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Viscosity iterative method for a new general system of variational inequalities in Banach spaces

Suwicha Imnang*

*Correspondence: suwicha.n@hotmail.com
Department of Mathematics and Statistics, Faculty of Science, Thaksin University, Phatthalung Campus, Phatthalung, 93110, Thailand
Centre of Excellence in Mathematics, CHE, Si Ayutthaya Road, Bangkok, 10400, Thailand

Abstract
In this paper, we study a new iterative method for finding a common element of the set of solutions of a new general system of variational inequalities for two different relaxed cocoercive mappings and the set of fixed points of a nonexpansive mapping in real 2-uniformly smooth and uniformly convex Banach spaces. We prove the strong convergence of the proposed iterative method without the condition of weakly sequentially continuous duality mapping. Our result improves and extends the corresponding results announced by many others.
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1 Introduction

Let X be a real Banach space and X^* be its dual space. Let C be a subset of X and let T be a self-mapping of C. We use $F(T)$ to denote the set of fixed points of T. The duality mapping $J : X \to 2^{X^*}$ is defined by $J(x) = \{x^* \in X^* | \langle x, x^*\rangle = \|x\|^2, \|x^*\| = \|x\|\}$, $\forall x \in X$. If X is a Hilbert space, then $J = I$, where I is the identity mapping. It is well-known that if X is smooth, then J is single-valued, which is denoted by j.

Recall that a mapping $f : C \to C$ is a *contraction* on C, if there exists a constant $\alpha \in (0, 1)$ such that $\|f(x) - f(y)\| \le \alpha\|x - y\|$, $\forall x, y \in C$. We use Π_C to denote the collection of all contractions on C. This is $\Pi_C = \{f | f : C \to C \text{ a contraction}\}$. A mapping $T : C \to C$ is said to be *nonexpansive* if $\|T(x) - T(y)\| \le \|x - y\|$, $\forall x, y \in C$. Let $A : C \to X$ be a nonlinear mapping. Then A is called

(i) *L-Lipschitz continuous* (or *Lipschitzian*) if there exists a constant $L \ge 0$ such that

$$\|Ax - Ay\| \le L\|x - y\|, \quad \forall x, y \in C;$$

(ii) *accretive* if there exists $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y)\rangle \ge 0, \quad \forall x, y \in C;$$

(iii) *α-inverse strongly accretive* if there exist $j(x - y) \in J(x - y)$ and $\alpha > 0$ such that

$$\langle Ax - Ay, j(x - y)\rangle \ge \alpha\|Ax - Ay\|^2, \quad \forall x, y \in C;$$

(iv) *relaxed* (c,d)*-cocoercive* if there exist $j(x-y) \in J(x-y)$ and two constants $c,d \geq 0$ such that

$$\langle Ax - Ay, j(x-y)\rangle \geq (-c)\|Ax - Ay\|^2 + d\|x-y\|^2, \quad \forall x,y \in C.$$

Let C be a nonempty closed convex subset of a real Hilbert space H. Recall that the classical variational inequality is to find $x^* \in C$ such that

$$\langle Ax^*, x - x^*\rangle \geq 0, \quad \forall x \in C, \tag{1.1}$$

where $A : C \to H$ is a nonlinear mapping. Variational inequality theory has emerged as an important tool in studying a wide class of obstacle, unilateral, free, moving, equilibrium problems arising in several branches of pure and applied sciences in a unified and general framework. The variational inequality problem has been extensively studied in the literature; see [1–8] and the references cited therein.

In 2006, Aoyama *et al.* [9] first considered the following generalized variational inequality problem in Banach spaces. Let $A : C \to X$ be an accretive operator. Find a point $x^* \in C$ such that

$$\langle Ax^*, j(x - x^*)\rangle \geq 0, \quad \forall x \in C. \tag{1.2}$$

Problem (1.2) is very interesting as it is connected with the fixed point problem for a nonlinear mapping and the problem of finding a zero point of an accretive operator in Banach spaces; see [10–13] and the references cited therein.

In 2010, Yao *et al.* [14] introduced the following system of general variational inequalities in Banach spaces. For given two operators $A,B : C \to X$, they considered the problem of finding $(x^*,y^*) \in C \times C$ such that

$$\begin{cases} \langle Ay^* + x^* - y^*, j(x - x^*)\rangle \geq 0, & \forall x \in C, \\ \langle Bx^* + y^* - x^*, j(x - y^*)\rangle \geq 0, & \forall x \in C, \end{cases} \tag{1.3}$$

which is called *the system of general variational inequalities in a real Banach space* and the set of solutions of problem (1.3) denoted by Ω_1. Yao *et al.* proved the following strong convergence theorem.

Theorem YNNLY *Let C be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space X which admits a weakly sequentially continuous duality mapping. Let Q_C be the sunny nonexpansive retraction from X onto C. Let the mappings $A,B : C \to X$ be α-inverse-strongly accretive with $\alpha \geq K^2$ and β-inverse-strongly accretive with $\beta \geq K^2$, respectively, with $\Omega_1 \neq \emptyset$. For a given $x_0 \in C$, let the sequence $\{x_n\}$ be generated iteratively by*

$$\begin{cases} y_n = Q_C(x_n - Bx_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n Q_C(y_n - Ay_n), \quad n \geq 0, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are three sequences in $(0,1)$. Suppose that the sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ satisfy the following conditions:

(i) $\alpha_n + \beta_n + \gamma_n = 1$, $\forall n \geq 0$;
(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n \leq 1$.
Then $\{x_n\}$ *converges strongly to* $Q'u$ *where* Q' *is the sunny nonexpansive retraction of* C *onto* Ω_1.

In 2011, Katchang and Kumam [15] introduced the following system of general variational inequalities in Banach spaces. For given two operators $A, B : C \to X$, they considered the problem of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \lambda Ay^* + x^* - y^*, j(x - x^*)\rangle \geq 0, & \forall x \in C, \\ \langle \mu Bx^* + y^* - x^*, j(x - y^*)\rangle \geq 0, & \forall x \in C, \end{cases} \tag{1.4}$$

which is called *the system of general variational inequalities* in a real Banach space and the set of solutions of problem (1.4) denoted by Ω_2. Katchang and Kumam proved the following strong convergence theorem.

Theorem KK *Let* C *be a nonempty closed convex subset of a uniformly convex and* 2*-uniformly smooth Banach space* X *which admits a weakly sequentially continuous duality mapping. Let* $S : C \to C$ *be a nonexpansive mapping and* Q_C *be a sunny nonexpansive retraction from* X *onto* C. *Let the mappings* $A, B : C \to X$ *be* β*-inverse-strongly accretive with* $\beta \geq \lambda K^2$ *and* γ*-inverse-strongly accretive with* $\gamma \geq \mu K^2$, *respectively, and let* K *be the* 2*-uniformly smooth constant of* X. *Let* f *be a contraction of* C *into itself with coefficient* $\alpha \in [0, 1)$. *Suppose that* $F := \Omega_2 \cap F(S) \neq \emptyset$. *For a given* $x_0 = x \in C$, *let the sequence* $\{x_n\}$ *be generated iteratively by*

$$\begin{cases} y_n = Q_C(x_n - \mu Bx_n), \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n SQ_C(y_n - \lambda Ay_n), \quad n \geq 0, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{\gamma_n\}$ *are three sequences in* $(0, 1)$. *Suppose that the sequences* $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{\gamma_n\}$ *satisfy the following conditions*:
(i) $\alpha_n + \beta_n + \gamma_n = 1$, $\forall n \geq 0$;
(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.
Then $\{x_n\}$ *converges strongly to* $\bar{x} = Q_F f(\bar{x})$ *and* $(\bar{x}, \bar{y})$ *is a solution of problem* (1.4), *where* $\bar{y} = Q_C(\bar{x} - \mu B\bar{x})$ *and* Q_F *is the sunny nonexpansive retraction of* C *onto* F.

