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On asymptotically strict pseudocontractions and equilibrium problems

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Abstract
In this paper, an equilibrium problem and a fixed point problem of an asymptotically strict pseudocontraction are investigated based on hybrid iterative algorithms. Strong convergence theorems of common solutions to the equilibrium problem and the fixed point problem are obtained in the framework of real Hilbert spaces.
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1 Introduction

Equilibrium problems have emerged as an effective and powerful tool for studying a wide class of problems which arise in economics, finance, image reconstruction, ecology, transportation, and network; see [1–12] and the references therein. Equilibrium problems include fixed point problems, variational inequality problems, variational inclusion problems, saddle point problems, the Nash equilibrium problem, complementarity problems and so on. For the solutions of equilibrium problems, there are several algorithms to solve the problem. The classical algorithm is the Krasnoselskii-Mann iterative algorithm. However, the Krasnoselskii-Mann iterative algorithm is weak convergence for the solutions of equilibrium problems. Haugazeau's projection method [13] recently has been considered for the approximation of solutions of equilibrium problems and fixed point problems. The advantage of the projection methods is that strong convergence of iterative sequences can be guaranteed without any compact assumptions. The aim of this paper is to study an equilibrium problem and a fixed point problem of an asymptotically strict pseudocontraction based on hybrid iterative algorithms and establish a strong convergence theorem of common solutions in the framework of Hilbert spaces.

2 Preliminaries

Throughout this paper, we always assume that H is a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$. Let C be a nonempty closed convex subset of H, let $A : C \to H$ be a monotone mapping and F be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers.

In this paper, we consider the following equilibrium problem.

$$\text{Find } x \in C \text{ such that } F(x,y) + \langle Ax, y-x\rangle \geq 0, \quad \forall y \in C. \tag{2.1}$$

The set of such an $x \in C$ is denoted by $\mathrm{EP}(F, A)$, *i.e.*,

$$\mathrm{EP}(F, A) = \{x \in C : F(x, y) + \langle Ax, y - x\rangle \geq 0, \forall y \in C\}.$$

If $A \equiv 0$, then the problem (2.1) is reduced to the following:

$$\text{Find } x \in C \text{ such that } F(x, y) \geq 0, \quad \forall y \in C. \tag{2.2}$$

The set of such an $x \in C$ is denoted by $\mathrm{EP}(F)$, *i.e.*,

$$\mathrm{EP}(F, A) = \{x \in C : F(x, y) \geq 0, \forall y \in C\}.$$

If $F \equiv 0$, the problem (2.1) is reduced to the classical variational inequality problem.

$$\text{Find } x \in C \text{ such that } \langle Ax, y - x\rangle \geq 0, \quad \forall y \in C. \tag{2.3}$$

To study the problems (2.1) and (2.2), we may assume that F satisfies the following conditions:

(A1) $F(x, x) = 0$ for all $x \in C$;
(A2) F is monotone, *i.e.*, $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;
(A3) for each $x, y, z \in C$,

$$\limsup_{t \downarrow 0} F\big(tz + (1 - t)x, y\big) \leq F(x, y);$$

(A4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and lower semi-continuous.

Recall that a mapping A is said to be monotone iff

$$\langle Ax - Ay, x - y\rangle \geq 0, \quad \forall x, y \in C.$$

A is said to be strongly monotone iff there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha \|x - y\|^2, \quad \forall x, y \in C.$$

For such a case, A is said to be α-strongly-monotone. A is said to be inverse-strongly monotone iff there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C.$$

For such a case, A is said to be α-inverse-strongly monotone.

Recall that a set-valued mapping $T : H \to 2^H$ is said to be monotone iff, for all $x, y \in H$, $f \in Tx$ and $g \in Ty$ imply $\langle x - y, f - g\rangle \geq 0$. $T : H \to 2^H$ is maximal if the graph $G(T)$ of T is not properly contained in the graph of any other monotone mapping.

It is known that a monotone mapping T is maximal if and only if, for any $(x, f) \in H \times H$, $\langle x - y, f - g\rangle \geq 0$ for all $(y, g) \in G(T)$ implies $f \in Tx$.

Let $S : C \to C$ be a nonlinear mapping. In this paper, we use $F(S)$ to denote the fixed point set of S.

Recall that S is said to be nonexpansive iff

$$\|Sx - Sy\| \le \|x - y\|, \quad \forall x, y \in C.$$

Recall that S is said to be asymptotically nonexpansive iff there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\left\|S^n x - S^n y\right\| \le k_n \|x - y\|, \quad \forall x, y \in C, n \ge 1.$$

The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [14] in 1972. Since 1972, a number of authors have studied the convergence problems of the iterative processes for such a class of mappings.

Recall that S is said to be strictly pseudocontractive iff there exists a constant $\kappa \in [0, 1)$ such that

$$\|Sx - Sy\|^2 \le \|x - y\|^2 + \kappa \left\|(I - S)x - (I - S)y\right\|^2, \quad \forall x, y \in C.$$

For such a case, S is also said to be a κ-strict pseudocontraction. The class of strict pseudocontractions is introduced by Browder and Petryshyn [15] in 1967. It is clear that every nonexpansive mapping is a 0-strict pseudocontraction. We also remark that if $\kappa = 1$, then S is said to be pseudocontractive.

Recall that S is said to be an asymptotically strict pseudocontraction iff there exist a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ and a constant $\kappa \in [0, 1)$ such that

$$\left\|S^n x - S^n y\right\|^2 \le k_n \|x - y\|^2 + \kappa \left\|\left(I - S^n\right)x - \left(I - S^n\right)y\right\|^2, \quad \forall x, y \in C, n \ge 1.$$

For such a case, S is also said to be an asymptotically κ-strict pseudocontraction. The class of asymptotically strict pseudocontractions is introduced by Qihou [16] in 1996. It is clear that every asymptotically nonexpansive mapping is an asymptotical 0-strict pseudocontraction. Every nonexpansive mapping is an asymptotically nonexpansive mapping with the sequence $\{1\}$. We also remark that if $\kappa = 1$, then S is said to be an asymptotically pseudocontractive mapping which was introduced by Schu [17] in 1991.

