Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

Toeplitz-type operators in weighted Morrey spaces

Peizhu Xie[1,2*] and Guangfu Cao[1,2]

*Correspondence: xiepeizhu82@163.com
[1]School of Mathematics and Information Science, Guangzhou University, Guangzhou, 510006, China
[2]Key Laboratory of Mathematics and Interdisciplinary Sciences of Guangdong Higher Education Institutes, Guangzhou University, Guangzhou, 510006, China

Abstract
Let $T_{j,1}$ and $T_{j,2}$ be singular integrals with non-smooth kernels, which are associated with an approximation of identity or $\pm I$ (I is the identity operator). Denote the Toeplitz-type operator by $T_b = \sum_{j=1}^{N} T_{j,1} M_b T_{j,2}$, where $M_b f(x) = b(x)f(x)$. In this paper, the estimates of the Toeplitz operator $T_b(f)$ related to singular integral operators with non-smooth kernels and $b \in BMO(\mathbb{R}^n)$ in weighted Morrey spaces is established.

MSC: 47B35

Keywords: Toeplitz operator; Morrey space; A_p weights

1 Introduction

The classical Morrey spaces were introduced by Morrey in [1] to investigate the local behavior of solutions to second-order elliptic partial differential equations. The boundedness of the Hardy-Littlewood maximal operator, the singular integral operator, the fractional integral operator and the commutator of these operators in Morrey spaces have been studied by many authors; see [2–5] and the references therein. In [6], Komori and Shirai studied the boundedness of these operators in weighted spaces.

It is well known that the commutator $[b, T]$ is defined by $[b, T](f) = T(bf) - bT(f)$, where T is a Calderón-Zygmund operator and $b \in BMO$. The commutator generated by the Calderón-Zygmund operators and a locally integrable function b can be regarded as a special case of the Toeplitz operator $T_b = \sum_{j=1}^{N} T_{j,1} M_b T_{j,2}$, where $T_{j,1}$ and $T_{j,2}$ are the Calderón-Zygmund operators or $\pm I$ (I is the identity operator), $M_b f(x) = b(x)f(x)$. When $b \in BMO$, Krantz and Li discussed the L^p boundedness of T_b on the homogeneous space, see [7, 8]. In [9], the authors studied the boundedness of T_b in Morrey spaces. In this paper, we study the boundedness of Toeplitz-type operators related to singular integral operators with non-smooth kernels in weighted Morrey spaces.

The singular integral operators with non-smooth kernels previously appeared in [10]. We say that T is a singular integral operator with non-smooth kernel if it satisfies the following conditions.

(i) There exists a class of operators A_t with kernels $a_t(x, y)$, which satisfy the condition (2.3) in Section 2, so that the kernels $k_t(x, y)$ of the operators $(T - A_t T)$ satisfy the condition

$$\left|k_t(x,y)\right| \le c\frac{t^{\gamma/m}}{|x-y|^{n+\gamma}}, \tag{1.1}$$

when $|x - y| \ge c_1 t^{1/m}$ for some $\gamma, m > 0$.

(ii) There exists a class of operators B_t with kernels $b_t(x,y)$, which satisfy the condition (2.3), such that $(T - TB_t)$ have associated kernels $K_t(x,y)$ and there exist positive constants c_2, c_3 such that

$$\int_{|x-y|\geq c_2 t^{1/m}} \left|K_t(x,y)\right| dx \leq c_3 \quad \text{for all } y \in \mathbb{R}^n. \tag{1.2}$$

Note that the classes of operators A_t and B_t play the role of a generalized approximation to the identity. It is not difficult to check that conditions (1.1) and (1.2) are consequences of the standard Calderón-Zygmund operator. See Proposition 2 in [10].

The paper is organized as follows. In Section 2, we recall some important estimates on BMO functions, maximal functions and sharp maximal functions. In Section 3, we prove the main result.

