Beg and Butt *Journal of Inequalities and Applications* 2013, **2013**:252
http://www.journalofinequalitiesandapplications.com/content/2013/1/252

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

Fixed point of set-valued graph contractive mappings

Ismat Beg[1*] and Asma Rashid Butt[2]

*Correspondence: begismat@yahoo.com
[1]Centre for Mathematics and Statistical Sciences, Lahore School of Economics, Lahore, Pakistan
Full list of author information is available at the end of the article

Abstract

Let (X,d) be a metric space and let F, H be two set-valued mappings on X. We obtained sufficient conditions for the existence of a common fixed point of the mappings F, H in the metric space X endowed with a graph G such that the set of vertices of G, $V(G) = X$ and the set of edges of G, $E(G) \subseteq X \times X$.

MSC: Primary 47H10; secondary 47H04; 47H07; 54C60; 54H25

Keywords: fixed point; directed graph; metric space; set-valued mapping

1 Introduction and preliminaries

Edelstein [1] generalized classical Banach's contraction mapping principle and Nadler [2] proved Banach's fixed point theorem for set-valued mappings. Recently several extensions of Nadler's theorem in different directions were obtained; see [3–15]. Beg and Azam [5] extended Edelstein's theorem by considering a pair of set-valued mappings with a general contractive condition. The aim of this paper is to study the existence of common fixed points for set-valued graph contractive mappings in metric spaces endowed with a graph G. Our results improve/generalize [1, 2, 16] and several other known results in the literature.

Let (X,d) be a complete metric space and let $CB(X)$ be a class of all nonempty closed and bounded subsets of X. For $A, B \in CB(X)$, let

$$D(A,B) := \max\Big\{\sup_{b\in B} d(b,A), \sup_{a\in A} d(a,B)\Big\},$$

where

$$d(a,B) := \inf_{b\in B} d(a,b).$$

Mapping D is said to be a *Hausdorff metric* induced by d.

Definition 1.1 Let $F : X \to X$ be a set-valued mapping, *i.e.*, $X \ni x \mapsto Fx$ is a subset of X. A point $x \in X$ is said to be a *fixed point* of the set-valued mapping F if $x \in Fx$.

Definition 1.2 A metric space (X,d) is called a ε-chainable metric space for some $\varepsilon > 0$ if given $x, y \in X$, there is $n \in N$ and a sequence $(x_i)_{i=0}^n$ such that

$$x_0 = x, \qquad x_n = y \quad \text{and} \quad d(x_{i-1}, x_i) < \varepsilon \quad \text{for } i = 1, \ldots, n.$$

Let $\operatorname{Fix} F := \{x \in X : x \in Fx\}$ denote the set of fixed points of the mapping F.

Definition 1.3 Let (X,d) be a metric space, $\varepsilon > 0$, $0 \leq \kappa < 1$ and $x, y \in X$. A mapping $f : X \to X$ is called (ε, κ) uniformly locally contractive if $0 < d(x,y) < \varepsilon \Rightarrow d(fx, fy) < \kappa d(x,y)$.

The following significant generalization of Banach's contraction principle [17, Theorem 2.1] was obtained by Edelstein [1].

Theorem 1.4 [1] *Let (X,d) be a ε-chainable complete metric space. If $f : X \to X$ is a (ε, κ) uniformly locally contractive mapping, then f has a unique fixed point.*

Afterwards, in 1969, Nadler [2] proved a set-valued extension of Banach's theorem and obtained the following result.

Theorem 1.5 [2] *Let (X,d) be a complete metric space and $F : X \to CB(X)$. If there exists $\kappa \in (0,1)$ such that*

$$D(Fx, Fy) \leq \kappa d(x,y) \quad \textit{for all } x, y \in X,$$

then F has a fixed point in X.

Nadler [2] also extended Edelstein's theorem for set-valued mappings.

Theorem 1.6 [2] *Let (X,d) be a ε-chainable complete metric space for some $\varepsilon > 0$ and let $F : X \to C(X)$ be a set-valued mapping such that Fx is a nonempty compact subset of X. If F satisfies the following condition*:

$$x, y \in X \quad \textit{and} \quad 0 < d(x,y) < \varepsilon \quad \Rightarrow \quad D(Fx, Fy) < \kappa d(x,y),$$

then F has a fixed point.