Recently, many authors considered the equilibrium problems (2.1), (2.2) and fixed point problems based on hybrid iterative methods; see, for instance, [18–30]. In this paper, motivated by these recent results, we consider the shrinking projection algorithm to solve the solutions of the equilibrium problem (2.1) and the fixed point problem of an asymptotically strict pseudocontraction. It is proved that the sequence generated in the purposed iterative process converges strongly to some common element in the solution set of the equilibrium problem (2.1) and in the fixed point set of an asymptotically strict pseudocontraction.

In order to prove our main results, we also need the following lemmas.

Lemma 2.1 [31] *Let C be a nonempty closed convex subset of H and let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying* (A1)-(A4). *Then, for any $r > 0$ and $x \in H$, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0, \quad \forall y \in C.$$

Further, define

$$T_r x = \left\{ z \in C : F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C \right\}$$

for all $r > 0$ *and* $x \in H$. *Then the following hold*:

(a) T_r *is single-valued*;

(b) T_r *is firmly nonexpansive, i.e., for any* $x, y \in H$,

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle;$$

(c) $F(T_r) = \mathrm{EP}(F)$;

(d) $\mathrm{EP}(F)$ *is closed and convex.*

Lemma 2.2 [32] *In a real Hilbert space, the following inequality holds*:

$$\|ax + (1-a)y\|^2 = a\|x\|^2 + (1-a)\|y\|^2 - a(1-a)\|x-y\|^2, \quad \forall a \in [0,1], x, y \in H.$$

Lemma 2.3 [33] *Let* H *be a real Hilbert space, let* C *be a nonempty closed convex subset of* H *and let* $S : C \to C$ *be an asymptotically* κ*-strict pseudocontraction with the sequence* $\{k_n\}$. *Then*

(a) $F(S)$ *is closed and convex*;

(b) S *is L-Lipschitz continuous.*

Lemma 2.4 [33] *Let* H *be a real Hilbert space, let* C *be a nonempty closed convex subset of* H *and let* $S : C \to C$ *be an asymptotically strict pseudocontraction. Then the mapping* $I - S$ *is demiclosed at zero, that is, if* $\{x_n\}$ *is a sequence in* C *such that* $x_n \rightharpoonup \bar{x}$ *and* $x_n - Sx_n \to 0$, *then* $\bar{x} \in F(S)$.

3 Main results

Theorem 3.1 *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* F_m *be a bifunction from* $C \times C$ *to* $\mathbb{R}$ *which satisfies* (A1)-(A4) *and let* $A_m : C \to H$ *be a* ξ_m*-inverse-strongly monotone mapping for each* $1 \leq m \leq N$, *where* $N \geq 1$ *is some positive integer. Let* $S : C \to C$ *be an asymptotically* κ*-strict pseudocontraction. Assume that* $\mathcal{F} := F(S) \cap \bigcap_{m=1}^N \mathrm{EP}(F_m, A_m)$ *is nonempty and bounded. Let* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *be sequences in* $[0,1]$ *and let* $\{r_{n,m}\}$ *be a positive sequence. Let* $\{\gamma_{n,m}\}$ *be a sequence in* $[0,1]$ *for each* $1 \leq m \leq N$ *such that* $\sum_{m=1}^N \gamma_{n,m} = 1$. *Let* $\{x_n\}$ *be a sequence generated in the following manner*:

$$\begin{cases} x_1 \in H, \\ C_1 = H, \\ F_m(u_{n,m}, u) + \langle A_m x_n, u - u_{n,m}\rangle + \frac{1}{r_{n,m}}\langle u - u_{n,m}, u_{n,m} - x_n\rangle \geq 0, \quad \forall u \in C, \\ y_n = \alpha_n x_n + (1-\alpha_n)(\beta_n \sum_{m=1}^N \gamma_{n,m} u_{n,m} + (1-\beta_n) S^n \sum_{m=1}^N \gamma_{n,m} u_{n,m}), \\ C_{n+1} = \{w \in C_n : \|y_n - w\|^2 \leq \|x_n - w\|^2 + \theta_n\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad n \geq 0, \end{cases}$$

where $\theta_n = (1-\alpha_n)(k_n - 1)\Theta_n$ *and* $\Theta_n = \sup\{\|x_n - w\|^2 : w \in \mathcal{F}\}$. *Assume that the control sequences* $\{\alpha_n\}, \{\beta_n\}, \{\gamma_{n,1}\}, \ldots, \{\gamma_{n,N}\}, \{r_{n,1}\}, \ldots,$ *and* $\{r_{n,N}\}$ *satisfy the following restrictions*:

(a) $0 \le \alpha_n \le a < 1$, $\kappa \le \beta_n \le b < 1$;
(b) $0 < c \le \gamma_{n,m} \le 1$ *and* $0 < d \le r_{n,m} \le e < 2\xi_m$ *for each* $1 \le m \le N$.

Then the sequence $\{x_n\}$ *converges strongly to some point* $\bar{x}$, *where* $\bar{x} = P_{\mathcal{F}}x_1$.