2 Definitions and preliminary results

Let $1 \leq p < \infty$, $0 < \kappa < 1$ and w be a weight. The weighted Morrey space is defined by

$$L^{p,\kappa}(w) := \left\{f \in L^p_{\mathrm{loc}}(w) : \|f\|_{L^{p,\kappa}(w)} < \infty\right\},$$

where

$$\|f\|_{L^{p,\kappa}(w)} = \sup_B \left(\frac{1}{w(B)^\kappa}\int_B |f|^p w\,dx\right)^{1/p},$$

and the supremum is taken over all balls B in $\mathbb{R}^n$. If $w = 1$ and $\kappa = \lambda/n$ with $0 < \lambda < n$, then $L^{p,\kappa}(w) = L^{p,\lambda}(\mathbb{R}^n)$, the classical Morrey spaces.

The standard Hardy-Littlewood maximal function $M_r f$, $1 \leq r < \infty$, is defined by

$$M_r f(x) = \sup_{B:x\in B}\left(\frac{1}{|B|}\int_B \left|f(y)\right|^r dy\right)^{1/r},$$

where the sup is taken over all balls containing x. If $r = 1$, $M_1 f$ will be denoted by Mf. The Fefferman-Stein sharp maximal function of f, $f^\sharp(x)$, is defined by

$$f^\sharp(x) = \sup_{B:x\in B}\frac{1}{|B|}\int_B \left|f(y) - f_B\right| dy,$$

where $f_B = \frac{1}{|B|}\int_B f\,dx$. We will say $f \in BMO(\mathbb{R}^n)$ if $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $f^\sharp(x) \in L^\infty$. If $f \in BMO$, the BMO semi-norm of f is given by

$$\|f\|_* = \sup_x f^\sharp(x) = \sup_x \sup_{x\in B}\frac{1}{|B|}\int_B \left|f(y) - f_B\right| dy.$$

A weight w is a non-negative locally integrable function. We say that $w \in A_p(\mathbb{R}^n)$, $1 < p < \infty$, if there exists a constant C such that for every ball $B \subset \mathbb{R}^n$,

$$\left(\frac{1}{|B|}\int_B w\,dx\right)\left(\frac{1}{|B|}\int_B w^{1-p'}\,dx\right)^{p-1} \leq C,$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. For $p = 1$, we say that $w \in A_1(\mathbb{R}^n)$ if there is a constant C such that for every ball $B \subset \mathbb{R}^n$,

$$\frac{1}{|B|}\int_B w\,dy \le Cw(x) \quad \text{for a.e. } x \in B,$$

or, equivalently, $M(w) \le Cw$ a.e. We denote $A_\infty(\mathbb{R}^n) = \bigcup_{1\le p<\infty} A_p(\mathbb{R}^n)$. For the above definition, see [11].

A family of operators A_t, $t > 0$, is said to be a 'generalized approximation to the identity' if, for every $t > 0$, A_t can be represented by kernels $a_t(x,y)$ in the following sense: For every function $f \in L^p(\mathbb{R}^n)$, $p \ge 1$, $A_t f(x) = \int_{\mathbb{R}^n} a_t(x,y)f(y)\,dy$, and the following condition holds:

$$\bigl|a_t(x,y)\bigr| \le h_t(x,y) = t^{-n/m}s\bigl(|x-y|^m t^{-1}\bigr), \tag{2.1}$$

in which m is a positive constant and s is a positive, bounded, decreasing function satisfying

$$\lim_{r\to\infty} r^{n+N+\epsilon}s\bigl(r^m\bigr) = 0 \tag{2.2}$$

for some $\epsilon > 0$.

Note that (2.2) implies that

$$\bigl|a_t(x,y)\bigr| \le t^{-n/m} \times \left(1 + \frac{|x-y|}{t^{1/m}}\right)^{-(n+\epsilon)}.$$

In [12], the sharp maximal function $M_A^\sharp f$ associated with a 'generalized approximation to the identity' $\{A_t, t > 0\}$ is defined by

$$M_A^\sharp f(x) = \sup_{x\in B}\frac{1}{|B|}\int_B \bigl|f(y) - A_{t_B}f(y)\bigr|\,dy, \tag{2.3}$$

where $t_B = r_B^m$, and $f \in L^p(\mathbb{R}^n)$ for some $p \ge 1$.