Consider a directed graph G such that the set of its vertices coincides with X (*i.e.*, $V(G) := X$) and the set of its edges $E(G) := \{(x,y) : (x,y) \in X \times X, x \neq y\}$. We assume that G has no parallel edges and weighted graph by assigning to each edge the distance between the vertices; for details about definitions in graph theory, see [18].

We can identify G as $(V(G), E(G))$. G^{-1} denotes the conversion of a graph G, the graph obtained from G by reversing the direction of its edges. $\widetilde{G}$ denotes the undirected graph obtained from G by ignoring the direction of edges of G. We consider $\widetilde{G}$ as a directed graph for which the set if its edges is symmetric, thus we have

$$E(\widetilde{G}) := E(G) \cup E\bigl(G^{-1}\bigr).$$

Definition 1.7 A *subgraph* of a graph G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ and for any edge $(x,y) \in E(H)$, $x, y \in V(H)$.

Definition 1.8 Let x and y be vertices in a graph G. A *path* in G from x to y of length n ($n \in N \cup \{0\}$) is a sequence $(x_i)_{i=0}^{n}$ of $n+1$ vertices such that $x_0 = x$, $x_n = y$ and $(x_{i-1}, x_i) \in E(G)$ for $i = 1, 2, \ldots, n$.

Definition 1.9 The number of edges in G constituting the path is called the *length of the path*.

Definition 1.10 A graph G is *connected* if there is a path between any two vertices of G.

If a graph G is not connected, then it is called *disconnected*. Moreover, G is weakly connected if $\widetilde{G}$ is connected.

Assume that G is such that $E(G)$ is symmetric, and x is a vertex in G, then the subgraph G_x consisting of all edges and vertices, which are contained in some path in G beginning at x, is called the component of G containing x. In this case the equivalence class $[x]_G$ defined on $V(G)$ by the rule R (uRv if there is a path from u to v) is such that $V(G_x) = [x]_G$.

Property A: For any sequence $(x_n)_{n\in N}$ in X, if $x_n \to x$ and $(x_n, x_{n+1}) \in E(G)$ for $n \in N$, then $(x_n, x) \in E(G)$.

Definition 1.11 Let (X, d) be a metric space and $F, H : X \to CB(X)$. The mappings F, H are said to be graph contractive if there exists $\kappa \in (0,1)$ such that

$$(x \neq y), \quad (x,y) \in E(G) \quad \Rightarrow \quad D(Fx, Hy) < \kappa d(x,y),$$

and if $u \in Fx$ and $v \in Hy$ are such that

$$d(u,v) < d(x,y),$$

then $(u,v) \in E(G)$.

Definition 1.12 A *partial order* is a binary relation $\preceq$ over a set X which satisfies the following conditions:

1. $x \preceq x$ (reflexivity);
2. if $x \preceq y$ and $y \preceq x$, then $x = y$ (antisymmetry);
3. if $x \preceq y$ and $y \preceq z$, then $x \preceq z$ (transitivity);

for all x, y and z in X.

A set with a partial order $\preceq$ is called a *partially ordered set*.

Let $(X, \preceq)$ be a partially ordered set and $x, y \in X$. Elements x and y are said to be *comparable elements* of X if either $x \preceq y$ or $y \preceq x$.

Let $\preceq$ be a partial order in X. Define the graph $G := G_1$ by

$$E(G_1) := \{(x,y) \in X \times X : x \preceq y, x \neq y\},$$

and $G := G_2$ by

$$E(G_2) := \{(x,y) \in X \times X : x \preceq y \vee y \preceq x, x \neq y\}.$$

The class of G_1-contractive mappings was considered in [19] and that of G_2-contractive mappings in [20].

The weak connectivity of G_1 or G_2 means, given $x, y \in X$, there is a sequence $(x_i)_{i=0}^n$ such that $x_0 = x$, $x_n = y$ and for all $i = 1, \dots, n$, x_{i-1} and x_i are comparable.

We shall make use of the following lemmas due to Nadler [2], Assad and Kirk [21] in the proof of our results in next section.