Proof First, we show that C_n is closed and convex for each $n \ge 1$. It is easy to see that C_n is closed for each $n \ge 1$. We only show that C_n is convex for each $n \ge 1$. Note that $C_1 = H$ is convex. Suppose that C_i is convex for some positive integer i. Next, we show that C_{i+1} is convex for the same i. Note that

$$\|y_i - w\|^2 \le \|x_i - w\|^2 + \theta_i$$

is equivalent to

$$2\langle x_i - y_i, w\rangle \le \|x_i\|^2 - \|y_i\|^2 + \theta_i. \tag{3.1}$$

Take w_1 and w_2 in C_{i+1} and put $\bar{w} = tw_1 + (1-t)w_2$. It follows that $w_1 \in C_i$, $w_2 \in C_i$,

$$2\langle x_i - y_i, w_1\rangle \le \|x_i\|^2 - \|y_i\|^2 + \theta_i \tag{3.2}$$

and

$$2\langle x_i - y_i, w_2\rangle \le \|x_i\|^2 - \|y_i\|^2 + \theta_i. \tag{3.3}$$

Combining (3.2) with (3.3), we can obtain that $2\langle x_i - y_i, \bar{w}\rangle \le \|x_i\|^2 - \|y_i\|^2 + \theta_i$, that is, $\|y_i - w\|^2 \le \|x_i - \bar{x}\|^2 + \theta_i$. In view of the convexity of C_i, we see that $\bar{w} \in C_i$. This shows that $\bar{w} \in C_{i+1}$. This concludes that C_n is closed and convex for each $n \ge 1$. Notice that $(I - r_{n,m}A_m)$ is nonexpansive. Indeed, for any $x, y \in C$, we find from the restriction (b) that

$$\begin{aligned}
&\big\|(I - r_{n,m}A_m)x - (I - r_{n,m}A_m)y\big\|^2 \\
&\quad = \big\|(x-y) - r_{n,m}(A_mx - A_my)\big\|^2 \\
&\quad = \|x-y\|^2 - 2r_{n,m}\langle x-y, A_mx - A_my\rangle + r_{n,m}^2\|A_mx - A_my\|^2 \\
&\quad \le \|x-y\|^2 - 2r_{n,m}\xi_m\|A_mx - A_my\|^2 + r_{n,m}^2\|A_mx - A_my\|^2 \\
&\quad = \|x-y\|^2 + r_{n,m}(r_{n,m} - 2\xi_m)\|A_mx - A_my\|^2 \\
&\quad \le \|x-y\|^2.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|u_{n,m} - p\|^2 &= \big\|T_{r_{n,m}}(x_n - r_{n,m}A_mx_n) - T_{r_{n,m}}(p - r_{n,m}A_mp)\big\|^2 \\
&\le \big\|(x_n - r_{n,m}A_mx_n) - (p - r_{n,m}A_mp)\big\|^2 \\
&\le \|x_n - p\|^2.
\end{aligned} \tag{3.4}$$

Put $z_n = \sum_{m=1}^{N} \gamma_{n,m}u_{n,m}$ for each $n \ge 1$. Next, we show that $\mathcal{F} \subset C_n$ for all $n \ge 1$. It is easy to see that $\mathcal{F} \subset C_1 = H$. Suppose that $\mathcal{F} \subset C_h$ for some integer $h \ge 1$. We intend to claim that

$\mathcal{F} \subset C_{h+1}$ for the same h. For any $p \in \mathcal{F} \subset C_h$, we have from Lemma 2.2 and the restriction (a) that

$$\begin{aligned}
&\|y_h - p\|^2 \\
&\quad = \big\|\alpha_h x_h + (1-\alpha_h)\big(\beta_h z_h + (1-\beta_h)S^h z_h\big) - p\big\|^2 \\
&\quad \le \alpha_h \|x_h - p\|^2 + (1-\alpha_h)\big\|\beta_h(z_h - p) + (1-\beta_h)\big(S^h z_h - S^h p\big)\big\|^2 \\
&\quad = \alpha_h \|x_h - p\|^2 + (1-\alpha_h)\big(\beta_h\|z_h - p\|^2 + (1-\beta_h)\big\|S^h z_h - S^h p\big\|^2 \\
&\qquad - \beta_h(1-\beta_h)\big\|z_h - p - \big(S^h z_h - S^h p\big)\big\|^2\big) \\
&\quad \le \alpha_h \|x_h - p\|^2 + (1-\alpha_h)\beta_h\|z_h - p\|^2 + (1-\alpha_h)(1-\beta_h)\big(k_h\|z_h - p\|^2 \\
&\qquad + \kappa\big\|z_h - p - \big(S^h z_h - S^h p\big)\big\|^2\big) - (1-\alpha_h)\beta_h(1-\beta_h)\big\|z_h - p - \big(S^h z_h - S^h p\big)\big\|^2 \\
&\quad \le \alpha_h \|x_h - p\|^2 + (1-\alpha_h)k_h\|z_h - p\|^2 \\
&\qquad - (1-\alpha_h)(1-\beta_h)(\beta_h - \kappa)\big\|z_h - p - \big(S^h z_h - S^h p\big)\big\|^2 \\
&\quad \le \alpha_h \|x_h - p\|^2 + (1-\alpha_h)k_h\|z_h - p\|^2 \\
&\quad \le \alpha_h \|x_h - p\|^2 + (1-\alpha_h)k_h \sum_{m=1}^{N} \gamma_{h,m}\big\|T_{r_{h,m}}(x_n - r_{h,m}A_m x_h) - p\big\|^2 \\
&\quad \le \alpha_h \|x_h - p\|^2 + (1-\alpha_h)k_h\|x_h - p\|^2 \\
&\quad \le \|x_h - p\|^2 + \theta_h.
\end{aligned}$$

This shows that $p \in C_{h+1}$. This proves that $\mathcal{F} \subset C_n$ for all $n \ge 1$. Since $x_n = P_{C_n}x_1$ and $x_{n+1} = P_{C_{n+1}}x_1 \in C_{n+1} \subset C_n$, we have that

$$\begin{aligned}
0 &\le \langle x_1 - x_n, x_n - x_{n+1}\rangle \\
&= \langle x_1 - x_n, x_n - x_1 + x_1 - x_{n+1}\rangle \\
&\le -\|x_1 - x_n\|^2 + \|x_1 - x_n\|\|x_1 - x_{n+1}\|.
\end{aligned} \tag{3.5}$$