The following results are proved in the context of spaces of homogeneous type in [13, 14] and [10].

Lemma 2.1

(i) *For every $p \in [1,\infty)$, there exists a constant C such that for every $f \in L^p(\mathbb{R}^n)$,*

$$A_t f(x) \le CMf(x);$$

(ii) *Assume that $b \in BMO$ and $M > 1$. Then, for every ball $B(x;r)$, we have*

$$|b_B - b_{MB}| \le C\|b\|_* \log M;$$

(iii) (*John-Nirenberg lemma*) *Let $1 \le p < \infty$ and $B \subset \mathbb{R}^n$, then $b \in BMO$ if and only if*

$$\frac{1}{|B|}\int_B |b - b_B|^p\,dx \le \|b\|_*^p.$$

Lemma 2.2 *For* $1 < p < \infty$, $0 < \kappa < 1$ *and* $w \in A_p$, *we have* $\|Mf\|_{L^{p,\kappa}(w)} \le C\|f\|_{L^{p,\kappa}(w)}$.

For the proof of this lemma, see [2, Theorem 3.2].

Lemma 2.3 *Let* $\{A_t, t > 0\}$ *be a 'generalized approximation to the identity' and let* $b \in BMO$. *Then, for every function* $f \in L^p(\mathbb{R}^n)$, $p > 1$, $x \in \mathbb{R}^n$ *and* $1 < r < \infty$, *we have*

$$\sup_{x\in B} \frac{1}{|B|} \int_B \left| A_{t_B}(b - b_B)f(y) \right| dy \le C\|b\|_* M_r f(x), \tag{2.4}$$

where $t_B = r_B^m$.

For the proof of this lemma, see Lemma 2.3 in [15].

Now, we have the following analogy of the classical Fefferman-Stein inequality [11, Chapter IV] for the sharp maximal function $M_A^\sharp f$. For the proof, see Proposition 4.1 in [12].

Lemma 2.4 *Take* $\lambda > 0$, $w \in A_\infty(\mathbb{R}^n)$, $f \in L^1_{\mathrm{loc}}$ *and a ball* B_0 *such that there exists* $x_0 \in B_0$ *with* $Mf(x_0) < \lambda$. *Then, for every* $0 < \eta < 1$, *there exist* $r, \gamma > 0$ (*independent of* λ, B_0, f, x_0) *and* C_w *which only depend on* w *such that*

$$w\left\{x \in B_0 : Mf(x) > D\lambda, M_A^\sharp f(x) \le \gamma\lambda\right\} \le C_w \eta^r w(B_0),$$

where $D > 1$ *is a fixed constant which depends only on the 'generalized approximation to the identity'* $\{A_t, t > 0\}$.

3 The main results

In this section, we consider the Toeplitz operator related to a singular integral with non-smooth kernel $T_b = \sum_{j=1}^M T_{j,1} M_b T_{j,2}$, where $T_{j,1}$ and $T_{j,2}$ are singular integrals with non-smooth kernels, which are associated with an approximation of identity or $\pm I$. For $i = 1, \dots, M$, $j = 1, 2$, we assume that if $T_{i,j} \ne \pm I$, then:

(a) $T_{i,j}$ are bounded operators on $L^2(\mathbb{R}^n)$.

(b) There exist 'generalized approximations of the identity' $\{B_t^{ij}, t > 0\}$ such that $(T_{i,j} - T_{i,j}B_t^{ij})$ have associated kernels $K_t^{ij}(x,y)$ and there exist positive constants C_1, C_2 such that

$$\int_{|x-y|>C_1 t^{1/m}} \left| K_t^{ij}(x,y) \right| dx \le C_2 \quad \text{for all } y \in \mathbb{R}^n.$$

(c) There exists a 'generalized approximation to the identity' $\{A_t, t > 0\}$ such that the kernels $k_t^{ij}(x,y)$ of the operators $(T_{i,j} - A_t T_{i,j})$ satisfy

$$\left| k_t^{ij} \right| \le C_4 \frac{t^{\alpha/m}}{|x-y|^{n+\alpha}} \frac{t^{\alpha/m}}{d(x,y)^\alpha}, \tag{3.1}$$

when $|x - y| \ge C_3 t^{1/m}$ for some $C_3, C_4, \alpha > 0$.