Lemma 1.13 *If $A, B \in CB(X)$ with $D(A,B) < \epsilon$, then for each $a \in A$ there exists an element $b \in B$ such that $d(a,b) < \epsilon$.*

Lemma 1.14 *Let $\{A_n\}$ be a sequence in $CB(X)$ and $\lim_{n\to\infty} D(A_n, A) = 0$ for $A \in CB(X)$. If $x_n \in A_n$ and $\lim_{n\to\infty} d(x_n, x) = 0$, then $x \in A$.*

2 Common fixed point

We begin with the following theorem that gives the existence of a common fixed point (not necessarily unique) in metric spaces endowed with a graph for the set-valued mappings. Further, we assume that (X,d) is a complete metric space and G is a directed graph such that $E(G)$ is symmetric.

Theorem 2.1 *Let $F, H : X \to CB(X)$ be graph contractive mappings and let the triple (X,d,G) have the property* A. *Set $X_F := \{x \in X : (x,u) \in E(G) \text{ for some } u \in Fx\}$. Then the following statements hold.*

1. *For any $x \in X_F$, $F, H|_{[x]_G}$ have a common fixed point.*
2. *If $X_F \neq \emptyset$ and G is weakly connected, then F, H have a common fixed point in X.*
3. *If $X' := \bigcup\{[x]_G : x \in X_F\}$, then $F, H|_{X'}$ have a common fixed point.*
4. *If $F \subseteq E(G)$, then F, H have a common fixed point.*

Proof 1. Let $x_0 \in X_F$, then there exists $x_1 \in Fx_0$ such that $(x_0, x_1) \in E(G)$. Since F, H are graph contractive mappings, we have

$$D(Fx_0, Hx_1) < \kappa d(x_0, x_1).$$

Using Lemma 1.13, we have the existence of $x_2 \in Hx_1$ such that

$$d(x_1, x_2) < \kappa d(x_0, x_1). \tag{1}$$

Again, because F, H are graph contractive $(x_1, x_2) \in E(G)$, also $(x_2, x_1) \in E(G)$, since $E(G)$ is symmetric, we have

$$D(Fx_2, Hx_1) < \kappa d(x_1, x_2) < \kappa^2 d(x_0, x_1),$$

and Lemma 1.13 gives the existence of $x_3 \in Fx_2$ such that

$$d(x_2, x_3) < \kappa^2 d(x_0, x_1). \tag{2}$$

Continuing in this way, we have $x_{2n+1} \in Fx_{2n}$ and $x_{2n+2} \in Hx_{2n+1}$, $n = 0, 1, 2, \ldots$. Also, $(x_n, x_{n+1}) \in E(G)$ such that

$$d(x_n, x_{n+1}) < \kappa^n d(x_0, x_1). \tag{3}$$

Next we show that (x_n) is a Cauchy sequence in X. Let $m > n$. Then

$$\begin{aligned} d(x_n,x_m) &\le d(x_n,x_{n+1}) + d(x_{n+1},x_{n+2}) + d(x_{n+2},x_{n+3}) + \cdots + d(x_{m-1},x_m) \\ &< \left[\kappa^n + \kappa^{n+1} + \kappa^{n+2} + \cdots + \kappa^{m-1}\right] d(x_0,x_1) \\ &= \kappa^n\left[1 + \kappa + \kappa^2 + \cdots + \kappa^{m-n-1}\right] d(x_0,x_1) \\ &= \kappa^n\left[\frac{1-\kappa^{m-n}}{1-\kappa}\right] d(x_0,x_1) \end{aligned}$$

because $\kappa \in (0,1)$, $1 - \kappa^{m-n} < 1$.

Therefore $d(x_n,x_m) \to 0$ as $n \to \infty$ implies that (x_n) is a Cauchy sequence and hence converges to some point (say) x in the complete metric space X.

Now we have to show that $x \in Fx \cap Hx$.

For n even: By property A, we have $(x_n,x) \in E(G)$. Therefore, by using graph contractivity, we have

$$D(Fx_n,Hx) < \kappa d(x_n,x).$$

Since $x_{n+1} \in Fx_n$ and $x_n \to x$, therefore by Lemma 1.14, $x \in Hx$.

For n odd: As $(x,x_n) \in E(G)$,

$$D(Fx,Hx_n) < \kappa d(x,x_n).$$

Now, by following the same arguments as above, $x \in Fx$.

Next as $(x_n,x_{n+1}) \in E(G)$, also $(x_n,x) \in E(G)$ for $n \in N$. We infer that $(x_0,x_1,\ldots,x_n,x)$ is a path in G and so $x \in [x_0]_G$.