It follows that

$$\|x_1 - x_n\| \le \|x_1 - x_{n+1}\|. \tag{3.6}$$

On the other hand, for any $p \in \mathcal{F} \subset C_n$, we see that $\|x_1 - x_n\| \le \|x_1 - p\|$. In particular, we have

$$\|x_1 - x_n\| \le \|x_1 - P_{\mathcal{F}}x_1\|.$$

This shows that the sequence $\{x_n\}$ is bounded. In view of (3.6), we see that $\lim_{n\to\infty}\|x_n - x_1\|$ exists. It follows from (3.5) that

$$\begin{aligned}
&\|x_n - x_{n+1}\|^2 \\
&\quad = \|x_n - x_1 + x_1 - x_{n+1}\|^2
\end{aligned}$$

$$
\begin{aligned}
&= \|x_n - x_1\|^2 + 2\langle x_n - x_1, x_1 - x_{n+1}\rangle + \|x_1 - x_{n+1}\|^2 \\
&= \|x_n - x_1\|^2 + 2\langle x_n - x_1, x_1 - x_n + x_n - x_{n+1}\rangle + \|x_1 - x_{n+1}\|^2 \\
&= \|x_n - x_1\|^2 - 2\|x_n - x_1\|^2 + 2\langle x_n - x_1, x_n - x_{n+1}\rangle + \|x_1 - x_{n+1}\|^2 \\
&\le \|x_1 - x_{n+1}\|^2 - \|x_n - x_1\|^2,
\end{aligned}
$$

which implies that

$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0. \tag{3.7}$$

In view of $x_{n+1} = P_{C_{n+1}}x_1 \in C_{n+1}$, we find that

$$\|y_n - x_{n+1}\|^2 \le \|x_n - x_{n+1}\|^2 + \theta_n.$$

This combines with (3.7) yielding that

$$\lim_{n\to\infty} \|y_n - x_{n+1}\| = 0. \tag{3.8}$$

Notice that $\|y_n - x_n\| \le \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\|$. Combining (3.7) with (3.8), we find that

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{3.9}$$

Note that

$$
\begin{aligned}
\|x_n - y_n\| &= \big\|x_n - \alpha_n x_n - (1-\alpha_n)\big(\beta_n z_n + (1-\beta_n)S^n z_n\big)\big\| \\
&= (1-\alpha_n)\big\|x_n - \big(\beta_n z_n + (1-\beta_n)S^n z_n\big)\big\|.
\end{aligned}
$$

In view of the restriction (a), we obtain from (3.9) that

$$\lim_{n\to\infty} \big\|x_n - \big(\beta_n z_n + (1-\beta_n)S^n z_n\big)\big\| = 0. \tag{3.10}$$

Notice that

$$
\begin{aligned}
\|u_{n,m} - p\|^2 &= \big\|T_{r_{n,m}}(I - r_{n,m}A_m)x_n - T_{r_{n,m}}(I - r_{n,m}A_m)p\big\|^2 \\
&\le \big\langle (I - r_{n,m}A_m)x_n - (I - r_{n,m}A_m)p, u_{n,m} - p\big\rangle \\
&= \frac{1}{2}\Big(\big\|(I - r_{n,m}A_m)x_n - (I - r_{n,m}A_m)p\big\|^2 + \|u_{n,m} - p\|^2 \\
&\quad - \big\|(I - r_{n,m}A_m)x_n - (I - r_{n,m}A_m)p - (u_{n,m} - p)\big\|^2\Big) \\
&\le \frac{1}{2}\Big(\|x_n - p\|^2 + \|u_{n,m} - p\|^2 - \big\|(x_n - u_{n,m}) - r_{n,m}(A_m x_n - A_m p)\big\|^2\Big) \\
&= \frac{1}{2}\Big(\|x_n - p\|^2 + \|u_{n,m} - p\|^2 - \|x_n - u_{n,m}\|^2 \\
&\quad + 2r_{n,m}\langle x_n - u_{n,m}, A_m x_n - A_m p\rangle - r_{n,m}^2\|A_m x_n - A_m p\|^2\Big)
\end{aligned}
$$

and hence

$$\begin{aligned}
&\|u_{n,m}-p\|^2\\
&\quad\le \|x_n-p\|^2-\|x_n-u_{n,m}\|^2+2r_{n,m}\langle x_n-u_{n,m},A_mx_n-A_mp\rangle-r_{n,m}^2\|A_mx_n-A_mp\|^2\\
&\quad\le \|x_n-p\|^2-\|x_n-u_{n,m}\|^2+2r_{n,m}\|x_n-u_{n,m}\|\|A_mx_n-A_m\|. \qquad (3.11)
\end{aligned}$$

It follows from Lemma 2.2 and the restriction (a) that

$$\begin{aligned}
&\|y_n-p\|^2\\
&\quad=\bigl\|\alpha_nx_n+(1-\alpha_n)\bigl(\beta_nz_n+(1-\beta_n)S^nz_n\bigr)-p\bigr\|^2\\
&\quad\le \alpha_n\|x_n-p\|^2+(1-\alpha_n)\bigl\|\beta_n(z_n-p)+(1-\beta_n)\bigl(S^nz_n-S^np\bigr)\bigr\|^2\\
&\quad=\alpha_h\|x_n-p\|^2+(1-\alpha_n)\bigl(\beta_n\|z_n-p\|^2+(1-\beta_n)\bigl\|S^nz_n-S^np\bigr\|^2\\
&\qquad-\beta_n(1-\beta_n)\bigl\|z_n-p-\bigl(S^nz_n-S^np\bigr)\bigr\|^2\bigr)\\
&\quad\le \alpha_n\|x_n-p\|^2+(1-\alpha_n)\beta_n\|z_n-p\|^2+(1-\alpha_n)(1-\beta_n)\bigl(k_n\|z_n-p\|^2\\
&\qquad+\kappa\bigl\|z_n-p-\bigl(S^nz_n-S^np\bigr)\bigr\|^2\bigr)-(1-\alpha_n)\beta_n(1-\beta_n)\bigl\|z_n-p-\bigl(S^nz_n-S^np\bigr)\bigr\|^2\\
&\quad\le \alpha_n\|x_n-p\|^2+(1-\alpha_n)k_n\|z_n-p\|^2\\
&\qquad-(1-\alpha_n)(1-\beta_n)(\beta_n-\kappa)\bigl\|z_n-p-\bigl(S^nz_n-S^np\bigr)\bigr\|^2\\
&\quad\le \alpha_n\|x_n-p\|^2+(1-\alpha_n)k_n\|z_n-p\|^2. \qquad (3.12)
\end{aligned}$$

