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On strong orthogonality and strictly convex normed linear spaces

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Abstract
We introduce the notion of a strongly orthogonal set relative to an element in the sense of Birkhoff-James in a normed linear space to find a necessary and sufficient condition for an element x of the unit sphere S_X to be an exposed point of the unit ball B_X. We then prove that a normed linear space is strictly convex iff for each element x of the unit sphere, there exists a bounded linear operator A on X which attains its norm only at the points of the form λx with $\lambda \in S_K$.

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1 Introduction

Suppose $(X, \|\cdot\|)$ is a normed linear space over the field K, real or complex. X is said to be strictly convex iff every element of the unit sphere $S_X = \{x \in X : \|x\| = 1\}$ is an extreme point of the unit ball $B_X = \{x \in X : \|x\| \leq 1\}$. There are many equivalent characterizations of the strict convexity of a normed space, some of them given in [1, 2] are as follows.

(i) If $x, y \in S_X$, then we have $\|x + y\| < 2$.

(ii) Every non-zero continuous linear functional attains a maximum on at most one point of the unit sphere.

(iii) If $\|x + y\| = \|x\| + \|y\|$, $x \neq 0$, then $y = cx$ for some $c \geq 0$.

The notion of strict convexity plays an important role in the studies of the geometry of Banach spaces. One may go through [1–11] for more information related to strictly convex spaces.

An element x is said to be orthogonal to y in X in the sense of Birkhoff-James [1, 8, 12], written as, $x \perp_B y$, iff

$$\|x\| \leq \|x + \lambda y\| \quad \text{for all scalars } \lambda.$$

If X is an inner product space, then $x \perp_B y$ implies $\|x\| < \|x + \lambda y\|$ for all scalars $\lambda \neq 0$. Motivated by this fact, we here introduce the notion of strong orthogonality as follows.

Strongly orthogonal in the sense of Birkhoff-James: In a normed linear space X, an element x is said to be strongly orthogonal to another element y in the sense of Birkhoff-James, written as $x \perp_{SB} y$, iff

$$\|x\| < \|x + \lambda y\| \quad \text{for all scalars } \lambda \neq 0.$$

If $x \perp_{SB} y$, then $x \perp_B y$, but the converse is not true. In $l_\infty(R^2)$ the element $(1,0)$ is orthogonal to $(0,1)$ in the sense of Birkhoff-James, but not strongly orthogonal.

Strongly orthogonal set relative to an element: A finite set of elements $S = \{x_1, x_2, \ldots, x_n\}$ is said to be a strongly orthogonal set relative to an element x_{i_0} contained in S in the sense of Birkhoff-James iff

$$\|x_{i_0}\| < \left\| x_{i_0} + \sum_{j=1,\ j\neq i_0}^{n} \lambda_j x_j \right\|$$

whenever not all λ_j's are 0.

An infinite set of elements is said to be a strongly orthogonal set relative to an element contained in the set in the sense of Birkhoff-James iff every finite subset containing that element is strongly orthogonal relative to that element in the sense of Birkhoff-James.

Strongly orthogonal set: A finite set of elements $\{x_1, x_2, \ldots, x_n\}$ is said to be a strongly orthogonal set in the sense of Birkhoff-James iff for each $i \in \{1, 2, \ldots, n\}$

$$\|x_i\| < \left\| x_i + \sum_{j=1, j\neq i}^{n} \lambda_j x_j \right\|$$

whenever not all λ_j's are 0.

An infinite set of elements is said to be a strongly orthogonal set in the sense of Birkhoff-James iff every finite subset of the set is a strongly orthogonal set in the sense of Birkhoff-James.

Clearly if a set is strongly orthogonal in the sense of Birkhoff-James, then it is strongly orthogonal relative to every element of the set in the sense of Birkhoff-James. If X has a Hamel basis which is strongly orthogonal in the sense of Birkhoff-James, then we call the Hamel basis a strongly orthogonal Hamel basis in the sense of Birkhoff-James, and if X has a Hamel basis which is strongly orthogonal relative to an element of the basis in the sense of Birkhoff-James, then we call the Hamel basis a strongly orthogonal Hamel basis relative to that element of the basis in the sense of Birkhoff-James. If, in addition, the norm of each element of a strongly orthogonal set is 1, then accordingly we call them orthonormal.

As, for example, the set $\{(1,0,\ldots,0), (0,1,0,\ldots,0), \ldots, (0,0,\ldots,1)\}$ is a strongly orthonormal Hamel basis in the sense of Birkhoff-James in $l_1(R^n)$, but not in $l_\infty(R^n)$.