The problem of finding solutions of (1.4) by using iterative methods has been studied by many others; see [16–19] and the references cited therein.

In this paper, we focus on the problem of finding $(x^*, y^*, z^*) \in C \times C \times C$ such that

$$\begin{cases} \langle \lambda_1 A_1 y^* + x^* - y^*, j(x - x^*)\rangle \geq 0, & \forall x \in C, \\ \langle \lambda_2 A_2 z^* + y^* - z^*, j(x - y^*)\rangle \geq 0, & \forall x \in C, \\ \langle \lambda_3 A_3 x^* + z^* - x^*, j(x - z^*)\rangle \geq 0, & \forall x \in C, \end{cases} \tag{1.5}$$

which is called *a new general system of variational inequalities in Banach spaces*, where $A_i : C \to X$ are three mappings, $\lambda_i > 0$ for all $i = 1, 2, 3$. In particular, if $A_3 = 0$ and $z^* = x^*$, then problem (1.5) reduces to problem (1.4). If we add up the requirement that $\lambda_i = 1$ for $i = 1,2$, then problem (1.5) reduces to problem (1.3).

In this paper, motivated and inspired by the idea of Katchang and Kumam [15] and Yao *et al.* [14], we introduce a new iterative method for finding a common element of the set of solutions of a new general system of variational inequalities in Banach spaces for two different relaxed cocoercive mappings and the set of fixed points of a nonexpansive mapping in real 2-uniformly smooth and uniformly convex Banach spaces. We prove the strong convergence of the proposed iterative algorithm without the condition of weakly sequentially continuous duality mapping. Our result improves and extends the corresponding results announced by many others.

2 Preliminaries

In this section, we recall the well-known results and give some useful lemmas that are used in the next section.

Let X be a Banach space and let $U = \{x \in X : \|x\| = 1\}$ be a unit sphere of X. X is said to be *uniformly convex* if for each $\epsilon \in (0,2]$, there exists a constant $\delta > 0$ such that for any $x, y \in U$,

$$\|x - y\| \geq \epsilon \quad \text{implies} \quad \left\| \frac{x+y}{2} \right\| \leq 1 - \delta.$$

The norm on X is said to be *Gâteaux differentiable* if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists for each $x, y \in U$ and in this case X is said to be *smooth*. X is said to have a *uniformly Frechet differentiable norm* if the limit (2.1) is attained uniformly for $x, y \in U$ and in this case X is said to be *uniformly smooth*. We define a function $\rho : [0,\infty) \to [0,\infty)$, called the *modulus of smoothness* of X, as follows:

$$\rho(\tau) = \sup\left\{ \frac{1}{2}\big(\|x + y\| + \|x - y\|\big) - 1 : x, y \in X, \|x\| = 1, \|y\| = \tau \right\}.$$

It is known that X is uniformly smooth if and only if $\lim_{\tau \to 0} \rho(\tau)/\tau = 0$. Let q be a fixed real number with $1 < q \leq 2$. Then a Banach space X is said to be *q-uniformly smooth* if there exists a constant $c > 0$ such that $\rho(\tau) \leq c\tau^q$ for all $\tau > 0$. For $q > 1$, the generalized duality mapping $J_q : X \to 2^{X^*}$ is defined by

$$J_q(x) = \big\{ f \in X^* : \langle x, f \rangle = \|x\|^q, \|f\| = \|x\|^{q-1} \big\}, \quad \forall x \in X.$$

In particular, if $q = 2$, the mapping J_2 is called the *normalized duality mapping* (or *duality mapping*), and usually we write $J_2 = J$. If X is a Hilbert space, then $J = I$. Further, we have the following properties of the generalized duality mapping J_q.

(1) $J_q(x) = \|x\|^{q-2} J_2(x)$ for all $x \in X$ with $x \neq 0$.
(2) $J_q(tx) = t^{q-1} J_q(x)$ for all $x \in X$ and $t \in [0,\infty)$.
(3) $J_q(-x) = -J_q(x)$ for all $x \in X$.

It is known that if X is smooth, then J is a single-valued function, which is denoted by j. Recall that the duality mapping j is said to be *weakly sequentially continuous* if for each $\{x_n\} \subset X$ with $x_n \to x$, we have $j(x_n) \to j(x)$ weakly-$*$. We know that if X admits a weakly sequentially continuous duality mapping, then X is smooth. For details, see [20].

Lemma 2.1 [21] *Let X be a q-uniformly smooth Banach space with $1 \le q \le 2$. Then*

$$\|x+y\|^q \le \|x\|^q + q\langle y, J_q(x)\rangle + 2\|Ky\|^q$$

for all $x, y \in X$, where K is the q-uniformly smooth constant of X.

Lemma 2.2 [22] *In a Banach space X, the following inequality holds*:

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, j(x+y)\rangle, \quad \forall x, y \in X,$$

where $j(x+y) \in J(x+y)$.

Lemma 2.3 [23] *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1-\gamma_n)a_n + \delta_n, \quad n \ge 1,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that
(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;
(ii) $\limsup_{n\to\infty} \delta_n/\gamma_n \le 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.
Then $\lim_{n\to\infty} a_n = 0$.

Let C be a nonempty closed convex subset of a smooth Banach space X and let D be a nonempty subset of C. A mapping $Q : C \to D$ is said to be *sunny* if

$$Q\bigl(Qx + t(x - Qx)\bigr) = Qx,$$

whenever $Qx + t(x - Qx) \in C$ for $x \in C$ and $t \ge 0$. A mapping $Q : C \to D$ is called a *retraction* if $Qx = x$ for all $x \in D$. Furthermore, Q is a *sunny nonexpansive retraction* from C onto D if Q is a retraction from C onto D, which is also sunny and nonexpansive. A subset D of C is called a sunny nonexpansive retraction of C if there exists a sunny nonexpansive retraction from C onto D.

It is well known that if X is a Hilbert space, then a sunny nonexpansive retraction Q_C is coincident with the metric projection from X onto C.

Lemma 2.4 [24] *Let C be a closed convex subset of a smooth Banach space X. Let D be a nonempty subset of C and $Q : C \to D$ be a retraction. Then the following are equivalent*:
(a) *Q is sunny and nonexpansive.*
(b) $\|Qx - Qy\|^2 \le \langle x - y, j(Qx - Qy)\rangle$, $\forall x, y \in C$.
(c) $\langle x - Qx, j(y - Qx)\rangle \le 0$, $\forall x \in C$, $y \in D$.

Lemma 2.5 [25] *If X is strictly convex and uniformly smooth and if $T : C \to C$ is a nonexpansive mapping having a nonempty fixed point set $F(T)$, then the set $F(T)$ is a sunny nonexpansive retraction of C.*

Lemma 2.6 [26] *Let* $\{x_n\}$ *and* $\{y_n\}$ *be bounded sequences in a Banach space* X *and let* $\{b_n\}$ *be a sequence in* $[0,1]$ *with* $0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty} b_n < 1$. *Suppose that* $x_{n+1} = (1-b_n)y_n + b_n x_n$ *for all integers* $n \geq 1$ *and* $\limsup_{n\to\infty}(\|y_{n+1}-y_n\| - \|x_{n+1}-x_n\|) \leq 0$. *Then* $\lim_{n\to\infty}\|y_n - x_n\| = 0$.

Lemma 2.7 [27] *Let* C *be a closed convex subset of a strictly convex Banach space* X. *Let* T_1 *and* T_2 *be two nonexpansive mappings from* C *into itself with* $F(T_1)\cap F(T_2) \neq \emptyset$. *Define a mapping* S *by*

$$Sx = \lambda T_1 x + (1-\lambda)T_2 x, \quad \forall x \in C,$$

where λ *is a constant in* $(0,1)$. *Then* S *is nonexpansive and* $F(S) = F(T_1)\cap F(T_2)$.