It follows that

$$\begin{aligned}
\|y_n-p\|^2&\le(1-\alpha_n)k_n\sum_{m=1}^N\gamma_{n,m}\bigl\|T_{r_{n,m}}(x_n-r_{n,m}A_mx_n)-T_{r_{n,m}}(p-r_{n,m}A_mp)\bigr\|^2\\
&\quad+\alpha_n\|x_n-p\|^2\\
&\le \alpha_n\|x_n-p\|^2+(1-\alpha_n)k_n\sum_{m=1}^N\gamma_{n,m}\bigl\|(x_n-r_{n,m}A_mx_n)-(p-r_{n,m}A_mp)\bigr\|^2\\
&\le \|x_n-p\|^2+(1-\alpha_n)(k_n-1)\|x_n-p\|^2\\
&\quad-r_{n,m}(2\xi_m-r_{n,m})(1-\alpha_n)k_n\sum_{m=1}^N\gamma_{n,m}\|A_mx_n-A_mp\|^2.
\end{aligned}$$

This implies that

$$\begin{aligned}
&r_{n,m}(2\xi_m-r_{n,m})(1-\alpha_n)k_n\gamma_{n,m}\|A_mx_n-A_mp\|^2\\
&\quad\le \|x_n-p\|^2-\|y_n-p\|^2+(1-\alpha_n)(k_n-1)\|x_n-p\|^2\\
&\quad\le \bigl(\|x_n-p\|+\|y_n-p\|\bigr)\|x_n-y_n\|+(1-\alpha_n)(k_n-1)\|x_n-p\|^2.
\end{aligned}$$

In view of the restrictions (a) and (b), we find from (3.9) that

$$\lim_{n\to\infty}\|A_mx_n-A_mp\|=0. \qquad (3.13)$$

On the other hand, we find from (3.11) and (3.12) that

$$\begin{aligned}\|y_n - p\|^2 &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)k_n\sum_{m=1}^{N}\gamma_{n,m}\|u_{n,m} - p\|^2\\ &\le \|x_n - p\|^2 + (1-\alpha_n)(k_n-1)\|x_n - p\|^2 - (1-\alpha_n)k_n\sum_{m=1}^{N}\gamma_{n,m}\|x_n - u_{n,m}\|^2\\ &\quad + 2r_{n,m}(1-\alpha_n)k_n\sum_{m=1}^{N}\gamma_{n,m}\|x_n - u_{n,m}\|\|A_m x_n - A_m p\|.\end{aligned}$$

This implies that

$$\begin{aligned}&(1-\alpha_n)k_n\gamma_{n,m}\|u_{n,m} - x_n\|^2\\ &\quad\le \|x_n - p\|^2 - \|y_n - p\|^2 + (1-\alpha_n)(k_n-1)\|x_n - p\|^2\\ &\qquad + 2r_{n,m}(1-\alpha_n)k_n\sum_{m=1}^{N}\gamma_{n,m}\|x_n - u_{n,m}\|\|A_m x_n - A_m p\|\\ &\quad\le \bigl(\|x_n - p\| + \|y_n - p\|\bigr)\|x_n - y_n\| + (1-\alpha_n)(k_n-1)\|x_n - p\|^2\\ &\qquad + 2r_{n,m}(1-\alpha_n)k_n\sum_{m=1}^{N}\gamma_{n,m}\|x_n - u_{n,m}\|\|A_m x_n - A_m p\|.\end{aligned}$$

In view of the restrictions (a) and (b), we obtain from (3.9) and (3.13) that

$$\lim_{n\to\infty}\|u_{n,m} - x_n\| = 0. \tag{3.14}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup q$. Next, we show that $q \in F(S)$. Note that

$$\|z_n - x_n\| = \left\|\sum_{m=1}^{N}\gamma_{n,m}u_{n,m} - x_n\right\| \le \sum_{m=1}^{N}\gamma_{n,m}\|u_{n,m} - x_n\|.$$

It follows from (3.14) that

$$\lim_{n\to\infty}\|z_n - x_n\| = 0. \tag{3.15}$$

On the other hand, we have

$$\begin{aligned}&\bigl\|x_n - \bigl(\beta_n x_n + (1-\beta_n)S^n x_n\bigr)\bigr\|\\ &\quad\le \bigl\|\bigl(\beta_n x_n + (1-\beta_n)S^n x_n\bigr) - \bigl(\beta_n z_n + (1-\beta_n)S^n z_n\bigr)\bigr\| + \bigl\|\bigl(\beta_n z_n + (1-\beta_n)S^n z_n\bigr) - x_n\bigr\|\\ &\quad\le \beta_n\|x_n - z_n\| + (1-\beta_n)\bigl\|S^n x_n - S^n z_n\bigr\| + \bigl\|\bigl(\beta_n z_n + (1-\beta_n)S^n z_n\bigr) - x_n\bigr\|\\ &\quad\le \beta_n\|x_n - z_n\| + (1-\beta_n)L\|x_n - z_n\| + \bigl\|\bigl(\beta_n z_n + (1-\beta_n)S^n z_n\bigr) - x_n\bigr\|\\ &\quad\le L\|x_n - z_n\| + \bigl\|\bigl(\beta_n z_n + (1-\beta_n)S^n z_n\bigr) - x_n\bigr\|.\end{aligned}$$