2. Since $X_F \neq \emptyset$, so there exists $x_0 \in X_F$, and since G is weakly connected, therefore $[x_0]_G = X$, and by 1, mappings F and H have a common fixed point in X.

3. It follows easily from 1 and 2.

4. $F \subseteq E(G)$ implies that all $x \in X$ are such that there exists some $u \in Fx$ with $(x,u) \in E(G)$ so $X_F = X$ and by 2 and 3. F, H have a fixed point. □

Remark 2.2 Replace X_F by $X_H := \{x \in X : (x,u) \in E(G) \text{ for some } u \in Hx\}$ in conditions 1-3 of Theorem 2.1, then the conclusion remains true. That is, if $X_F \cup X_H \neq \emptyset$, then we have $\operatorname{Fix} F \cap \operatorname{Fix} H \neq \emptyset$, which follows easily from 1-3. Similarly, in condition 4, we can replace $F \subseteq E(G)$ by $H \subseteq E(G)$.

Corollary 2.3 is a direct consequence of Theorem 2.1(1).

Corollary 2.3 *Let (X,d) be a complete metric space and let the triple (X,d,G) have the* property A. *If G is weakly connected, then graph contractive mappings $F,H : X \to CB(X)$ such that $(x_0,x_1) \in E(G)$ for some $x_1 \in Fx_0$ have a common fixed point.*

Corollary 2.4 *Let (X,d) be a ε-chainable complete metric space for some $\varepsilon > 0$. Let $F,H : X \to CB(X)$ be such that there exists $\kappa \in (0,1)$ with*

$$0 < d(x,y) < \varepsilon \quad \Rightarrow \quad D(Fx,Hx) < \kappa d(x,y).$$

Then F and H have a common fixed point.

Proof Consider the graph G as $V(G) := X$ and

$$E(G) := \{(x,y) \in X \times X : 0 < d(x,y) < \varepsilon\}. \tag{4}$$

The ε-chainability of (X,d) means G is connected. If $(x,y) \in E(G)$, then

$$D(Fx,Hy) < \kappa d(x,y) < \kappa\varepsilon < \varepsilon$$

and by using Lemma 1.13, for each $u \in Fx$, we have the existence of $v \in Hy$ such that $d(u,v) < \varepsilon$, which implies $(u,v) \in E(G)$. Hence F and H are graph contractive mappings. Also, (X,d,G) has *property* A. Indeed, if $x_n \to x$ and $d(x_n,x_{n+1}) < \varepsilon$ for $n \in N$, then $d(x_n,x) < \varepsilon$ for sufficiently large n, therefore $(x_n,x) \in E(G)$. So, by Theorem 2.1(2), F and H have a common fixed point. □

Theorem 2.5 *Let $F : X \to CB(X)$ be a graph contractive mapping and let the triple (X,d,G) have the property* A. *Set $X_F := \{x \in X : (x,u) \in E(G)$ for some $u \in Fx\}$. Then the following statements hold.*

1. *For any $x \in X_F$, $F|_{[x]_G}$ has a fixed point.*
2. *If $X_F \neq \emptyset$ and G is weakly connected, then F has a fixed point in X.*
3. *If $X' := \bigcup\{[x]_G : x \in X_F\}$, then $F|_{X'}$ has a fixed point.*
4. *If $F \subseteq E(G)$, then F has a fixed point.*
5. *If $X_F \neq \emptyset$, then* Fix $F \neq \emptyset$.

Proof Statements 1-4 can be proved by taking $F = H$ in Theorem 2.1 and 5 obtained from Remark 2.2.

Note that the assumption that $E(G)$ is symmetric is not needed in our Theorem 2.5. □

Remark 2.6

1. If we assume G is such that $E(G) := X \times X$, then clearly G is connected and our Theorem 2.5(2) improves Nadler's theorem, and further if F is single-valued, then we improve the Banach contraction theorem.
2. If F is a single-valued mapping, then Theorem 2.5(2, 5) with the graph G_1 improves [19, Theorem 2.2].
3. If F is a single-valued mapping, then Theorem 2.5(2, 5) with the graph G_2 improves [20, Theorem 2.1].
4. If $F = H$ is a single-valued mapping, then Theorem 2.1 and Theorem 2.5 partially generalize [22, Theorem 3.2].
5. If we take $F = H$ as single-valued mappings in Corollary 2.4, then we have [1, Theorem 5.2].
6. If we take $F = H$, then Corollary 2.4 becomes Theorem 1.5 due to [2].