Lemma 2.8 [28] *Let* X *be a real smooth and uniformly convex Banach space and let* $r > 0$. *Then there exists a strictly increasing, continuous and convex function* $g : [0,2r] \to \mathbb{R}$ *such that* $g(0) = 0$ *and* $g(\|x-y\|) \leq \|x\|^2 - 2\langle x, j(y)\rangle + \|y\|^2$ *for all* $x, y \in B_r$.

Lemma 2.9 [23] *Let* X *be a uniformly smooth Banach space, let* C *be a closed convex subset of* X, *let* $T : C \to C$ *be a nonexpansive mapping with* $F(T) \neq \emptyset$ *and let* $f \in \Pi_C$. *Then the sequence* $\{x_t\}$ *defined by* $x_t = tf(x_t) + (1-t)Tx_t$ *converges strongly to a point in* $F(T)$ *as* $t \to 0$. *If we define a mapping* $Q : \Pi_C \to F(T)$ *by* $Q(f) := \lim_{t\to 0} x_t$, $\forall f \in \Pi_C$, *then* $Q(f)$ *solves the following variational inequality*:

$$\big\langle (I-f)Q(f), j\big(Q(f)-p\big)\big\rangle \leq 0, \quad \forall f \in \Pi_C, p \in F(T).$$

Lemma 2.10 [17] *Let* C *be a nonempty closed convex subset of a real* 2-*uniformly smooth Banach space* X. *Let the mapping* $A : C \to X$ *be relaxed* (c,d)-*cocoercive and* L_A-*Lipschitzian. Then we have*

$$\big\|(I-\lambda A)x - (I-\lambda A)y\big\|^2 \leq \|x-y\|^2 + 2\big(\lambda c L_A^2 - \lambda d + K^2\lambda^2 L_A^2\big)\|x-y\|^2,$$

where $\lambda > 0$ *and* K *is the* 2-*uniformly smooth constant of* X. *In particular, if* $0 < \lambda \leq \frac{d-cL_A^2}{K^2 L_A^2}$, *then* $I - \lambda A$ *is a nonexpansive mapping.*

In order to prove our main result, the next lemma is crucial for proving the main theorem.

Lemma 2.11 *Let* C *be a nonempty closed convex subset of a real* 2-*uniformly smooth Banach space* X *with the* 2-*uniformly smooth constant* K. *Let* Q_C *be the sunny nonexpansive retraction from* X *onto* C *and let* $A_i : C \to X$ *be a relaxed* (c_i, d_i)-*cocoercive and* L_i-*Lipschitzian mapping for* $i = 1,2,3$. *Let* $G : C \to C$ *be a mapping defined by*

$$\begin{aligned} G(x) = Q_C\big[&Q_C\big(Q_C(x-\lambda_3A_3x) - \lambda_2A_2Q_C(x-\lambda_3A_3x)\big) \\ &- \lambda_1A_1Q_C\big(Q_C(x-\lambda_3A_3x) - \lambda_2A_2Q_C(x-\lambda_3A_3x)\big)\big], \quad \forall x \in C. \end{aligned}$$

If $0 < \lambda_i \leq \frac{d_i - c_iL_i{}^2}{K^2 L_i{}^2}$ *for all* $i = 1,2,3$, *then* $G : C \to C$ *is nonexpansive.*

Proof For all $x, y \in C$, by Lemma 2.10, we have

$$\begin{aligned}
\|G(x) - G(y)\| &= \big\|Q_C\big[Q_C\big(Q_C(I - \lambda_3 A_3)x - \lambda_2 A_2 Q_C(I - \lambda_3 A_3)x\big) \\
&\quad - \lambda_1 A_1 Q_C\big(Q_C(I - \lambda_3 A_3)x - \lambda_2 A_2 Q_C(I - \lambda_3 A_3)x\big)\big] \\
&\quad - Q_C\big[Q_C\big(Q_C(I - \lambda_3 A_3)y - \lambda_2 A_2 Q_C(I - \lambda_3 A_3)y\big) \\
&\quad - \lambda_1 A_1 Q_C\big(Q_C(I - \lambda_3 A_3)y - \lambda_2 A_2 Q_C(I - \lambda_3 A_3)y\big)\big]\big\| \\
&\le \big\|Q_C\big(Q_C(I - \lambda_3 A_3)x - \lambda_2 A_2 Q_C(I - \lambda_3 A_3)x\big) \\
&\quad - \lambda_1 A_1 Q_C\big(Q_C(I - \lambda_3 A_3)x - \lambda_2 A_2 Q_C(I - \lambda_3 A_3)x\big) \\
&\quad - \big[Q_C\big(Q_C(I - \lambda_3 A_3)y - \lambda_2 A_2 Q_C(I - \lambda_3 A_3)y\big) \\
&\quad - \lambda_1 A_1 Q_C\big(Q_C(I - \lambda_3 A_3)y - \lambda_2 A_2 Q_C(I - \lambda_3 A_3)y\big)\big]\big\| \\
&= \big\|(I - \lambda_1 A_1)Q_C(I - \lambda_2 A_2)Q_C(I - \lambda_3 A_3)x \\
&\quad - (I - \lambda_1 A_1)Q_C(I - \lambda_2 A_2)Q_C(I - \lambda_3 A_3)y\big\| \\
&\le \|x - y\|,
\end{aligned}$$

which implies that G is nonexpansive. □

Lemma 2.12 [29] *Let C be a nonempty closed convex subset of a real smooth Banach space X. Let Q_C be the sunny nonexpansive retraction from X onto C. Let $A_i : C \to X$ be three possibly nonlinear mappings. For given $x^*, y^*, z^* \in C$, (x^*, y^*, z^*) is a solution of problem* (1.5) *if and only if $x^* \in F(G)$, $y^* = Q_C(z^* - \lambda_2 A_2 z^*)$ and $z^* = Q_C(x^* - \lambda_3 A_3 x^*)$, where G is the mapping defined as in Lemma* 2.11.

3 Main results

We are now in a position to state and prove our main result.

Theorem 3.1 *Let X be a uniformly convex and 2-uniformly smooth Banach space with the 2-uniformly smooth constant K, let C be a nonempty closed convex subset of X and Q_C be a sunny nonexpansive retraction from X onto C. Let the mappings $A_i : C \to X$ be relaxed (c_i, d_i)-cocoercive and L_i-Lipschitzian with $0 < \lambda_i < \frac{d_i - c_i L_i^2}{K^2 L_i^2}$ for all $i = 1, 2, 3$. Let f be a contractive mapping with the constant $\alpha \in (0, 1)$ and let $S : C \to C$ be a nonexpansive mapping such that $\Omega = F(S) \cap F(G) \neq \emptyset$, where G is the mapping defined as in Lemma* 2.11. *For a given $x_1 \in C$, let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be the sequences generated by*

$$\begin{cases}
z_n = Q_C(x_n - \lambda_3 A_3 x_n), \\
y_n = Q_C(z_n - \lambda_2 A_2 z_n), \\
x_{n+1} = a_n f(x_n) + b_n x_n + (1 - a_n - b_n) S Q_C(y_n - \lambda_1 A_1 y_n), \quad n \ge 1,
\end{cases} \tag{3.1}$$

where $\{a_n\}$ and $\{b_n\}$ are two sequences in $(0, 1)$ such that

(C1) $\lim_{n\to\infty} a_n = 0$ *and* $\sum_{n=1}^{\infty} a_n = \infty$;

(C2) $0 < \liminf_{n\to\infty} b_n \le \limsup_{n\to\infty} b_n < 1$.

