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Study on the existence of solutions for a generalized functional integral equation in L^1 spaces

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Abstract
Using a nonlinear alternative theorem of Krasnosel'skii type proved recently by Smaïl Djebali and Zahira Sahnoun, we investigate, in this paper, the existence of solutions for a generalized mixed-type functional integral equation in L^1 space. We also present some examples of the integral equation to confirm the efficiency of our results.
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1 Introduction

The functional integral equations describe many physical phenomena in various areas of natural science, mathematical physics, mechanics and population dynamics [1–4]. The theory of integral equations is developing rapidly with the help of tools in functional analysis, topology and fixed-point theory (see, for instance, [5–8]) and it serves as a useful tool, in turn, for other branches of mathematics, for example, for differential equations (see [9, 10]). A fixed point theorem, frequently used to solve integral equations, is a theorem proved by Krasnosel'skii in 1958 (see, for instance, [11, 12]). The Krasnosel'skii theorem asserts that $A + B$ has a fixed point in a closed, convex nonempty subset M of X if A, B satisfy the following conditions:

- A is compact and continuous;
- B is a strict contraction;
- $AM + BM \subseteq M$.

However, the Krasnosel'skii fixed point theorem sometimes turns out to be restrictive for some equations due to the weak topology of the problem. In order to use this result and its variant, one has to find a self-mapped closed convex set M so that $A + B$ maps M into itself or the weaker one: $x = Ax + By$ $(y \in M) \Rightarrow x \in M$. From the application point of view, this condition is also generally strict and is hard to achieve. To relax these conditions, a new effort is made in [13] by establishing a new variant of nonlinear Krasnosel'skii type fixed point theorem for nonself maps.

Let us first recall the nonlinear alternative Krasnosel'skii fixed point theorem established in [13], which plays a central role in our discussion.

Theorem 1 *Let $S \ni 0$ be an open subset of a Banach space X and let $\overline{S}$ be the closure of S. Let $A : S \to X$ and $B : X \to X$ be two mappings satisfying*:

- *A is continuous, $A(\overline{S})$ is relatively weakly compact, and A verifies the condition* H1.
- *B is a contraction and verifies the condition* H2.

Then either the equation $Ax + Bx = x$ admits a solution in $\overline{S}$, or there exists an element $x \in \partial S$ (∂S denotes the boundary of S) such that $x = \lambda Ax + \lambda B(\frac{x}{\lambda})$ for some $\lambda \in (0,1)$, where conditions H1 *and* H2 *are given in Section* 2.

The advantage of Theorem 1 lies in that in applying Theorem 1, one does not need to verify that the involved operator maps a closed convex subset onto itself.

In this paper, we utilize the alternative Theorem 1 and employ the concept of measure of weak noncompactness defined in [14] to study the solvability of a nonlinear generalized mixed-type operator equation of the form

$$y(t) = g\big(t, y(t)\big) + Tf_1\left(t, \int_\Omega u\big(t, s, f_2\big(s, Ay(s)\big)\big)\, ds\right), \tag{1}$$

where $t \in \Omega$, g is a function satisfying a contraction condition with respect to the second variable, y belongs to $L^1(\Omega, X)$, the space of Lebesgue integrable functions on a measurable subset Ω of $\mathcal{R}^n$ with values in a finite-dimensional Banach space X, while T and A are bounded linear operators on $L^1(\Omega, X)$. Suppose that N_f is the superposition operator generated by the function f given by $(N_f x)(t) = f(t, x(t))$, $t \in \Omega$, and U is a nonlinear Urysohn integral operator defined by $(Ux)(t) = \int_\Omega u(t, s, x(s))\, ds$, $s, t \in \Omega$, then Eq. (1) may be written in the form

$$y(t) = g\big(t, y(t)\big) + (TN_{f_1} UN_{f_2} Ay)(t). \tag{2}$$

The outline of this paper is as follows. In Section 2, we introduce some basic facts and use them to obtain our aims in Section 3. In the last section, we present some examples that verify the application of this kind of nonlinear integral equation.