From (3.10) and (3.15), we find that

$$\lim_{n\to\infty}\left\|x_n-\left(\beta_n x_n+(1-\beta_n)S^n x_n\right)\right\|=0. \tag{3.16}$$

Note that

$$\begin{aligned}\left\|S^n x_n-x_n\right\| &\le \left\|S^n x_n-\left(\beta_n x_n+(1-\beta_n)S^n x_n\right)\right\|+\left\|\left(\beta_n x_n+(1-\beta_n)S^n x_n\right)-x_n\right\|\\ &\le \beta_n\left\|S^n x_n-x_n\right\|+\left\|\left(\beta_n x_n+(1-\beta_n)S^n x_n\right)-x_n\right\|,\end{aligned}$$

which yields that

$$(1-\beta_n)\left\|S^n x_n-x_n\right\|\le\left\|\left(\beta_n x_n+(1-\beta_n)S^n x_n\right)-x_n\right\|.$$

In view of the restriction (a), we find from (3.16) that

$$\lim_{n\to\infty}\left\|S^n x_n-x_n\right\|=0. \tag{3.17}$$

Note that

$$\begin{aligned}\|x_n-Sx_n\| &\le \|x_n-x_{n+1}\|+\left\|x_{n+1}-S^{n+1}x_{n+1}\right\|\\ &\quad+\left\|S^{n+1}x_{n+1}-S^{n+1}x_n\right\|+\left\|S^{n+1}x_n-Sx_n\right\|\\ &\le (1+L)\|x_n-x_{n+1}\|+\|x_{n+1}-S^{n+1}x_{n+1}\|+L\left\|S^n x_n-x_n\right\|.\end{aligned}$$

It follows from (3.7) and (3.17) that

$$\lim_{n\to\infty}\|Sx_n-x_n\|=0.$$

With the aid of Lemma 2.4, we find that $q\in F(S)$. Next, we prove $q\in\bigcap_{m=1}^N \mathrm{EP}(F_m)$. In view of (3.14), we find that $u_{n_i,m}\rightharpoonup q$ for each $1\le m\le N$. From (3.14) and the restriction (b), we see that

$$\lim_{n\to\infty}\frac{\|u_{n,m}-x_n\|}{r_{n,m}}=0,\quad \forall 1\le m\le N. \tag{3.18}$$

Notice that

$$F_m(u_{n,m},u)+\langle A_m x_n,u-u_{n,m}\rangle+\frac{1}{r_{n,m}}\langle u-u_{n,m},u_{n,m}-x_n\rangle\ge 0,\quad \forall u\in C.$$

From (A2), we see that

$$\langle A_m x_n,u-u_{n,m}\rangle+\frac{1}{r_{n,m}}\langle u-u_{n,m},u_{n,m}-x_n\rangle\ge F_m(u,u_{n,m}).$$

Replacing n by n_i, we arrive at

$$\langle A_m x_{n_i},u-u_{n_i,m}\rangle+\left\langle u-u_{n_i,m},\frac{u_{n_i,m}-x_{n_i}}{r_{n_i,m}}\right\rangle\ge F_m(u,u_{n_i,m}). \tag{3.19}$$

For any t with $0 < t \leq 1$ and $u \in C$, let $\rho_t = tu + (1-t)q$. Since $u \in C$ and $q \in C$, we have $\rho_t \in C$. It follows from (3.19) that

$$\begin{aligned}
&\langle \rho_t - u_{n_i,m}, A_m\rho_t\rangle \\
&\quad\geq \langle \rho_t - u_{n_i,m}, A_m\rho_t\rangle - \langle A_m x_{n_i}, \rho_t - u_{n_i,m}\rangle - \left\langle \rho_t - u_{n_i,m}, \frac{u_{n_i,m} - x_{n_i}}{r_{n_i,m}}\right\rangle + F_m(\rho_t, u_{n_i,m}) \\
&\quad= \langle \rho_t - u_{n_i,m}, A_m\rho_t - A_m u_{n,i,m}\rangle + \langle \rho_t - u_{n_i,m}, A_m u_{n,i,m} - A_m x_{n_i}\rangle \\
&\qquad - \left\langle \rho_t - u_{n_i,m}, \frac{u_{n_i,m} - x_{n_i}}{r_{n_i,m}}\right\rangle + F(\rho_t, u_{n_i,m}).
\end{aligned} \tag{3.20}$$

Since A_m is Lipschitz continuous, we find from (3.14) that $A_m u_{n,i,m} - A_m x_{n_i} \to 0$ as $i \to \infty$. From the monotonicity of A_m, we get that

$$\langle \rho_t - u_{n_i,m}, A_m\rho_t - A_m u_{n,i,m}\rangle \geq 0.$$

In view of (A4), we find from (3.20) that

$$\langle \rho_t - q, A_m\rho_t\rangle \geq F(\rho_t, q). \tag{3.21}$$

With the aid of (A1), (A4), we obtain from (3.21) that

$$\begin{aligned}
0 &= F(\rho_t, \rho_t) \leq tF(\rho_t, u) + (1-t)F(\rho_t, q) \\
&\leq tF(\rho_t, u) + (1-t)\langle \rho_t - q, A_m\rho_t\rangle \\
&= tF(\rho_t, u) + (1-t)t\langle u - q, A_m\rho_t\rangle,
\end{aligned}$$

which implies that

$$F(\rho_t, u) + (1-t)\langle u - q, A_m\rho_t\rangle \geq 0.$$

Letting $t \to 0$ in the above inequality, we arrive at

$$F(q, u) + \langle u - q, A_m q\rangle \geq 0.$$

This shows that $q \in \mathrm{EP}(F_m, A_m)$, $\forall 1 \leq m \leq N$. This completes the proof that $q \in \mathcal{F}$. Put $\bar{x} = P_{\mathcal{F}}x_1$, we obtain that

$$\begin{aligned}
\|x_1 - \bar{x}\| &\leq \|x_1 - q\| \leq \liminf_{i\to\infty} \|x_1 - x_{n_i}\| \\
&\leq \limsup_{i\to\infty} \|x_1 - x_{n_i}\| \leq \|x_1 - \bar{x}\|,
\end{aligned}$$

which yields that

$$\lim_{i\to\infty} \|x_1 - x_{n_i}\| = \|x_1 - q\| = \|x_1 - \bar{x}\|.$$

It follows that $\{x_{n_i}\}$ converges strongly to $\bar{x}$. Therefore, we can conclude that the sequence $\{x_n\}$ converges strongly to $\bar{x} = P_{\mathcal{F}}x_1$. This completes the proof. □

Based on Theorem 3.1, we have the following results.