Then $\{x_n\}$ converges strongly to $q \in \Omega$, which solves the following variational inequality:

$$\langle q - f(q), j(q - p)\rangle \le 0, \quad \forall f \in \Pi_C, p \in \Omega.$$

Proof Step 1. We show that $\{x_n\}$ is bounded.

Let $x^* \in \Omega$ and $t_n = Q_C(y_n - \lambda_1 A_1 y_n)$. It follows from Lemma 2.12 that

$$\begin{aligned} x^* = Q_C\big[&Q_C\big(Q_C\big(x^* - \lambda_3 A_3 x^*\big) - \lambda_2 A_2 Q_C\big(x^* - \lambda_3 A_3 x^*\big)\big) \\ &- \lambda_1 A_1 Q_C\big(Q_C\big(x^* - \lambda_3 A_3 x^*\big) - \lambda_2 A_2 Q_C\big(x^* - \lambda_3 A_3 x^*\big)\big)\big]. \end{aligned}$$

Put $y^* = Q_C(z^* - \lambda_2 A_2 z^*)$ and $z^* = Q_C(x^* - \lambda_3 A_3 x^*)$. Then $x^* = Q_C(y^* - \lambda_1 A_1 y^*)$ and

$$x_{n+1} = a_n f(x_n) + b_n x_n + (1 - a_n - b_n) S t_n.$$

From Lemma 2.10, we have $I - \lambda_i A_i$ $(i = 1, 2, 3)$ is nonexpansive. Therefore

$$\begin{aligned} \|t_n - x^*\| &= \|Q_C(y_n - \lambda_1 A_1 y_n) - Q_C\big(y^* - \lambda_1 A_1 y^*\big)\| \le \|y_n - y^*\| \\ &= \|Q_C(z_n - \lambda_2 A_2 z_n) - Q_C\big(z^* - \lambda_2 A_2 z^*\big)\| \le \|z_n - z^*\| \\ &= \|Q_C(x_n - \lambda_3 A_3 x_n) - Q_C\big(x^* - \lambda_3 A_3 x^*\big)\| \le \|x_n - x^*\| \end{aligned} \tag{3.2}$$

and $\|St_n - x^*\| \le \|t_n - x^*\|$. It follows from (3.2) that

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|a_n f(x_n) + b_n x_n + (1 - a_n - b_n) S t_n - x^*\| \\ &\le a_n \|f(x_n) - x^*\| + b_n \|x_n - x^*\| + (1 - a_n - b_n)\|t_n - x^*\| \\ &\le a_n \|f(x_n) - x^*\| + (1 - a_n)\|x_n - x^*\| \\ &\le a_n \alpha \|x_n - x^*\| + a_n \|f\big(x^*\big) - x^*\| + (1 - a_n)\|x_n - x^*\| \\ &= a_n \|f\big(x^*\big) - x^*\| + \big(1 - a_n(1 - \alpha)\big)\|x_n - x^*\|. \end{aligned}$$

By induction, we have

$$\|x_{n+1} - x^*\| \le \max\left\{\frac{\|f(x^*) - x^*\|}{1 - \alpha}, \|x_1 - x^*\|\right\}.$$

Therefore, $\{x_n\}$ is bounded. Hence $\{y_n\}$, $\{z_n\}$, $\{t_n\}$, $\{A_1 y_n\}$, $\{A_2 z_n\}$, $\{St_n\}$, $\{f(x_n)\}$ and $\{A_3 x_n\}$ are also bounded.

Step 2. We show that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.

By nonexpansiveness of Q_C and $I - \lambda_i A_i$ $(i = 1, 2, 3)$, we have

$$\begin{aligned} \|t_{n+1} - t_n\| &= \|Q_C(y_{n+1} - \lambda_1 A_1 y_{n+1}) - Q_C(y_n - \lambda_1 A_1 y_n)\| \\ &\le \|y_{n+1} - y_n\| \\ &= \|Q_C(z_{n+1} - \lambda_2 A_2 z_{n+1}) - Q_C(z_n - \lambda_2 A_2 z_n)\| \\ &\le \|z_{n+1} - z_n\| \\ &= \|Q_C(x_{n+1} - \lambda_3 A_3 x_{n+1}) - Q_C(x_n - \lambda_3 A_3 x_n)\| \\ &\le \|x_{n+1} - x_n\|. \end{aligned} \tag{3.3}$$

Let $w_n = \frac{x_{n+1}-b_nx_n}{1-b_n}$, $n \in \mathbb{N}$. Then $x_{n+1} = b_nx_n + (1-b_n)w_n$ for all $n \in \mathbb{N}$ and

$$\begin{aligned} w_{n+1} - w_n &= \frac{x_{n+2} - b_{n+1}x_{n+1}}{1-b_{n+1}} - \frac{x_{n+1} - b_nx_n}{1-b_n} \\ &= \frac{a_{n+1}f(x_{n+1}) + (1-a_{n+1}-b_{n+1})St_{n+1}}{1-b_{n+1}} - \frac{a_nf(x_n) + (1-a_n-b_n)St_n}{1-b_n} \\ &= \frac{a_{n+1}}{1-b_{n+1}}\bigl(f(x_{n+1}) - St_{n+1}\bigr) + \frac{a_n}{1-b_n}\bigl(St_n - f(x_n)\bigr) + St_{n+1} - St_n. \end{aligned} \tag{3.4}$$

By (3.3), (3.4) and nonexpansiveness of S, we have

$$\|w_{n+1} - w_n\| - \|x_{n+1} - x_n\| \le \frac{a_{n+1}}{1-b_{n+1}}\bigl\|f(x_{n+1}) - St_{n+1}\bigr\| + \frac{a_n}{1-b_n}\bigl\|St_n - f(x_n)\bigr\|.$$

By this together with (C1) and (C2), we obtain that

$$\limsup_{n\to\infty} \|w_{n+1} - w_n\| - \|x_{n+1} - x_n\| \le 0.$$

Hence, by Lemma 2.6, we get $\|x_n - w_n\| \to 0$ as $n \to \infty$. Consequently,

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} (1-b_n)\|w_n - x_n\| = 0. \tag{3.5}$$

Step 3. We show that $\lim_{n\to\infty} \|Sx_n - x_n\| = 0$.
Since

$$x_{n+1} - x_n = a_n\bigl(f(x_n) - x_n\bigr) + (1-a_n-b_n)(St_n - x_n),$$

therefore

$$\|St_n - x_n\| \to 0 \quad \text{as } n \to \infty. \tag{3.6}$$

Next, we prove that $\lim_{n\to\infty} \|x_n - t_n\| = 0$. From Lemma 2.1 and nonexpansiveness of Q_C, we have

$$\begin{aligned} \|z_n - z^*\|^2 &= \bigl\|Q_C(x_n - \lambda_3A_3x_n) - Q_C\bigl(x^* - \lambda_3A_3x^*\bigr)\bigr\|^2 \\ &\le \bigl\|x_n - x^* - \lambda_3\bigl(A_3x_n - A_3x^*\bigr)\bigr\|^2 \\ &\le \|x_n - x^*\|^2 - 2\lambda_3\bigl\langle A_3x_n - A_3x^*, j\bigl(x_n - x^*\bigr)\bigr\rangle \\ &\quad + 2K^2\lambda_3^2\|A_3x_n - A_3x^*\|^2 \\ &\le \|x_n - x^*\|^2 - 2\lambda_3\bigl(-c_3\|A_3x_n - A_3x^*\|^2 + d_3\|x_n - x^*\|^2\bigr) \\ &\quad + 2K^2\lambda_3^2\|A_3x_n - A_3x^*\|^2 \\ &\le \|x_n - x^*\|^2 + 2\lambda_3c_3\|A_3x_n - A_3x^*\|^2 - \frac{2\lambda_3d_3}{L_3^2}\|A_3x_n - A_3x^*\|^2 \\ &\quad + 2K^2\lambda_3^2\|A_3x_n - A_3x^*\|^2 \\ &= \|x_n - x^*\|^2 - 2\lambda_3\left(\frac{d_3}{L_3^2} - c_3 - K^2\lambda_3\right)\|A_3x_n - A_3x^*\|^2 \end{aligned} \tag{3.7}$$