In $l_2(R^3)$ the set $\{(1,0,0), (0,1,0), (0,1,1)\}$ is strongly orthogonal relative to $(1,0,0)$ in the sense of Birkhoff-James, but not relative to $(0,1,1)$.

In this paper we give another characterization of strictly convex normed linear spaces by using the Hahn-Banach theorem and the notion of a strongly orthogonal Hamel basis relative to an element in the sense of Birkhoff-James. More precisely, we explore the relation between the existence of a strongly orthogonal Hamel basis relative to an element with the unit norm in the sense of Birkhoff-James in a normed space and that of an extreme point of the unit ball in the space. We also prove that a normed linear space is strictly convex iff for each point x of the unit sphere, there exists a bounded linear operator A on X which attains its norm only at the points of the form λx with $\lambda \in S_K$.

2 Main results

We first obtain a sufficient condition for an element in the unit sphere to be an extreme point of the unit ball in an arbitrary normed linear space.

Theorem 2.1 *Let X be a normed linear space and $x_0 \in S_X$. If there exists a Hamel basis of X containing x_0 which is strongly orthonormal relative to x_0 in the sense of Birkhoff-James, then x_0 is an extreme point of B_X.*

Proof Let $D = \{x_0, x_\alpha : \alpha \in \Lambda\}$ be a strongly orthonormal Hamel basis relative to x_0 in the sense of Birkhoff-James.

If possible, suppose that x_0 is not an extreme point of B_X, then $x_0 = tz_1 + (1-t)z_2$ where $0 < t < 1$ and $\|z_1\| = \|z_2\| = 1$.

So, there exists $\alpha_1, \alpha_2, \dots, \alpha_n$ in Λ such that

$$z_1 = \beta_0 x_0 + \sum_{j=1}^{n} \beta_j x_{\alpha_j} \quad \text{and} \quad z_2 = \gamma_0 x_0 + \sum_{j=1}^{n} \gamma_j x_{\alpha_j}$$

for some scalars β_j, γ_j ($j = 0, 1, 2, \dots, n$).

If $\beta_0 = 0$ and $\gamma_0 = 0$, then $x_0 = tz_1 + (1-t)z_2$ implies that

$$x_0 = \sum_{j=1}^{n} \bigl(t\beta_j + (1-t)\gamma_j\bigr) x_{\alpha_j},$$

which contradicts the fact that every finite subset of D is linearly independent. So, the case $\beta_0 = 0$ and $\gamma_0 = 0$ is ruled out.

If $\beta_0 \neq 0$, $\gamma_0 = 0$, then as $\{x_0, x_{\alpha_1}, x_{\alpha_2}, \dots, x_{\alpha_n}\}$ is a strongly orthonormal set relative to x_0 in the sense of Birkhoff-James, so we get

$$1 = \|z_1\| = |\beta_0| \left\| x_0 + \sum_{j=1}^{n} \frac{\beta_j}{\beta_0} x_{\alpha_j} \right\| \geq |\beta_0|.$$

Now

$$x_0 = t\beta_0 x_0 + \sum_{j=1}^{n} \bigl(t\beta_j + (1-t)\gamma_j\bigr) x_{\alpha_j}$$

and so $t\beta_0 = 1$, which is not possible as $|\beta_0| \leq 1$ and $0 < t < 1$.

Similarly $\beta_0 = 0$, $\gamma_0 \neq 0$ is also ruled out.

Thus we have $\beta_0 \neq 0$ and $\gamma_0 \neq 0$.

Our claim is that at least one of $|\beta_0|$, $|\gamma_0|$ must be less than 1.

If possible, suppose that $|\beta_0| > 1$. Then

$$\left\| \beta_0 x_0 + \sum_{j=1}^{n} \beta_j x_{\alpha_j} \right\| = |\beta_0| \left\| x_0 + \sum_{j=1}^{n} \frac{\beta_j}{\beta_0} x_{\alpha_j} \right\| \geq |\beta_0| > 1.$$

This contradicts $\|z_1\| = 1$. Thus $|\beta_0| \leq 1$. Similarly $|\gamma_0| \leq 1$. We next show that $|\beta_0| = 1$ and $|\gamma_0| = 1$ cannot hold simultaneously.

Case 1. X is a real normed linear space.

Then $|\beta_0| = 1$ implies that

$$1 = \|z_1\| = |\beta_0| \left\| x_0 + \sum_{j=1}^{n} \frac{\beta_j}{\beta_0} x_{\alpha_j} \right\| > \|x_0\|,$$

unless $\beta_i = 0$ $\forall i = 1, 2, \dots, n$.