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
IB gave the idea. ARB wrote the initial draft. IB and ARB finalized the manuscript. All authors read and approved the final manuscript. Correspondence was mainly done by IB.

Author details
[1]Centre for Mathematics and Statistical Sciences, Lahore School of Economics, Lahore, Pakistan. [2]Department of Mathematics, University of Engineering and Technology, Lahore, Pakistan.

Received: 26 November 2012 Accepted: 30 April 2013 Published: 17 May 2013

References

1. Edelstein, M: An extension of Banach's contraction principle. Proc. Am. Math. Soc. **12**, 07-10 (1961)
2. Nadler, SB: Multivalued contraction mappings. Pac. J. Math. **30**, 475-488 (1969)
3. Azam, A, Arshad, M: Fixed points of a sequence of locally contractive multivalued maps. Comput. Math. Appl. **57**, 96-100 (2009)
4. Azam, A, Beg, I: Common fixed points of fuzzy maps. Math. Comput. Model. **49**, 1331-1336 (2009)
5. Beg, I, Azam, A: Fixed points of multivalued locally contractive mappings. Boll. Unione Mat. Ital., A (7) **4**, 227-233 (1990)
6. Beg, I, Butt, AR: Fixed point for set valued mappings satisfying an implicit relation in partially ordered metric spaces. Nonlinear Anal. **71**, 3699-3704 (2009)
7. Beg, I, Butt, AR: Fixed points for weakly compatible mappings satisfying an implicit relation in partially ordered metric spaces. Carpath. J. Math. **25**(1), 01-12 (2009)
8. Beg, I, Butt, AR: Common fixed point for generalized set valued contractions satisfying an implicit relation in partially ordered metric spaces. Math. Commun. **15**(1), 65-76 (2010)
9. Beg, I, Nashine, HK: End-point results for multivalued mappings in partially ordered metric spaces. Int. J. Math. Math. Sci. **2012**, Article ID 580250 (2012)
10. Daffer, PZ: Fixed points of generalized contractive multivalued mappings. J. Math. Anal. Appl. **192**, 655-666 (1995)
11. Daffer, PZ, Kaneko, H, Li, W: On a conjecture of S. Reich. Proc. Am. Math. Soc. **124**, 3159-3162 (1996)
12. Feng, Y, Liu, S: Fixed point theorems for multivalued contractive mappings and multivaled Caristi type mappings. J. Math. Anal. Appl. **317**, 103-112 (2006)
13. Klim, D, Wardowski, D: Fixed point theorems for set-valued contractions in complete metric spaces. J. Math. Anal. Appl. **334**, 132-139 (2007)
14. Reich, S: Fixed points of contractive functions. Boll. Unione Mat. Ital. **5**(4), 26-42 (1972)
15. Qing, CY: On a fixed point problem of Reich. Proc. Am. Math. Soc. **124**, 3085-3088 (1996)
16. Beg, I, Butt, AR, Radojevic, S: Contraction principle for set valued mappings on a metric space with a graph. Comput. Math. Appl. **60**, 1214-1219 (2010)
17. Kirk, WA, Goebel, K: Topics in Metric Fixed Point Theory. Cambridge University Press, Cambridge (1990)
18. Johnsonbaugh, R: Discrete Mathematics. Prentice Hall, New York (1997)
19. Nieto, JJ, Rodríguez-López, R: Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations. Order **22**, 223-239 (2005)
20. Ran, ACM, Reurings, MCB: A fixed point theorem in partially ordered sets and some applications to matrix equations. Proc. Am. Math. Soc. **132**, 1435-1443 (2004)
21. Assad, NA, Kirk, WA: Fixed point theorems for setvalued mappings of contractive type. Pac. J. Math. **43**, 533-562 (1972)
22. Jachymski, J: The contraction principle for mappings on a metric space with a graph. Proc. Am. Math. Soc. **1**(136), 1359-1373 (2008)

doi:10.1186/1029-242X-2013-252
Cite this article as: Beg and Butt: **Fixed point of set-valued graph contractive mappings.** *Journal of Inequalities and Applications* 2013 **2013**:252.