and

$$\begin{aligned}\|y_n - y^*\|^2 &= \|Q_C(z_n - \lambda_2 A_2 z_n) - Q_C(z^* - \lambda_2 A_2 z^*)\|^2 \\ &\le \|z_n - z^* - \lambda_2(A_2 z_n - A_2 z^*)\|^2 \\ &\le \|z_n - z^*\|^2 - 2\lambda_2\langle A_2 z_n - A_2 z^*, j(z_n - z^*)\rangle \\ &\quad + 2K^2\lambda_2^2\|A_2 z_n - A_2 z^*\|^2 \\ &\le \|z_n - z^*\|^2 - 2\lambda_2(-c_2\|A_2 z_n - A_2 z^*\|^2 + d_2\|z_n - z^*\|^2) \\ &\quad + 2K^2\lambda_2^2\|A_2 z_n - A_2 z^*\|^2 \\ &\le \|z_n - z^*\|^2 + 2\lambda_2 c_2\|A_2 z_n - A_2 z^*\|^2 - \frac{2\lambda_2 d_2}{L_2^2}\|A_2 z_n - A_2 z^*\|^2 \\ &\quad + 2K^2\lambda_2^2\|A_2 z_n - A_2 z^*\|^2 \\ &= \|z_n - z^*\|^2 - 2\lambda_2\left(\frac{d_2}{L_2^2} - c_2 - K^2\lambda_2\right)\|A_2 z_n - A_2 z^*\|^2. \end{aligned} \tag{3.8}$$

Similarly, we have

$$\begin{aligned}\|t_n - x^*\|^2 &= \|Q_C(y_n - \lambda_1 A_1 y_n) - Q_C(y^* - \lambda_1 A_1 y^*)\|^2 \\ &\le \|y_n - y^* - \lambda_1(A_1 y_n - A_1 y^*)\|^2 \\ &\le \|y_n - y^*\|^2 - 2\lambda_1\langle A_1 y_n - A_1 y^*, j(y_n - y^*)\rangle \\ &\quad + 2K^2\lambda_1^2\|A_1 y_n - A_1 y^*\|^2 \\ &\le \|y_n - y^*\|^2 - 2\lambda_1(-c_1\|A_1 y_n - A_1 y^*\|^2 + d_1\|y_n - y^*\|^2) \\ &\quad + 2K^2\lambda_1^2\|A_1 y_n - A_1 y^*\|^2 \\ &\le \|y_n - y^*\|^2 + 2\lambda_1 c_1\|A_1 y_n - A_1 y^*\|^2 - \frac{2\lambda_1 d_1}{L_1^2}\|A_1 y_n - A_1 y^*\|^2 \\ &\quad + 2K^2\lambda_1^2\|A_1 y_n - A_1 y^*\|^2 \\ &= \|y_n - y^*\|^2 - 2\lambda_1\left(\frac{d_1}{L_1^2} - c_1 - K^2\lambda_1\right)\|A_1 y_n - A_1 y^*\|^2. \end{aligned} \tag{3.9}$$

Substituting (3.7) and (3.8) into (3.9), we have

$$\begin{aligned}\|t_n - x^*\|^2 \le{}& \|x_n - x^*\|^2 - 2\lambda_3\left(\frac{d_3}{L_3^2} - c_3 - K^2\lambda_3\right)\|A_3 x_n - A_3 x^*\|^2 \\ &- 2\lambda_2\left(\frac{d_2}{L_2^2} - c_2 - K^2\lambda_2\right)\|A_2 z_n - A_2 z^*\|^2 \\ &- 2\lambda_1\left(\frac{d_1}{L_1^2} - c_1 - K^2\lambda_1\right)\|A_1 y_n - A_1 y^*\|^2. \end{aligned} \tag{3.10}$$

By the convexity of $\|\cdot\|^2$, we obtain

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= \|a_n f(x_n) + b_n x_n + (1 - a_n - b_n)St_n - x^*\|^2 \\ &\le a_n\|f(x_n) - x^*\|^2 + b_n\|x_n - x^*\|^2 + (1 - a_n - b_n)\|St_n - x^*\|^2 \\ &\le a_n\|f(x_n) - x^*\|^2 + b_n\|x_n - x^*\|^2 + (1 - a_n - b_n)\|t_n - x^*\|^2. \end{aligned} \tag{3.11}$$

Substituting (3.10) into (3.11), we have

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &\le a_n\|f(x_n)-x^*\|^2 + b_n\|x_n-x^*\|^2 \\
&\quad + (1-a_n-b_n)\Bigg(\|x_n-x^*\|^2 - 2\lambda_3\bigg(\frac{d_3}{L_3^2}-c_3-K^2\lambda_3\bigg)\|A_3x_n-A_3x^*\|^2 \\
&\quad - 2\lambda_2\bigg(\frac{d_2}{L_2^2}-c_2-K^2\lambda_2\bigg)\|A_2z_n-A_2z^*\|^2 \\
&\quad - 2\lambda_1\bigg(\frac{d_1}{L_1^2}-c_1-K^2\lambda_1\bigg)\|A_1y_n-A_1y^*\|^2\Bigg) \\
&= a_n\|f(x_n)-x^*\|^2 + (1-a_n)\|x_n-x^*\|^2 \\
&\quad - (1-a_n-b_n)2\lambda_3\bigg(\frac{d_3}{L_3^2}-c_3-K^2\lambda_3\bigg)\|A_3x_n-A_3x^*\|^2 \\
&\quad - (1-a_n-b_n)2\lambda_2\bigg(\frac{d_2}{L_2^2}-c_2-K^2\lambda_2\bigg)\|A_2z_n-A_2z^*\|^2 \\
&\quad - (1-a_n-b_n)2\lambda_1\bigg(\frac{d_1}{L_1^2}-c_1-K^2\lambda_1\bigg)\|A_1y_n-A_1y^*\|^2,
\end{aligned}$$

which implies

$$\begin{aligned}
&(1-a_n-b_n)2\lambda_3\bigg(\frac{d_3}{L_3^2}-c_3-K^2\lambda_3\bigg)\|A_3x_n-A_3x^*\|^2 \\
&\quad + (1-a_n-b_n)2\lambda_2\bigg(\frac{d_2}{L_2^2}-c_2-K^2\lambda_2\bigg)\|A_2z_n-A_2z^*\|^2 \\
&\quad + (1-a_n-b_n)2\lambda_1\bigg(\frac{d_1}{L_1^2}-c_1-K^2\lambda_1\bigg)\|A_1y_n-A_1y^*\|^2 \\
&\le a_n\|f(x_n)-x^*\|^2 + \|x_n-x^*\|^2 - \|x_{n+1}-x^*\|^2 \\
&\le a_n\|f(x_n)-x^*\|^2 + \|x_n-x_{n+1}\|\big(\|x_n-x^*\| + \|x_{n+1}-x^*\|\big).
\end{aligned}$$

By the conditions (C1), (C2), (3.5) and $0<\lambda_i<\frac{d_i-c_iL_i{}^2}{K^2L_i{}^2}$ for each $i=1,2,3$, we obtain

$$\begin{aligned}
&\lim_{n\to\infty}\|A_3x_n-A_3x^*\| = 0, \qquad \lim_{n\to\infty}\|A_2z_n-A_2z^*\| = 0 \quad \text{and} \\
&\lim_{n\to\infty}\|A_1y_n-A_1y^*\| = 0.
\end{aligned} \tag{3.12}$$