Thus $|\beta_0| = 1 \Rightarrow z_1 = \beta_0 x_0 \Rightarrow z_1 = \pm x_0 \Rightarrow x_0 = z_1 = z_2$ or $t = 0$, which is not possible. Thus $|\beta_0| \neq 1$. Similarly $|\gamma_0| \neq 1$.

Case 2. X is a complex normed linear space.

Then $|\beta_0| = 1$ implies that

$$1 = \|z_1\| = |\beta_0| \left\| x_0 + \sum_{j=1}^{n} \frac{\beta_j}{\beta_0} x_{\alpha_j} \right\| > \|x_0\|,$$

unless $\beta_i = 0\ \forall i = 1, 2, \dots, n$.

Thus $|\beta_0| = 1 \Rightarrow z_1 = \beta_0 x_0 \Rightarrow z_1 = e^{i\theta} x_0$, similarly $|\gamma_0| = 1 \Rightarrow z_2 = e^{i\phi} x_0$. Then $x_0 = te^{i\theta} x_0 + (1-t)e^{i\phi} x_0 \Rightarrow x_0 = z_1 = z_2$, which is not possible. Thus $|\beta_0| = 1$ and $|\gamma_0| = 1$ cannot hold simultaneously.

So, at least one of $|\beta_0|, |\gamma_0|$ is less than 1.

Now $x_0 = tz_1 + (1-t)z_2$ implies

$$t\beta_0 + (1-t)\gamma_0 = 1, \qquad t\beta_j + (1-t)\gamma_j = 0 \quad \forall j = 1, 2, \dots, n.$$

But $|\beta_0| < 1$ or $|\gamma_0| < 1$ implies

$$1 = \left| t\beta_0 + (1-t)\gamma_0 \right| \leq t|\beta_0| + (1-t)|\gamma_0| < 1,$$

which is not possible.

Thus x_0 is an extreme point of B_X. This completes the proof. □

The converse of the above theorem is, however, not always true. If x_0 is an extreme point of B_X, then there may or may not exist a strongly orthonormal Hamel basis relative to x_0 in the sense of Birkhoff-James.

Example 2.2 (i) Consider $(R^2, \|\cdot\|)$ where the unit sphere S is given by $S = \{(x,y) \in R^2 : x = \pm 1 \text{ and } -1 \leq y \leq 1\} \cup \{(x,y) \in R^2 : x^2 - 2y + y^2 = 0 \text{ and } y \geq 1\} \cup \{(x,y) \in R^2 : x^2 + 2y + y^2 = 0 \text{ and } y \leq -1\}$. Then $(1,1)$ is an extreme point of the unit ball, but there exists no strongly orthonormal Hamel basis relative to $(1,1)$ in the sense of Birkhoff-James.

(ii) Consider $(R^2, \|\cdot\|)$ where the unit sphere S is given by $S = \{(x,y) \in R^2 : x = \pm 1 \text{ and } -1 \leq y \leq 1\} \cup \{(x,y) \in R^2 : x^2 + 2y - 3 = 0 \text{ and } y \geq 1\} \cup \{(x,y) \in R^2 : x^2 - 2y - 3 = 0 \text{ and } y \leq -1\}$. Then $(1,1)$ is an extreme point of the unit ball and $\{(1,1), (-1,1)\}$ is a strongly orthonormal basis relative to $(1,1)$ in the sense of Birkhoff-James.

(iii) In $l_\infty(R^3)$ the extreme points of the unit ball are of the form $(\pm 1, \pm 1, \pm 1)$, and for the extreme point $(1,1,1)$, we can find a strongly orthonormal basis relative to $(1,1,1)$ in the sense of Birkhoff-James which is $\{(1,1,1), (1,0,-1), (0,1,-1)\}$.

In the first two examples, the extreme point $(1,1)$ is such that every neighborhood of $(1,1)$ contains both extreme as well as non-extreme points, whereas in the third case the extreme point $(1,1,1)$ is an isolated extreme point.

An element x in the boundary of a convex set S is called an exposed point of S iff there exists a hyperplane of support H to S through x such that $H \cap S = \{x\}$. The notion of exposed points can be found in [5, 13–15]. We next prove that if the extreme point x_0 is an exposed point of B_X, then there exists a Hamel basis of X containing x_0 which is strongly orthonormal relative to x_0 in the sense of Birkhoff-James.

