_g^\theta(\Delta^m) \subset S_g(\Delta^m)$ implies $\limsup q_r < \infty$, the same technique of Lemma 3 of [20] can be used. Now suppose that $\limsup q_r = \infty$. Consider the same space defined above and the sequence defined by

$$\Delta x_i = \begin{cases} h_r + i, & k_{r_j-1} < i \le 2k_{r_j-1}, j = 1, 2, \ldots, \\ 0, & \text{otherwise.} \end{cases}$$

Then we get

$$g(\Delta x_i) = \begin{cases} 1, & k_{r_j-1} < i \le 2k_{r_j-1}, j = 1, 2, \ldots, \\ 0, & \text{otherwise.} \end{cases}$$

Then $x \in N_g^\theta(\Delta^m)$, but $x \notin \sigma_g^1(\Delta^m)$. By Theorem 3.1(i) we conclude that $x \in S_g^\theta(\Delta^m)$, but by Corollary 3.1 that $x \notin S_g(\Delta^m)$. Hence, $S_g^\theta(\Delta^m) \not\subset S_g(\Delta^m)$. This completes the proof. □

Definition 3.5 Let f be a modulus function. Then a sequence $x = (x_k)$ is lacunary strongly p-Cesaro summable to L with respect to f in (X, g) if

$$\lim_r \frac{1}{h_r} \sum_{k \in I_r} \bigl[f\bigl(g\bigl(\Delta^m x_k - L\bigr)\bigr)\bigr]^{p_k} = 0.$$

In this case, we write $x_k \to L(N_g^\theta(f, \Delta^m, p))$. If we take $p_k = 1$ for all $k \in \mathbb{N}$, we say $x_k \to L(N_g^\theta(f, \Delta^m))$.

Lemma 3.1 *Let f be a modulus function and let $0 < \delta < 1$. Then for each $x > \delta$ we have $f(x) \le 2f(1)\delta^{-1}x$ [21].*

Theorem 3.3 *Let f be a modulus function and (X, g) be a paranormed space. Then $N_g^\theta(\Delta^m) \subset N_g^\theta(f, \Delta^m)$.*

Proof Let $x \in N_g^\theta(\Delta^m)$. Then we have $\tau_r = \frac{1}{h_r} \sum_{k \in I_r} g(\Delta^m x_k - L) \to 0$ as $r \to \infty$ for some L.

Let $\varepsilon > 0$ and choose δ with $0 < \delta < 1$ such that $f(u) < \varepsilon$ for u with $0 \le u \le \delta$. Then we can write

$$\begin{aligned} \frac{1}{h_r} \sum_{k \in I_r} f\bigl(g\bigl(\Delta^m x_k - L\bigr)\bigr) &= \frac{1}{h_r} \sum_{\substack{k \in I_r \\ g(\Delta^m x_k - L) \le \delta}} f\bigl(g\bigl(\Delta^m x_k - L\bigr)\bigr) \\ &\quad + \frac{1}{h_r} \sum_{\substack{k \in I_r \\ g(\Delta^m x_k - L) > \delta}} f\bigl(g\bigl(\Delta^m x_k - L\bigr)\bigr) \\ &\le \frac{1}{h_r}(h_r \delta) + \frac{1}{h_r} 2f(1)\delta^{-1} h_r \tau_r \end{aligned}$$

from Lemma 3.1. Therefore $x \in N_g^\theta(f, \Delta^m)$. □

Theorem 3.4 *Let* $0 < \inf_k p_k \leq p_k \leq \sup_k p_k < \infty$. *Then* $S_g^\theta(\Delta^m) = N_g^\theta(f, \Delta^m, p)$ *if and only if* f *is bounded*.

Proof Following the technique applied for establishing Theorem 3.16 of [22], we can prove the theorem. □

Competing interests
The author declares that they have no competing interests.

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