Let $r=\sup_{n\ge1}\{\|x_n-x^*\|,\|z_n-z^*\|,\|y_n-y^*\|,\|t_n-x^*\|\}$. By Lemma 2.4(b) and Lemma 2.8, we obtain

$$\begin{aligned}
\|t_n-x^*\|^2 &= \|Q_C(y_n-\lambda_1A_1y_n)-Q_C(y^*-\lambda_1A_1y^*)\|^2 \\
&\le \langle y_n-\lambda_1A_1y_n-(y^*-\lambda_1A_1y^*), j(t_n-x^*)\rangle \\
&= \langle y_n-y^*, j(t_n-x^*)\rangle - \lambda_1\langle A_1y_n-A_1y^*, j(t_n-x^*)\rangle \\
&\le \frac{1}{2}\big[\|y_n-y^*\|^2 + \|t_n-x^*\|^2 - g\big(\|y_n-y^*-(t_n-x^*)\|\big)\big] \\
&\quad + \lambda_1\langle A_1y^*-A_1y_n, j(t_n-x^*)\rangle,
\end{aligned}$$

which implies

$$\begin{aligned}\|t_n - x^*\|^2 &\le \|y_n - y^*\|^2 - g(\|y_n - y^* - (t_n - x^*)\|) \\ &\quad + 2\lambda_1\langle A_1y^* - A_1y_n, j(t_n - x^*)\rangle \\ &\le \|y_n - y^*\|^2 - g(\|y_n - y^* - (t_n - x^*)\|) \\ &\quad + 2\lambda_1\|A_1y^* - A_1y_n\|\|t_n - x^*\|. \end{aligned} \tag{3.13}$$

And

$$\begin{aligned}\|y_n - y^*\|^2 &= \|Q_C(z_n - \lambda_2A_2z_n) - Q_C(z^* - \lambda_2A_2z^*)\|^2 \\ &\le \langle z_n - \lambda_2A_2z_n - (z^* - \lambda_2A_2z^*), j(y_n - y^*)\rangle \\ &= \langle z_n - z^*, j(y_n - y^*)\rangle - \lambda_2\langle A_2z_n - A_2z^*, j(y_n - y^*)\rangle \\ &\le \frac{1}{2}\bigl[\|z_n - z^*\|^2 + \|y_n - y^*\|^2 - g(\|z_n - z^* - (y_n - y^*)\|)\bigr] \\ &\quad + \lambda_2\langle A_2z^* - A_2z_n, j(y_n - y^*)\rangle, \end{aligned}$$

which implies

$$\begin{aligned}\|y_n - y^*\|^2 &\le \|z_n - z^*\|^2 - g(\|z_n - z^* - (y_n - y^*)\|) \\ &\quad + 2\lambda_2\langle A_2z^* - A_2z_n, j(y_n - y^*)\rangle \\ &\le \|z_n - z^*\|^2 - g(\|z_n - z^* - (y_n - y^*)\|) \\ &\quad + 2\lambda_2\|A_2z^* - A_2z_n\|\|y_n - y^*\|. \end{aligned} \tag{3.14}$$

Similarly, we have

$$\begin{aligned}\|z_n - z^*\|^2 &= \|Q_C(x_n - \lambda_3A_3x_n) - Q_C(x^* - \lambda_3A_3x^*)\|^2 \\ &\le \langle x_n - \lambda_3A_3x_n - (x^* - \lambda_3A_3x^*), j(z_n - z^*)\rangle \\ &= \langle x_n - x^*, j(z_n - z^*)\rangle - \lambda_3\langle A_3x_n - A_3x^*, j(z_n - z^*)\rangle \\ &\le \frac{1}{2}\bigl[\|x_n - x^*\|^2 + \|z_n - z^*\|^2 - g(\|x_n - x^* - (z_n - z^*)\|)\bigr] \\ &\quad + \lambda_3\langle A_3x^* - A_3x_n, j(z_n - z^*)\rangle, \end{aligned}$$

which implies

$$\begin{aligned}\|z_n - z^*\|^2 &\le \|x_n - x^*\|^2 - g(\|x_n - x^* - (z_n - z^*)\|) \\ &\quad + 2\lambda_3\langle A_3x^* - A_3x_n, j(z_n - z^*)\rangle \\ &\le \|x_n - x^*\|^2 - g(\|x_n - x^* - (z_n - z^*)\|) \\ &\quad + 2\lambda_3\|A_3x^* - A_3x_n\|\|z_n - z^*\|. \end{aligned} \tag{3.15}$$

From (3.11), (3.13), (3.14) and (3.15), we have

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &\le a_n\|f(x_n)-x^*\|^2 + b_n\|x_n-x^*\|^2 + (1-a_n-b_n)\|t_n-x^*\|^2 \\
&\le a_n\|f(x_n)-x^*\|^2 + b_n\|x_n-x^*\|^2 \\
&\quad + (1-a_n-b_n)\big[\|y_n-y^*\|^2 - g\big(\|y_n-y^*-(t_n-x^*)\|\big) \\
&\quad + 2\lambda_1\|A_1y^*-A_1y_n\|\|t_n-x^*\|\big] \\
&\le a_n\|f(x_n)-x^*\|^2 + b_n\|x_n-x^*\|^2 \\
&\quad + (1-a_n-b_n)\big[\|z_n-z^*\|^2 - g\big(\|z_n-z^*-(y_n-y^*)\|\big) \\
&\quad + 2\lambda_2\|A_2z^*-A_2z_n\|\|y_n-y^*\| - g\big(\|y_n-y^*-(t_n-x^*)\|\big) \\
&\quad + 2\lambda_1\|A_1y^*-A_1y_n\|\|t_n-x^*\|\big] \\
&\le a_n\|f(x_n)-x^*\|^2 + b_n\|x_n-x^*\|^2 \\
&\quad + (1-a_n-b_n)\big[\|x_n-x^*\|^2 - g\big(\|x_n-x^*-(z_n-z^*)\|\big) \\
&\quad + 2\lambda_3\|A_3x^*-A_3x_n\|\|z_n-z^*\| - g\big(\|z_n-z^*-(y_n-y^*)\|\big) \\
&\quad + 2\lambda_2\|A_2z^*-A_2z_n\|\|y_n-y^*\| - g\big(\|y_n-y^*-(t_n-x^*)\|\big) \\
&\quad + 2\lambda_1\|A_1y^*-A_1y_n\|\|t_n-x^*\|\big] \\
&= a_n\|f(x_n)-x^*\|^2 + (1-a_n)\|x_n-x^*\|^2 \\
&\quad + (1-a_n-b_n)\big(2\lambda_1\|A_1y^*-A_1y_n\|\|t_n-x^*\|\big) \\
&\quad + (1-a_n-b_n)\big(2\lambda_2\|A_2z^*-A_2z_n\|\|y_n-y^*\|\big) \\
&\quad + (1-a_n-b_n)\big(2\lambda_3\|A_3x^*-A_3x_n\|\|z_n-z^*\|\big) \\
&\quad - (1-a_n-b_n)g\big(\|y_n-y^*-(t_n-x^*)\|\big) \\
&\quad - (1-a_n-b_n)g\big(\|z_n-z^*-(y_n-y^*)\|\big) \\
&\quad - (1-a_n-b_n)g\big(\|x_n-x^*-(z_n-z^*)\|\big),
\end{aligned}$$

which implies

$$\begin{aligned}
&(1-a_n-b_n)g\big(\|y_n-y^*-(t_n-x^*)\|\big) + (1-a_n-b_n)g\big(\|z_n-z^*-(y_n-y^*)\|\big) \\
&\qquad + (1-a_n-b_n)g\big(\|x_n-x^*-(z_n-z^*)\|\big) \\
&\quad \le a_n\|f(x_n)-x^*\|^2 + \|x_n-x^*\|^2 - \|x_{n+1}-x^*\|^2 \\
&\qquad + (1-a_n-b_n)\big(2\lambda_1\|A_1y^*-A_1y_n\|\|t_n-x^*\|\big) \\
&\qquad + (1-a_n-b_n)\big(2\lambda_2\|A_2z^*-A_2z_n\|\|y_n-y^*\|\big) \\
&\qquad + (1-a_n-b_n)\big(2\lambda_3\|A_3x^*-A_3x_n\|\|z_n-z^*\|\big) \\
&\quad \le a_n\|f(x_n)-x^*\|^2 + \|x_n-x_{n+1}\|\big(\|x_n-x^*\| + \|x_{n+1}-x^*\|\big) \\
&\qquad + (1-a_n-b_n)\big(2\lambda_1\|A_1y^*-A_1y_n\|\|t_n-x^*\|\big) \\
&\qquad + (1-a_n-b_n)\big(2\lambda_2\|A_2z^*-A_2z_n\|\|y_n-y^*\|\big) \\
&\qquad + (1-a_n-b_n)\big(2\lambda_3\|A_3x^*-A_3x_n\|\|z_n-z^*\|\big).
\end{aligned}$$