It is proved in [10] that if T is an operator satisfying (a) and (b) above, then T is of weak $(1,1)$ and of strong type (p,p) for $1 < p \le 2$. In addition, if (c) is also satisfied, the operator T is bounded on $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$. Moreover, if $w \in A_p$, then T is bounded on $L^p(w)$ (see [12]).

In order to study the boundedness of T_b in weighted Morrey spaces, we need the following result.

Lemma 3.1 *Let $w \in A_\infty$, $0 < \kappa < 1$ and $1 < p < \infty$. Then, for every $f \in L^1_{\mathrm{loc}}$ with $Mf \in L^{p,\kappa}(w)$, there exists a constant C_w, which only depends on w, such that*

$$\|Mf\|_{L^{p,\kappa}(w)} \le C_w \big\|M_A^\sharp f\big\|_{L^{p,\kappa}(w)}. \tag{3.2}$$

Proof Let B be a ball in $\mathbb{R}^n$. Set $E_\lambda = \{x \in B : Mf(x) > \lambda\}$. Then from the Whitney decomposition theorem, we know that there exist mutually disjoint cubes Q_k such that $E_\lambda = \bigcup_k Q_k$ and $10Q_k \cap B \setminus E_\lambda \neq \varnothing$. Denote B_k to be the ball with the same center as Q_k and $r_{B_k} = \frac{1}{2}$ diameter Q_k. Let $\widetilde{B}_k = 10B_k$. Then there exists an $x_k \in \widetilde{B}_k \cap B \setminus E_\lambda$, that is, $Mf(x_k) \le \lambda$. Let us use Lemma 2.4. There are C_w; $r > 0$ and $D > 1$ such that, if $0 < \eta < 1$ (to be chosen later), we can find $\gamma > 0$ in such a way that

$$w\big\{x \in \widetilde{B}_k : Mf(x) > D\lambda, M_A^\sharp f(x) \le \gamma\lambda\big\} \le C_w \eta^r w(\widetilde{B}_k).$$

Set $U_\lambda = \{x \in B : Mf(x) > D\lambda, M_A^\sharp f(x) \le \gamma\lambda\}$ and so $U_\lambda \subset E_\lambda = \bigcup_k Q_k \subset \bigcup_k \widetilde{B}_k$ since $D > 1$. Then

$$\begin{aligned} w(U_\lambda) &\le \sum_k w\big\{x \in \widetilde{B}_k : Mf(x) > D\lambda, M_A^\sharp f(x) \le \gamma\lambda\big\} \\ &\le C_w \eta^r \sum_k w(\widetilde{B}_k) \\ &\le C\eta^r \sum_k w(Q_k) = C\eta^r w(E_\lambda) \\ &= C\eta^r w\big\{x \in B : Mf(x) > \lambda\big\}, \end{aligned}$$

where we used the fact that A_∞ weights are doubling measures and C is a constant that only depends on the weight. One can prove that

$$\begin{aligned} \int_B |Mf|^p w\,dx &= D^p \int_0^\infty p\lambda^{p-1} w\big\{x \in B : Mf(x) > D\lambda\big\}\,d\lambda \\ &\le D^p \int_0^\infty p\lambda^{p-1}\big(w(U_\lambda) + w\big\{x \in B : M_A^\sharp f(x) > \gamma\lambda\big\}\big)\,d\lambda \\ &\le CD^p\eta^r \int_B |Mf|^p w\,dx + \frac{D^p}{\gamma^p}\int_B \big|M_A^\sharp f\big|^p w\,dx. \end{aligned}$$

Let us choose η such that $CD^p\eta^r = 1/2$. The former inequality turns out to be

$$\int_B |Mf|^p w\,dx \le 2\frac{D^p}{\gamma^p}\int_B \big|M_A^\sharp f\big|^p w\,dx.$$

This implies that

$$\|Mf\|_{L^{p,\kappa}(w)} \le C\|M_A^\sharp f\|_{L^{p,\kappa}(w)}.$$

The proof of this lemma is completed. □

The aim of this section is to prove the following theorem.