Theorem 2.3 *Let X be a normed linear space and $x_0 \in S_X$ be an exposed point of B_X. Then there exists a Hamel basis of X containing x_0 which is strongly orthonormal relative to x_0 in the sense of Birkhoff-James.*

Proof As x_0 is an exposed point of B_X, so there exists a hyperplane of support H to B_X through x_0 such that $H \cap B_X = \{x_0\}$. Then we can find a linear functional f on X such that $H = \{x \in X : f(x) = 1\}$. Let $H_0 = \{x \in X : f(x) = 0\}$. Then H_0 is a subspace of X. Let $D = \{x_\alpha : \alpha \in \Lambda\}$ be a Hamel basis of H_0 with $\|x_\alpha\| = 1$. Clearly $\{x_0\} \cup D$ is a Hamel basis of X. We claim that $\{x_0\} \cup D$ is a strongly orthonormal set relative to x_0 in the sense of Birkhoff-James.

Consider a finite subset $\{x_{\alpha_1}, x_{\alpha_2}, \dots, x_{\alpha_{n-1}}\}$ of D and let $(\lambda_1, \lambda_2, \dots, \lambda_{n-1}) \neq (0, 0, \dots, 0)$. Now if $z = x_0 + \sum_{j=1}^{n-1} \lambda_j x_{\alpha_j}$, then

$$\begin{aligned} f(z) &= f\left(x_0 + \sum_{j=1}^{n-1} \lambda_j x_{\alpha_j}\right) = f(x_0) = 1 \\ &\Rightarrow \quad z \in H, \\ &\Rightarrow \quad z \notin B_X, \quad \text{as } H \cap B_X = \{x_0\}. \end{aligned}$$

So $\|x_0 + \sum_{j=1}^{n-1} \lambda_j x_{\alpha_j}\| > 1 = \|x_0\|$. Thus $\{x_0\} \cup D$ is a Hamel basis containing x_0 which is strongly orthonormal relative to x_0 in the sense of Birkhoff-James.

This completes the proof. □

We next prove the following theorem.

Theorem 2.4 *Let X be a normed linear space and $x_0 \in S_X$. If there exists a Hamel basis of X containing x_0 which is strongly orthonormal relative to x_0 in the sense of Birkhoff-James, then there exists a bounded invertible linear operator A on X such that $\|A\| = \|Ax_0\| > \|Ay\|$ for all y in S_X with $y \neq \lambda x_0$, $\lambda \in S_K$.*

Proof Let $\{x_0, x_\alpha : \alpha \in \Lambda\}$ be a Hamel basis of X which is strongly orthonormal relative to x_0 in the sense of Birkhoff-James.

Define a linear operator A on X by $A(x_0) = x_0$ and $A(x_\alpha) = \frac{1}{2} x_\alpha$ $\forall \alpha \in \Lambda$.

Clearly A is invertible. Take any $z \in X$ such that $\|z\| = 1$. Then $z = \lambda_0 x_0 + \sum_{j=1}^{n-1} \lambda_j x_{\alpha_j}$ for some scalars λ_j's and λ_0.

If $\lambda_0 = 0$, then $Az = \frac{1}{2} z$ and so

$$\|Ax_0\| = 1 > \frac{1}{2} = \|Az\|.$$

If $\lambda_0 \neq 0$, then as $\{x_0, x_\alpha : \alpha \in \Lambda\}$ is a strongly orthonormal Hamel basis relative to x_0 in the sense of Birkhoff-James, so we get

$$1 = \|z\| = \left\| \lambda_0 x_0 + \sum_{j=1}^{n-1} \lambda_j x_{\alpha_j} \right\| \geq |\lambda_0|.$$

Hence we get

$$\begin{aligned}\|Az\| &= \left\| \lambda_0 x_0 + \frac{1}{2}\sum_{j=1}^{n-1} \lambda_j x_{\alpha_j} \right\| \\ &= \left\| \frac{1}{2}\left(\lambda_0 x_0 + \sum_{j=1}^{n-1} \lambda_j x_{\alpha_j}\right) + \frac{1}{2}\lambda_0 x_0 \right\| \\ &\le \frac{1}{2}\|z\| + \frac{1}{2}|\lambda_0| \\ &\le 1 = \|Ax_0\|.\end{aligned}$$

This proves that $\|A\| \le 1$. Also $\|Az\| = 1$ iff $|\lambda_0| = 1$ and $\lambda_j = 0$ $\forall j = 1, 2, \ldots, n-1$.

Thus $\|Az\| = 1$ iff $z = \lambda_0 x_0$ with $\lambda_0 \in S_K$. This completes the proof. □

We now prove the following theorem.