By the conditions (C1), (C2), (3.5) and (3.12), we obtain

$$\lim_{n\to\infty} g\bigl(\bigl\|y_n - y^* - \bigl(t_n - x^*\bigr)\bigr\|\bigr) = 0, \qquad \lim_{n\to\infty} g\bigl(\bigl\|z_n - z^* - \bigl(y_n - y^*\bigr)\bigr\|\bigr) = 0 \quad \text{and}$$
$$\lim_{n\to\infty} g\bigl(\bigl\|x_n - x^* - \bigl(z_n - z^*\bigr)\bigr\|\bigr) = 0.$$

It follows from the properties of g that

$$\lim_{n\to\infty} \bigl\|y_n - y^* - \bigl(t_n - x^*\bigr)\bigr\| = 0, \qquad \lim_{n\to\infty} \bigl\|z_n - z^* - \bigl(y_n - y^*\bigr)\bigr\| = 0 \quad \text{and}$$
$$\lim_{n\to\infty} \bigl\|x_n - x^* - \bigl(z_n - z^*\bigr)\bigr\| = 0.$$

Therefore

$$\begin{aligned}\|x_n - t_n\| &\le \bigl\|x_n - z_n - \bigl(x^* - z^*\bigr)\bigr\| + \bigl\|z_n - y_n - \bigl(z^* - y^*\bigr)\bigr\| \\ &\quad + \bigl\|y_n - t_n - \bigl(y^* - x^*\bigr)\bigr\| \to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{3.16}$$

By (3.6) and (3.16), we have

$$\begin{aligned}\|Sx_n - x_n\| &\le \|Sx_n - St_n\| + \|St_n - x_n\| \\ &\le \|x_n - t_n\| + \|St_n - x_n\| \to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{3.17}$$

Define a mapping $W : C \to C$ as

$$Wx = \eta Sx + (1 - \eta)Gx, \quad \forall x \in C,$$

where η is a constant in $(0,1)$. Then it follows from Lemma 2.7 that $F(W) = F(G) \cap F(S)$ and W is nonexpansive. From (3.16) and (3.17), we have

$$\begin{aligned}\|x_n - Wx_n\| &= \bigl\|x_n - \bigl(\eta Sx_n + (1 - \eta)Gx_n\bigr)\bigr\| \\ &= \bigl\|\eta(x_n - Sx_n) + (1 - \eta)(x_n - Gx_n)\bigr\| \\ &\le \eta\|x_n - Sx_n\| + (1 - \eta)\|x_n - Gx_n\| \\ &= \eta\|x_n - Sx_n\| + (1 - \eta)\|x_n - t_n\| \to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{3.18}$$

Step 4. We claim that

$$\limsup_{n\to\infty} \bigl\langle f(q) - q, j(x_n - q)\bigr\rangle \le 0, \tag{3.19}$$

where $q = \lim_{t\to 0} x_t$ with x_t being the fixed point of the contraction

$$x \mapsto tf(x) + (1 - t)Wx.$$

From Lemma 2.9, we have $q \in F(W) = F(G) \cap F(S) = \Omega$ and

$$\bigl\langle (I - f)q, j(q - p)\bigr\rangle \le 0, \quad \forall f \in \Pi_C, p \in \Omega.$$

Since $x_t = tf(x_t) + (1-t)Wx_t$, we have

$$\begin{aligned}\|x_t - x_n\| &= \bigl\|tf(x_t) + (1-t)Wx_t - x_n\bigr\| \\ &= \bigl\|(1-t)(Wx_t - x_n) + t\bigl(f(x_t) - x_n\bigr)\bigr\|.\end{aligned}$$

It follows from (3.18) and Lemma 2.2 that

$$\begin{aligned}\|x_t - x_n\|^2 &= \bigl\|(1-t)(Wx_t - x_n) + t\bigl(f(x_t) - x_n\bigr)\bigr\|^2 \\ &\le (1-t)^2\|Wx_t - x_n\|^2 + 2t\bigl\langle f(x_t) - x_n, j(x_t - x_n)\bigr\rangle \\ &\le (1-t)^2\bigl(\|Wx_t - Wx_n\| + \|Wx_n - x_n\|\bigr)^2 \\ &\quad + 2t\bigl\langle f(x_t) - x_n, j(x_t - x_n)\bigr\rangle \\ &= (1-t)^2\bigl(\|Wx_t - Wx_n\|^2 + 2\|Wx_t - Wx_n\|\|Wx_n - x_n\| + \|Wx_n - x_n\|^2\bigr) \\ &\quad + 2t\bigl\langle f(x_t) - x_t, j(x_t - x_n)\bigr\rangle + 2t\bigl\langle x_t - x_n, j(x_t - x_n)\bigr\rangle \\ &\le \bigl(1 - 2t + t^2\bigr)\|x_t - x_n\|^2 + (1-t)^2\bigl(2\|x_t - x_n\|\|Wx_n - x_n\| + \|Wx_n - x_n\|^2\bigr) \\ &\quad + 2t\bigl\langle f(x_t) - x_t, j(x_t - x_n)\bigr\rangle + 2t\|x_t - x_n\|^2 \\ &= \bigl(1 + t^2\bigr)\|x_t - x_n\|^2 + f_n(t) + 2t\bigl\langle f(x_t) - x_t, j(x_t - x_n)\bigr\rangle,\end{aligned} \tag{3.20}$$

where $f_n(t) = (1-t)^2(2\|x_t - x_n\| + \|Wx_n - x_n\|)\|Wx_n - x_n\| \to 0$ as $n \to \infty$. It follows from (3.20) that

$$\bigl\langle x_t - f(x_t), j(x_t - x_n)\bigr\rangle \le \frac{t}{2}\|x_t - x_n\|^2 + \frac{f_n(t)}{2t}. \tag{3.21}$$

Let $n \to \infty$ in (3.21), we obtain that

$$\limsup_{n\to\infty}\bigl\langle x_t - f(x_t), j(x_t - x_n)\bigr\rangle \le \frac{t}{2}M, \tag{3.22}$$

where $M > 0$ is a constant such that $M \ge \|x_t - x_n\|^2$ for all $t \in (0,1)$ and $n \ge 1$. Let $t \to 0$ in (3.22), we obtain

$$\limsup_{t\to 0}\limsup_{n\to\infty}\bigl\langle x_t - f(x_t), j(x_t - x_n)\bigr\rangle \le 0. \tag{3.23}$$