Theorem 3.2 *Let $T_{i,j}$ be operators satisfying the above conditions* (a), (b) *and* (c) *or* $\pm I$. *Let* $1 < p < \infty$, $0 < \kappa < 1$ *and* $w \in A_p$. *Suppose that* $T_1(f) = 0$ *when* $f \in L^{p,\kappa}(w)$. *If* $b \in BMO(\mathbb{R}^n)$, *then there exists a constant C such that*

$$\|T_b(f)\|_{L^{p,\kappa}(w)} \le C\left(\sum_{j=1}^M \|T_{j,1}\|_{L^2(\mathbb{R}^n)}\right)\left(\sum_{j=1}^M \|T_{j,2}\|_{L^2(\mathbb{R}^n)}\right)\|b\|_*\|f\|_{L^{p,\kappa}(w)} \tag{3.3}$$

for all $f \in L^{p,\kappa}(w)$.

Proof Without loss of generality, we may assume that

$$\|T_{j,i}\|_{L^2(\mathbb{R}^n)} \le 1 \quad \text{for all } 1 \le j \le M, i = 1, 2.$$

For $w \in A_p$, it is well known that there exists $t > 1$ such that $w \in A_{p/t}$. Then we can choose two real numbers r and s larger than 1 such that $1 < rs < p < \infty$ and $w \in A_{p/(rs)}$. We will prove that there exists a constant C such that

$$M_A^\sharp(T_bf)(x) \le \sum_{j=1}^M C\|b\|_* M_{rs}(T_{j,2}f)(x) \tag{3.4}$$

for all $x \in \mathbb{R}^n$.

We now prove (3.4). For an arbitrary fixed $x \in \mathbb{R}^n$, choose a ball $B(x_0; r) = \{y \in \mathbb{R}^n : |x_0 - y| < r\}$ which contains x. We have that $T_1(f) = 0$, and so $T_{b_B}(f) = b_BT_1(f) = 0$. Thus

$$T_b(f) = T_{(b-b_B)\chi_{2B}}(f) + T_{(b-b_B)\chi_{(2B)^c}}(f)$$

and

$$A_{t_B}(T_bf) = A_{t_B}(T_{(b-b_B)\chi_{2B}}f) + A_{t_B}(T_{(b-b_B)\chi_{(2B)^c}}f),$$

where $t_B = r_B^m$. Then

$$\begin{aligned}
&\frac{1}{|B|}\int_B |T_b(f)(y) - A_{t_B}(T_bf)(y)|\,dy \\
&\quad\le \frac{1}{|B|}\int_B |T_{(b-b_B)\chi_{2B}}(f)(y)|\,dy + \frac{1}{|B|}\int_B |A_{t_B}(T_{(b-b_B)\chi_{2B}}f)(y)|\,dy \\
&\qquad + \frac{1}{|B|}\int_B |T_{(b-b_B)\chi_{(2B)^c}}(f)(y) - A_{t_B}(T_{(b-b_B)\chi_{(2B)^c}}f)(y)|\,dy \\
&\quad =: I + II + III.
\end{aligned}$$