Corollary 3.2 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies* (A1)-(A4) *and let $A : C \to H$ be a ξ-inverse-strongly monotone mapping. Let $S : C \to C$ be an asymptotically κ-strict pseudocontraction. Assume that $\mathcal{F} := F(S) \cap \mathrm{EP}(F, A)$ is nonempty and bounded. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0,1]$ and $\{r_n\}$ be a positive sequence. Let $\{x_n\}$ be a sequence generated in the following manner*:

$$\begin{cases} x_1 \in H, \\ C_1 = H, \\ F(u_n, u) + \langle Ax_n, u - u_n\rangle + \frac{1}{r_n}\langle u - u_n, u_n - x_n\rangle \geq 0, \quad \forall u \in C, \\ y_n = \alpha_n x_n + (1 - \alpha_n)(\beta_n u_n + (1 - \beta_n)S^n u_n), \\ C_{n+1} = \{w \in C_n : \|y_n - w\|^2 \leq \|x_n - w\|^2 + \theta_n\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad n \geq 0, \end{cases}$$

where $\theta_n = (1 - \alpha_n)(k_n - 1)\Theta_n$ and $\Theta_n = \sup\{\|x_n - w\|^2 : w \in \mathcal{F}\}$. Assume that the control sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{r_n\}$ satisfy the following restrictions:

(a) $0 \leq \alpha_n \leq a < 1$, $0 < \kappa \leq \beta_n \leq b < 1$;

(b) $0 < d \leq r_n \leq e < 2\xi$.

Then the sequence $\{x_n\}$ converges strongly to some point $\bar{x}$, where $\bar{x} = P_{\mathcal{F}} x_1$.

Proof Putting $F_m \equiv F$, $A_m \equiv A$ and $r_{n,m} \equiv r_n$ in Theorem 3.1, we see that $\sum_{m=1}^{N} \gamma_{n,m} u_{n,m} \equiv u_n$. With the aid of Theorem 3.1, we can easily conclude the desired conclusion. □

If S is asymptotically nonexpansive, then Corollary 3.2 is reduced to the following.

Corollary 3.3 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies* (A1)-(A4) *and let $A : C \to H$ be a ξ-inverse-strongly monotone mapping. Let $S : C \to C$ be an asymptotically nonexpansive mapping. Assume that $\mathcal{F} := F(S) \cap \mathrm{EP}(F, A)$ is nonempty and bounded. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0,1]$ and let $\{r_n\}$ be a positive sequence. Let $\{x_n\}$ be a sequence generated in the following manner*:

$$\begin{cases} x_1 \in H, \\ C_1 = H, \\ F(u_n, u) + \langle Ax_n, u - u_n\rangle + \frac{1}{r_n}\langle u - u_n, u_n - x_n\rangle \geq 0, \quad \forall u \in C, \\ y_n = \alpha_n x_n + (1 - \alpha_n)(\beta_n u_n + (1 - \beta_n)S^n u_n), \\ C_{n+1} = \{w \in C_n : \|y_n - w\|^2 \leq \|x_n - w\|^2 + \theta_n\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad n \geq 0, \end{cases}$$

where $\theta_n = (1 - \alpha_n)(k_n - 1)\Theta_n$ and $\Theta_n = \sup\{\|x_n - w\|^2 : w \in \mathcal{F}\}$. Assume that the control sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{r_n\}$ satisfy the following restrictions:

(a) $0 \leq \alpha_n \leq a < 1$, $0 \leq \beta_n \leq b < 1$;

(b) $0 < d \leq r_n \leq e < 2\xi$.

Then the sequence $\{x_n\}$ converges strongly to some point $\bar{x}$, where $\bar{x} = P_{\mathcal{F}} x_1$.

Putting $\beta_n \equiv 0$ in Corollary 3.3, we have the following.

Corollary 3.4 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies* (A1)-(A4) *and let $A : C \to H$ be a ξ-inverse-strongly monotone mapping. Let $S : C \to C$ be an asymptotically nonexpansive mapping. Assume that $\mathcal{F} := F(S) \cap \mathrm{EP}(F, A)$ is nonempty and bounded. Let $\{\alpha_n\}$ be a sequence in* $[0,1]$ *and $\{r_n\}$ be a positive sequence. Let $\{x_n\}$ be a sequence generated in the following manner*:

$$\begin{cases} x_1 \in H, \\ C_1 = H, \\ F(u_n, u) + \langle Ax_n, u - u_n\rangle + \frac{1}{r_n}\langle u - u_n, u_n - x_n\rangle \ge 0, \quad \forall u \in C, \\ y_n = \alpha_n x_n + (1 - \alpha_n)S^n u_n, \\ C_{n+1} = \{w \in C_n : \|y_n - w\|^2 \le \|x_n - w\|^2 + \theta_n\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad n \ge 0, \end{cases}$$

where $\theta_n = (1 - \alpha_n)(k_n - 1)\Theta_n$ and $\Theta_n = \sup\{\|x_n - w\|^2 : w \in \mathcal{F}\}$. Assume that the control sequences $\{\alpha_n\}$ and $\{r_n\}$ satisfy the following restrictions:

(a) $0 \le \alpha_n \le a < 1$;

(b) $0 < d \le r_n \le e < 2\xi$.