On the other hand, we have

$$\begin{aligned}\bigl\langle f(q) - q, j(x_n - q)\bigr\rangle &= \bigl\langle f(q) - q, j(x_n - q)\bigr\rangle - \bigl\langle f(q) - q, j(x_n - x_t)\bigr\rangle \\ &\quad + \bigl\langle f(q) - q, j(x_n - x_t)\bigr\rangle - \bigl\langle f(q) - x_t, j(x_n - x_t)\bigr\rangle \\ &\quad + \bigl\langle f(q) - x_t, j(x_n - x_t)\bigr\rangle - \bigl\langle f(x_t) - x_t, j(x_n - x_t)\bigr\rangle \\ &\quad + \bigl\langle f(x_t) - x_t, j(x_n - x_t)\bigr\rangle \\ &= \bigl\langle f(q) - q, j(x_n - q) - j(x_n - x_t)\bigr\rangle + \bigl\langle x_t - q, j(x_n - x_t)\bigr\rangle \\ &\quad + \bigl\langle f(q) - f(x_t), j(x_n - x_t)\bigr\rangle + \bigl\langle f(x_t) - x_t, j(x_n - x_t)\bigr\rangle.\end{aligned}$$

It follows that

$$\begin{aligned}\limsup_{n\to\infty}\langle f(q)-q,j(x_n-q)\rangle \le{}& \limsup_{n\to\infty}\langle f(q)-q,j(x_n-q)-j(x_n-x_t)\rangle\\ &+\|x_t-q\|\limsup_{n\to\infty}\|x_n-x_t\|+\alpha\|x_t-q\|\limsup_{n\to\infty}\|x_n-x_t\|\\ &+\limsup_{n\to\infty}\langle f(x_t)-x_t,j(x_n-x_t)\rangle.\end{aligned}$$

Noticing that j is norm-to-norm uniformly continuous on a bounded subset of C, it follows from (3.23) and $\lim_{t\to 0} x_t = q$ that

$$\limsup_{n\to\infty}\langle f(q)-q,j(x_n-q)\rangle = \limsup_{t\to 0}\limsup_{n\to\infty}\langle f(q)-q,j(x_n-q)\rangle \le 0.$$

Hence (3.19) holds.

Step 5. Finally, we show that $x_n \to q$ as $n\to\infty$.

From (3.2), we have

$$\begin{aligned}\|x_{n+1}-q\|^2 &= \langle x_{n+1}-q,j(x_{n+1}-q)\rangle\\ &= \langle a_n\bigl(f(x_n)-q\bigr)+b_n(x_n-q)+(1-a_n-b_n)(St_n-q),j(x_{n+1}-q)\rangle\\ &= a_n\langle f(x_n)-f(q),j(x_{n+1}-q)\rangle + b_n\langle x_n-q,j(x_{n+1}-q)\rangle\\ &\quad+(1-a_n-b_n)\langle St_n-q,j(x_{n+1}-q)\rangle + a_n\langle f(q)-q,j(x_{n+1}-q)\rangle\\ &\le a_n\alpha\|x_n-q\|\|x_{n+1}-q\| + b_n\|x_n-q\|\|x_{n+1}-q\|\\ &\quad+(1-a_n-b_n)\|St_n-q\|\|x_{n+1}-q\| + a_n\langle f(q)-q,j(x_{n+1}-q)\rangle\\ &\le a_n\alpha\|x_n-q\|\|x_{n+1}-q\| + b_n\|x_n-q\|\|x_{n+1}-q\|\\ &\quad+(1-a_n-b_n)\|x_n-q\|\|x_{n+1}-q\| + a_n\langle f(q)-q,j(x_{n+1}-q)\rangle\\ &= \bigl(1-a_n(1-\alpha)\bigr)\|x_n-q\|\|x_{n+1}-q\| + a_n\langle f(q)-q,j(x_{n+1}-q)\rangle\\ &\le \frac{1-a_n(1-\alpha)}{2}\bigl(\|x_n-q\|^2+\|x_{n+1}-q\|^2\bigr) + a_n\langle f(q)-q,j(x_{n+1}-q)\rangle\\ &\le \frac{1-a_n(1-\alpha)}{2}\|x_n-q\|^2+\frac{1}{2}\|x_{n+1}-q\|^2 + a_n\langle f(q)-q,j(x_{n+1}-q)\rangle,\end{aligned}$$

which implies

$$\|x_{n+1}-q\|^2 \le \bigl(1-a_n(1-\alpha)\bigr)\|x_n-q\|^2 + a_n(1-\alpha)\frac{2\langle f(q)-q,j(x_{n+1}-q)\rangle}{1-\alpha}.$$

It follows from Lemma 2.3, (3.19) and condition (C1) that $\{x_n\}$ converges strongly to q. This completes the proof. □

Example 3.2 Let $X=\mathbb{R}$ and $C=[0,1]$. Define the mappings $S,f: C\to C$ and $A_1,A_2,A_3: C\to X$ as follows:

$$S(x)=\frac{x}{3},\qquad f(x)=\frac{x}{2}+3,\qquad A_1(x)=x,\qquad A_2(x)=2x\quad\text{and}\quad A_3(x)=3x.$$

Then it is obvious that S is nonexpansive, f is contractive with a constant $\alpha = \frac{1}{2}$, A_1 is relaxed $(\frac{1}{2}, 1)$-cocoercive and 1-Lipschitzian, A_2 is relaxed $(\frac{1}{4}, 2)$-cocoercive and 2-Lipschitzian and A_3 is relaxed $(\frac{1}{9}, 3)$-cocoercive and 3-Lipschitzian. In this case, we have $\Omega = F(S) \cap F(G) = \{0\}$. In the terms of Theorem 3.1, we choose the parameters λ_1, λ_2, λ_3. Then the sequence $\{x_n\}$ generated by (3.1) converges to $q = 0 \in \Omega$, which solves the following variational inequality:

$$\langle q - f(q), j(q-p) \rangle \le 0, \quad \forall p \in \Omega.$$

Let $A_3 = 0$ in Theorem 3.1, we obtain the following result.

Corollary 3.3 *Let X be a uniformly convex and 2-uniformly smooth Banach space with the 2-uniformly smooth constant K, let C be a nonempty closed convex subset of X and Q_C a sunny nonexpansive retraction from X onto C. Let the mappings $A_i : C \to X$ be relaxed (c_i, d_i)-cocoercive and L_i-Lipschitzian with $0 < \lambda_i < \frac{d_i - c_i L_i^2}{K^2 L_i^2}$, for all $i = 1, 2$. Let f be a contractive mapping with the constant $\alpha \in (0, 1)$ and let $S : C \to C$ be a nonexpansive mapping such that $F = F(S) \cap \Omega_2 \neq \emptyset$, where Ω_2 is the set of solutions of problem* (1.4). *For a given $x_1 \in C$, let $\{x_n\}$ and $\{y_n\}$ be the sequences generated by*

$$\begin{cases} y_n = Q_C(x_n - \lambda_2 A_2 x_n), \\ x_{n+1} = a_n f(x_n) + b_n x_n + (1 - a_n - b_n) S Q_C(y_n - \lambda_1 A_1 y_n), \quad n \ge 1, \end{cases}$$

where $\{a_n\}$ and $\{b_n\}$ are two sequences in $(0, 1)$ *such that*

(C1) $\lim_{n\to\infty} a_n = 0$ *and* $\sum_{n=1}^{\infty} a_n = \infty$;

(C2) $0 < \liminf_{n\to\infty} b_n \le \limsup_{n\to\infty} b_n < 1$.

Then $\{x_n\}$ converges strongly to $q \in F$, which solves the following variational inequality:

$$\langle q - f(q), j(q-p) \rangle \le 0, \quad \forall f \in \Pi_C, p \in F.$$

Remark 3.4 (i) Since L^p for all $p \ge 2$ is uniformly convex and 2-uniformly smooth, we see that Theorem 3.1 is applicable to L^p for all $p \ge 2$.

(ii) The problem of finding solutions for a finite number of variational inequalities can use the same idea of a new general system of variational inequalities in Banach spaces.

Competing interests

The author declares that they have no competing interests.

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