Theorem 2.5 *Let X be a normed linear space and $x_0 \in S_X$. If there exists a bounded linear operator $A : X \to X$ which attains its norm only at the points of the form λx_0 with $\lambda \in S_K$, then x_0 is an exposed point of B_X.*

Proof Assume, without loss of generality, that $\|A\| = 1$ and by the Hahn-Banach theorem, there exists $f \in S_{X^*}$ such that $f(A(x_0)) = 1$. Clearly $\|foA\| = 1$ as $f \in S_{X^*}$ and $\|A\| = \|Ax_0\| = 1$. If $y \in S_X$ is such that $|foA(y)| = 1$, then $\|Ay\| = 1$.

Now $\|A\| = 1$ and A attains its norm only at the points of the form λx_0 with $\lambda \in S_K$, so $y \in \{\lambda x_0 : \lambda \in S_K\}$.

Thus foA attains its norm only at the points of the form λx_0 with $\lambda \in S_K$. Considering the hyperplane $H = \{x \in X : foA(x) = 1\}$, it is easy to verify that $H \cap B_X = \{x_0\}$ and so x_0 is an exposed point of B_X. □

Thus we obtained complete characterizations of exposed points, which is stated clearly in the following theorem.

Theorem 2.6 *For a normed linear space X and a point $x \in S_X$, the following are equivalent*:

1. *x is an exposed point of B_X.*
2. *There exists a Hamel basis of X containing x which is strongly orthonormal relative to x in the sense of Birkhoff-James.*
3. *There exists a bounded linear operator A on X which attains its norm only at the points of the form λx with $\lambda \in S_K$.*

We next give a characterization of a strictly convex space as follows.

Theorem 2.7 *For a normed linear space X, the following are equivalent*:

1. *X is strictly convex.*
2. *For each $x \in S_X$, there exists a Hamel basis of X containing x which is strongly orthonormal relative to x in the sense of Birkhoff-James.*

3. *For each $x \in S_X$, there exists a bounded linear operator A on X which attains its norm only at the points of the form λx with $\lambda \in S_K$.*

Proof The proof follows from previous theorem and the fact that a normed linear space X is strictly convex iff every element of S_X is an exposed point of B_X. □

Remark 2.8 Even though the notions of strong Birkhoff-James orthogonality and Birkhoff-James orthogonality coincide in a Hilbert space, they do not characterize Hilbert spaces as $(R^n, \|\cdot\|_p)$ $(1 < p < \infty, p \neq 2)$ is not a Hilbert space, but the notions of strong Birkhoff-James orthogonality and Birkhoff-James orthogonality coincide there.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors did not provide this information.

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References
1. James, RC: Orthogonality and linear functionals in normed linear spaces. Trans. Am. Math. Soc. **61**, 265-292 (1947)
2. Šmuljan, V: On some geometrical properties of the unit sphere in spaces of the type (B). Mat. Sb. **48**, 90-94 (1938)
3. Beesack, PR, Hughes, E, Ortel, M: Rotund complex normed linear spaces. Proc. Am. Math. Soc. **75**, 42-44 (1979)
4. Day Mahlon, M: Strict convexity and smoothness of normed spaces. Trans. Am. Math. Soc. **78**, 516-528 (1955)
5. Day Mahlon, M: Normed Linear Spaces. Springer, Berlin (1973)
6. Gudder, S, Strawther, D: Strictly convex normed linear spaces. Proc. Am. Math. Soc. **59**, 263-267 (1976)
7. James, RC: Inner products in normed linear spaces. Bull. Am. Math. Soc. **53**, 559-566 (1947)
8. James, RC: Orthogonality in normed linear spaces. Duke Math. J. **12**, 291-302 (1945)
9. Petryshyn, W: A characterization of strict convexity of Banach spaces and other uses of duality mappings. J. Funct. Anal. **6**, 282-291 (1970)
10. Strawther, D, Gudder, S: A characterization of strictly convex Banach spaces. Proc. Am. Math. Soc. **47**, 268 (1975)
11. Torrance, E: Strictly convex spaces via semi-inner-product space orthogonality. Proc. Am. Math. Soc. **26**, 108-110 (1970)
12. Birkhoff, G: Orthogonality in linear metric spaces. Duke Math. J. **1**, 169-172 (1935)
13. Klee, VL Jr.: Extremal structure of convex sets. I. Arch. Math. **8**, 234-240 (1957)
14. Klee, VL Jr.: Extremal structure of convex sets. II. Math. Z. **69**, 90-104 (1958)
15. Valentine, FA: Convex Sets. McGraw-Hill, New York (1964)

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