Let r' be the dual of r such that $1/r + 1/r' = 1$. By Lemma 2.1 and the boundedness of $T_{j,1}$, we have

$$
\begin{aligned}
I &\le \left(\frac{1}{|B|}\int_{\mathbb{R}^n}\left|T_{(b-b_B)\chi_{2B}}(f)(y)\right|^s dy\right)^{1/s} \\
&\le \sum_{j=1}^{M} C\left(\frac{1}{|B|}\int_{2B}\left|\big(b(y)-b_B\big)T_{j,2}(f)(y)\right|^s dy\right)^{1/s} \\
&\le \sum_{j=1}^{M} C\left(\frac{1}{|B|}\int_{2B}\left|b(y)-b_B\right|^{sr'} dy\right)^{1/(sr')}\left(\frac{1}{|B|}\int_{2B}\left|T_{j,2}(f)(y)\right|^{sr} dy\right)^{1/(sr)} \\
&\le \sum_{j=1}^{M} C\|b\|_* M_{rs}(T_{j,2}f)(x).
\end{aligned}
$$

Similarly, by Lemma 2.1 and the boundedness of $T_{j,1}$, we obtain

$$
\begin{aligned}
II &\le \frac{1}{|B|}\int_{B}\left|M(T_{(b-b_B)\chi_{2B}}f)(y)\right| dy \le \left(\frac{1}{|B|}\int_{\mathbb{R}^n}\left|M(T_{(b-b_B)\chi_{2B}}f)(y)\right|^s dy\right)^{1/s} \\
&\le \left(\frac{1}{|B|}\int_{\mathbb{R}^n}\left|T_{(b-b_B)\chi_{2B}}(f)(y)\right|^s dy\right)^{1/s} \\
&\le \sum_{j=1}^{M} C\|b\|_* M_{rs}(T_{j,2}f)(x).
\end{aligned}
$$

We now consider the term III. There are two cases:
(1) Suppose that $T_{j,1} \neq \pm I$ $(j = 1, \ldots, M)$, then using the assumption (c), we have

$$
\begin{aligned}
III &\le \sum_{j=1}^{M}\frac{1}{|B|}\int_{B}\int_{(2B)^c}\left|k_{t_B}^{j,1}(y,z)\right|\left|\big(b(z)-b_B\big)T_{j,2}(f)(z)\right| dz\, dy \\
&\le C\sum_{j=1}^{M}\sum_{k=1}^{\infty}\int_{2^k r_B \le |x_0-z| < 2^{k+1} r_B}\frac{r_B^{\alpha}}{|x_0-z|^{n+\alpha}}\left|\big(b(z)-b_B\big)T_{j,2}(f)(z)\right| dz \\
&\le C\sum_{j=1}^{M}\sum_{k=1}^{\infty}2^{-k\alpha}\frac{1}{|B(x_0;2^k r_B)|}\int_{|x_0-z|<2^{k+1}r_B}\left|\big(b(z)-b_B\big)T_{j,2}(f)(z)\right| dz \\
&\le C\sum_{j=1}^{M}\sum_{k=1}^{\infty}2^{-k\alpha}\frac{1}{|B(x_0;2^k r_B)|}\int_{|x_0-z|<2^{k+1}r_B}\left|b(z)-b_{2^{k+1}B}\right|\left|T_{j,2}(f)(z)\right| dz \\
&\quad + C\sum_{j=1}^{M}\sum_{k=1}^{\infty}2^{-k\alpha}\left|b_{2^{k+1}B}-b_B\right|\frac{1}{|B(x_0;2^k r_B)|}\int_{|x_0-z|<2^{k+1}r_B}\left|T_{j,2}(f)(z)\right| dz \\
&\le C\|b\|_*\sum_{j=1}^{M}\sum_{k=1}^{\infty}2^{-k\alpha}M_{rs}(T_{j,2}f)(x) + C\|b\|_*\sum_{j=1}^{M}\sum_{k=1}^{\infty}2^{-k\alpha}(k+1)M(T_{j,2}f)(x) \\
&\le C\|b\|_*\sum_{j=1}^{M}M_{rs}(T_{j,2}f)(x).
\end{aligned}
$$