2 Preliminaries

2.1 The weak MNC

We always use $(X, \|\cdot\|)$ to denote a Banach space with the norm $\|\cdot\|$. Denote by $B(X)$ the collection of all nonempty bounded subsets of X and by $W(X)$ the subset of $B(X)$ consisting of all weakly compact subsets of X. Let B_r be the closed ball in X centered at origin with radius r. The measure of weak noncompactness introduced by De Blasi is a map $\omega : B(X) \to [0, \infty)$ defined by

$$\omega(M) = \inf\big\{r > 0 \mid \text{there exists a } W \in W(X) \text{ with } M \subseteq W + B_r\big\}$$

for each $M \in B(X)$.

The following Lemma 1 comes from [14].

Lemma 1 *Let $M_1, M_2 \in B(X)$. Then we have*:

(i) *$M_1 \subseteq M_2$ implies $\omega(M_1) \leq \omega(M_2)$.*
(ii) *$\omega(M_1) = 0$ if and only if M_1 is relatively weakly compact.*

(iii) $\omega(\overline{M_1^w}) = \omega(M_1)$ (*where* $\overline{M_1^w}$ *is the weak closure of* M_1).
(iv) $\omega(M_1 \cup M_2) = \max\{\omega(M_1), \omega(M_2)\}$.
(v) $\omega(\lambda M_1) = |\lambda|\omega(M_1)$ *for all* $\lambda \in \mathcal{R}$.
(vi) $\omega(\mathrm{co}(M_1)) = \omega(M_1)$ ($\mathrm{co}(M_1)$ *refers to the convex hull of* M_1).
(vii) $\omega(M_1 + M_2) \le \omega(M_1) + \omega(M_2)$.

The map $\omega(\cdot)$ *is called the De Blasi measure of weak noncompactness.*

In [15], it is shown that in the L^1 space, $\omega(\cdot)$ is of the following simple form:

$$\omega(M) = \limsup_{\varepsilon\to 0}\left\{\sup_{\psi\in M}\left[\int_D \left\|\psi(t)\right\|_X dt | D\subset\Omega, \mathrm{meas}(D)\le\varepsilon\right]\right\} \tag{3}$$

for all bounded $M \subset L^1(\Omega, X)$, where $\mathrm{meas}(\cdot)$ represents the Lebesgue measure, X is a finite-dimensional Banach space.

Let J be a nonlinear operator from X into itself. In what follows, we need the following two conditions:

H1. *If* $(x_n)_{n\in N}$ *is a weakly convergent sequence in* X*, then* $(Jx_n)_{n\in N}$ *has a strongly convergent subsequence in* X*;*

H2. *If* $(x_n)_{n\in N}$ *is a weakly convergent sequence in* X*, then* $(Jx_n)_{n\in N}$ *has a weakly convergent subsequence in* X*.*

2.2 The superposition operator

In this subsection, we introduce the superposition (Nemytskii's) operator. Let Ω be a bounded domain of $\mathcal{R}^n$ and let X and Y be two separable Banach spaces. $m(\Omega, X)$ denotes the set of all measurable functions $\psi : \Omega \to X$. Consider a function $f : \Omega \times X \to Y$. We say that f satisfies *Carathéodory conditions* if

(i) for any $x \in X$, the map $t \to f(t,x)$ is measurable from Ω to Y;
(ii) for almost all $t \in \Omega$, the map $x \to f(t,x)$ is continuous from X to Y.

Definition 1 (Nemytskii's operator) Let $f : \Omega \times X \to Y$ be a Carathéodory function, Nemytskii's operator associated with f, $N_f : m(\Omega, X) \to m(\Omega, Y)$ is defined by $N_f x(t) = f(t, x(t))$, $\forall t \in \Omega$.

The superposition operator enjoys several nice properties. Specifically, we have the following results.