Then the sequence $\{x_n\}$ converges strongly to some point $\bar{x}$, where $\bar{x} = P_{\mathcal{F}} x_1$.

Corollary 3.5 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A_m : C \to H$ be a ξ_m-inverse-strongly monotone mapping, for each $1 \le m \le N$, where $N \ge 1$ is some positive integer. Let $S : C \to C$ be an asymptotically κ-strict pseudocontraction. Assume that $\mathcal{F} := F(S) \cap \bigcap_{m=1}^N \mathrm{VI}(C, A_m)$ is nonempty and bounded. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in* $[0,1]$ *and $\{r_{n,m}\}$ be a positive sequence. Let $\{\gamma_{n,m}\}$ be a sequence in* $[0,1]$ *for each $1 \le m \le N$ such that $\sum_{m=1}^N \gamma_{n,m} = 1$. Let $\{x_n\}$ be a sequence generated in the following manner*:

$$\begin{cases} x_1 \in H, \\ C_1 = H, \\ u_{n,m} = P_C(x_n - r_{n,m}A_m x_n), \\ y_n = \alpha_n x_n + (1 - \alpha_n)(\beta_n \sum_{m=1}^N \gamma_{n,m}u_{n,m} + (1 - \beta_n)S^n \sum_{m=1}^N \gamma_{n,m}u_{n,m}), \\ C_{n+1} = \{w \in C_n : \|y_n - w\|^2 \le \|x_n - w\|^2 + \theta_n\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad n \ge 0, \end{cases}$$

where $\theta_n = (1 - \alpha_n)(k_n - 1)\Theta_n$ and $\Theta_n = \sup\{\|x_n - w\|^2 : w \in \mathcal{F}\}$. Assume that the control sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_{n,1}\}, \dots, \{\gamma_{n,N}\}$, $\{r_{n,1}\}, \dots$ and $\{r_{n,N}\}$ satisfy the following restrictions:

(a) $0 \le \alpha_n \le a < 1$, $\kappa \le \beta_n \le b < 1$;

(b) $0 < c \le \gamma_{n,m} \le 1$ *and* $0 < d \le r_{n,m} \le e < 2\xi_m$ *for each* $1 \le m \le N$.

Then the sequence $\{x_n\}$ converges strongly to some point $\bar{x}$, where $\bar{x} = P_{\mathcal{F}} x_1$.

Proof In Theorem 3.1, put $F_m(x, y) = 0$ for all $x, y \in C$. From

$$\langle A_m x_n, u - u_{n,m}\rangle + \frac{1}{r_{n,m}}\langle u - u_{n,m}, u_{n,m} - x_n\rangle,$$

we have

$$\langle u - u_{n,m}, x_n - u_{n,m} - r_{n,m}A_m x_n\rangle \geq 0, \quad \forall u \in C.$$

This implies that

$$u_{n,m} = P_C(x_n - r_{n,m}A_m x_n).$$

In view of Theorem 3.1, we can immediately obtain the desired conclusion. This completes the proof. □

Corollary 3.6 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F_m be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies* (A1)-(A4) *for each $1 \leq m \leq N$, where $N \geq 1$ is some positive integer. Let $S : C \to C$ be an asymptotically κ-strict pseudocontraction. Assume that $\mathcal{F} := F(S) \cap \bigcap_{m=1}^{N} \mathrm{EP}(F_m)$ is nonempty and bounded. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0,1]$ and $\{r_{n,m}\}$ be a positive sequence. Let $\{\gamma_{n,m}\}$ be a sequence in $[0,1]$ for each $1 \leq m \leq N$ such that $\sum_{m=1}^{N} \gamma_{n,m} = 1$. Let $\{x_n\}$ be a sequence generated in the following manner*:

$$\begin{cases} x_1 \in H, \\ C_1 = H, \\ F_m(u_{n,m}, u) + \frac{1}{r_{n,m}}\langle u - u_{n,m}, u_{n,m} - x_n\rangle \geq 0, \quad \forall u \in C, \\ y_n = \alpha_n x_n + (1-\alpha_n)(\beta_n \sum_{m=1}^{N} \gamma_{n,m}u_{n,m} + (1-\beta_n)S^n \sum_{m=1}^{N} \gamma_{n,m}u_{n,m}), \\ C_{n+1} = \{w \in C_n : \|y_n - w\|^2 \leq \|x_n - w\|^2 + \theta_n\}, \\ x_{n+1} = P_{C_{n+1}}x_1, \quad n \geq 0, \end{cases}$$

where $\theta_n = (1-\alpha_n)(k_n - 1)\Theta_n$ and $\Theta_n = \sup\{\|x_n - w\|^2 : w \in \mathcal{F}\}$. Assume that the control sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_{n,1}\}, \ldots, \{\gamma_{n,N}\}, \{r_{n,1}\}, \ldots$ and $\{r_{n,N}\}$ satisfy the following restrictions:

(a) $0 \leq \alpha_n \leq a < 1$, $\kappa \leq \beta_n \leq b < 1$;

(b) $0 < c \leq \gamma_{n,m} \leq 1$ *and* $0 < d \leq r_{n,m} \leq e < \infty$ *for each* $1 \leq m \leq N$.

Then the sequence $\{x_n\}$ converges strongly to some point $\bar{x}$, where $\bar{x} = P_{\mathcal{F}}x_1$.

Proof In Theorem 3.1, put $A_m = 0$. Then, for any $\xi_m > 0$, we see that

$$\langle x - y, A_m x - A_m y\rangle \geq \xi_m \|A_m x - A_m y\|^2, \quad \forall x, y \in C.$$

Let $\{r_{n,m}\}$ be a sequence satisfying the restriction $d \leq r_n \leq e$, where $d, e \in (0, \infty)$. Then we can obtain the desired conclusion easily from Theorem 3.1. This completes the proof. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors contributed equally and significantly in writing this paper. Both authors read and approved the final manuscript.

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