(2) Suppose that there are i identity or $-I$ operators in $\{T_{j,1}\}$. Without loss of generality, we assume that $T_{1,1},\ldots,T_{i,1}$ are identity operators, then

$$\begin{aligned}
III &\le \sum_{j=1}^{i}\frac{1}{|B|}\int_B\bigl|\bigl(b(z)-b_B\bigr)\chi_{(2B)^c}T_{j,2}(f)(z)\bigr|\,dz \\
&\quad+\sum_{j=1}^{i}\frac{1}{|B|}\int_B\bigl|A_{t_B}\bigl((b-b_B)\chi_{(2B)^c}T_{j,2}(f)\bigr)(z)\bigr|\,dz \\
&\quad+\sum_{j=i+1}^{M}\frac{1}{|B|}\int_B\bigl|T_{j,1}M_{(b-b_B)\chi_{(2B)^c}}T_{j,2}(f)(z)-A_{t_B}\bigl(T_{j,1}M_{(b-b_B)\chi_{(2B)^c}}T_{j,2}(f)\bigr)(z)\bigr|\,dz \\
&= III_1+III_2+III_3.
\end{aligned}$$

It is obvious that $III_1=0$.

By (2.4) and Lemma 2.1, we obtain

$$\begin{aligned}
III_2 &\le \sum_{j=1}^{i}\frac{1}{|B|}\int_B\int_{(2B)^c}\bigl|h_{t_B}(z,y)\bigl(b(y)-b_B\bigr)T_{j,2}(f)(y)\bigr|\,dy\,dz \\
&= \sum_{j=1}^{i}\sum_{k=1}^{\infty}\frac{1}{|B|}\int_B\int_{(2^{k+1}B)\setminus(2^kB)}\bigl|h_{t_B}(z,y)\bigl(b(y)-b_B\bigr)T_{j,2}(f)(y)\bigr|\,dy\,dz \\
&\le C\sum_{j=1}^{i}\sum_{k=1}^{\infty}2^{kn}s\bigl(2^{(k-1)m}\bigr)\int_{2^{k+1}B}\bigl|\bigl(b(y)-b_B\bigr)T_{j,2}(f)(y)\bigr|\,dy\,dz \\
&\le C\|b\|_*\sum_{j=1}^{i}\sum_{k=1}^{\infty}(k+1)2^{kn}s\bigl(2^{(k-1)m}\bigr)M_{rs}(T_{j,2}f)(x) \\
&\le C\|b\|_*\sum_{j=1}^{i}M_{rs}(T_{j,2}f)(x).
\end{aligned}$$

Moreover, from case (1) it follows that

$$III_3\le C\|b\|_*\sum_{j=i+1}^{M}M_{rs}(T_{j,2}f)(x).$$

So, $III\le C\|b\|_*\sum_{j=1}^{M}M_{rs}(T_{j,2}f)(x)$.

Combining the above estimates of I, II and III, we obtain (3.4).

From (3.4), we know that if T is an operator satisfying (a), (b) and (c), then there exists $1<s<p$ such that $w\in A_{p/s}$ and

$$M_A^{\sharp}(Tf)(x)\le CM_sf(x). \tag{3.5}$$

For the proof of (3.5), one can also see [12, Proposition 5.4]. Then combining (3.5), Lemmas 2.2 and 3.1, we have

$$\|Tf\|_{L^{p,\kappa}(w)}\le C\bigl\|M_A^{\sharp}(Tf)\bigr\|_{L^{p,\kappa}(w)}\le\|M_sf\|_{L^{p,\kappa}(w)}\le\|f\|_{L^{p,\kappa}(w)}.$$

Combining this, (3.4), Lemmas 2.2 and 3.1, we have

$$\|T_b f\|_{L^{p,\kappa}(w)} \leq C \left\| M_A^{\sharp}(T_b f) \right\|_{L^{p,\kappa}(w)} \leq \sum_{j=1}^{M} C\|b\|_* \left\| M_{rs}(T_{j,2} f) \right\|_{L^{p,\kappa}(w)} \leq C\|b\|_* \|f\|_{L^{p,\kappa}(w)}$$

for all $f \in L^{p,\kappa}(w)$. The proof of this theorem is completed. □

Corollary 3.3 *Let T be operators satisfying the above conditions* (a), (b) *and* (c). *Let $w \in A_p$, $1 < p < \infty$. If $b \in BMO(\mathbb{R}^n)$, then there exists a constant C such that*

$$\left\| [b, T] f \right\|_{L^{p,\kappa}(w)} \leq C\|b\|_* \|f\|_{L^{p,\kappa}(w)} \tag{3.6}$$

for all $f \in L^{p,\kappa}(w)$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors completed the paper together. They also read and approved the final manuscript.

Acknowledgements
The authors would like to thank the referee for carefully reading the manuscript and for making several useful suggestions. This research was supported by the National Natural Science Foundation of China (Grant No. 11271092), Natural Science Foundation of Guangdong Province (Grant No. s2011010005367), Specialized Research Fund for the Doctoral Program of Higher Education (Grant No. 20114410110001, 20124410120002) and SRF of Guangzhou Education Bureau (Grant No. 2012A088).

Received: 4 July 2012 Accepted: 30 April 2013 Published: 17 May 2013

References
1. Morrey, CB: On the solutions of quasi-linear elliptic partial differential equations. Trans. Am. Math. Soc. **43**, 126-166 (1938)
2. Chiarenza, F, Frasca, M: Morrey spaces and Hardy-Littlewood maximal function. Rend. Mat. Appl. **7**, 273-279 (1987)
3. Nakai, E: Hardy-Littlewood maximal operator, singular integral operators and the Riesz potentials on generalized Morrey spaces. Math. Nachr. **166**, 95-103 (1994)
4. Palagachev, DK, Softova, LG: Singular integral operators, Morrey spaces and fine regularity of solutions to PDE's. Potential Anal. **20**, 237-263 (2004)
5. Softova, L: Singular integrals and commutators in generalized Morrey spaces. Acta Math. Sin. Engl. Ser. **22**, 757-766 (2006) (English summary)
6. Komori, Y, Shirai, S: Weighted Morrey spaces and a singular integral operator. Math. Nachr. **282**, 219-231 (2009)
7. Krantz, S, Li, S: Boundedness and compactness of integral operators on spaces of homogeneous type and applications, I. J. Math. Anal. Appl. **258**, 629-641 (2001)
8. Lu, SZ, Mo, HX: Toeplitz-type operators on Lebesgue spaces. Acta Math. Sci., Ser. B, Engl. Ed. **29**, 140-150 (2009)
9. Xie, PZ, Cao, GF: Toeplitz-type operators on homogeneous type spaces. Chin. Ann. Math., Ser. A **32**, 219-228 (2011) (Chinese)
10. Duong, XT, McIntosh, A: Singular integral operators with non-smooth kernels on irregular domains. Rev. Mat. Iberoam. **15**, 233-265 (1999)
11. Stein, EM: Harmonic Analysis: Real Variable Methods, Orthogonality and Oscillatory Integrals. Princeton University Press, Princeton (1993)
12. Martell, JM: Sharp maximal functions associated with approximations of the identity in spaces of homogeneous type and applications. Stud. Math. **161**, 113-145 (2004)
13. Burger, N: Espace des fonctions à variation moyenne bornée sur un espace de nature homogène. C. R. Acad. Sci. Paris, Sér. A-B **286**, 139-141 (1978)
14. Bramanti, M, Cerutti, M: Commutators of singular integrals on homogeneous spaces. Boll. Unione Mat. Ital. **10**, 843-883 (1996)
15. Duong, XT, Yan, LX: Commutators of BMO functions and singular integral operators with non-smooth kernels. Bull. Aust. Math. Soc. **67**, 187-200 (2003)

doi:10.1186/1029-242X-2013-253
Cite this article as: Xie and Cao: **Toeplitz-type operators in weighted Morrey spaces.** *Journal of Inequalities and Applications* 2013 **2013**:253.