Lemma 2 [16] *Let* X *and* Y *be two separable Banach spaces. If* f *is a Carathéodory function, then Nemytskii's operator* N_f *maps continuously* $L^1(\Omega, X)$ *into* $L^1(\Omega, Y)$ *if and only if there exist a constant* $b > 0$ *and a function* $a(\cdot) \in L^1_+(\Omega)$ *such that*

$$\left\|f(t,x)\right\|_Y \le a(t) + b\|x\|_X,$$

where $L^1_+(\Omega)$ *stands for the positive cone of the space* $L^1(\Omega)$.

Lemma 3 [17] *Let* X, Y *be two finite-dimensional Banach spaces and let* Ω *be a bounded domain of* $\mathcal{R}^n$. *If* $f : \Omega \times X \to Y$ *is a Carathéodory function and* N_f *maps* $L^1(\Omega, X)$ *into* $L^1(\Omega, Y)$, *then* N_f *satisfies the condition* H2.

We give a fixed point lemma for bilinear forms.

Lemma 4 *Let X be a Banach space and let $B : X \times X \to X$ be a bilinear map. Let $\|\cdot\|_X$ denote the norm in X. If for all $x_1, x_2 \in X$, $\|B(x_1, x_2)\|_X \le \eta \|x_1\|_X \|x_2\|_X$. Then for all $y \in X$ satisfying $4\eta\|y\|_X < 1$, the equation $x = y + B(x, x)$ has a solution $x \in X$ satisfying and uniquely defined by the condition $\|x\|_X \le \|y\|_X$.*

Remark 1 The proof of this lemma also shows that $x = \lim_{k\to\infty} x_k$, where the approximate solutions x_k are defined by $x_0 = y$ and $x_k = y + B(x_{k-1}, x_{k-1})$. Moreover, $\|x_k\|_X \le 2\|y\|_X$ for all k.

3 Main results

In this section, we investigate the solvability of the nonlinear functional integral Eq. (1) in the space $L^1(\Omega, X)$ by applying Theorem 1.

First notice that Eq. (1) may be written in the abstract form

$$y = \mathcal{A}y + \mathcal{B}y,$$

where $(\mathcal{B}y)(t) = g(t, y(t))$, and $\mathcal{A} = TN_{f_1}UN_{f_2}A$ is the composition of the linear operator T and A with the nonlinear Urysohn integral operator U and the two superposition operators N_{f_1}, N_{f_2} generated by f_1, f_2, respectively, where $N_{f_i}y(t) = f_i(t, y(t))$, $i = 1, 2$. Our aim is to prove that $\mathcal{A} + \mathcal{B}$ has a fixed point in $L^1(\Omega, X)$. To do so, we assume that the following conditions are satisfied:

(a) The function $g : \Omega \times X \to X$ is a measurable function, $g(\cdot, 0) \in L^1(\Omega, X)$ and g is a contraction with respect to the second variable, *i.e.*, there exists an $L \in [0, 1)$ such that $\|g(t, x) - g(t, y)\| \le L\|x - y\|$ for almost all $t \in \Omega$ and all $x, y \in X$.

(b) $f_i : \Omega \times X \to X$, $i = 1, 2$ satisfy *Carathéodory conditions* and N_{f_i}, $i = 1, 2$, act from $L^1(\Omega, X)$ into itself continuously.

(c) The operators T and A are linear and bounded on $L^1(\Omega, X)$.

(d) The Urysohn operator U defined as before maps continuously $L^1(\Omega, X)$ into $L^1(\Omega, X)$.

(e) $\|u(t, s, x)\| \le \kappa(t, s)\{\xi(s) + \mu\|x\|\}$ for $(t, s) \in \Omega^2$ and $x \in X$, where ξ belongs to $L^1_+(\Omega)$, μ is a nonnegative constant and $\kappa : \Omega \times \Omega \to \mathcal{R}^+$ is a measurable function such that its associated integral operator K defined by

$$(K\rho)(t) = \int_\Omega \kappa(t, s)\rho(s)\,ds, \qquad \rho \in L^1(\Omega), t \in \Omega, \tag{4}$$

is continuous and maps $L^1(\Omega)$ into itself.

(f) There exists a constant $N > 0$ independent of $\lambda^* \in (0, 1)$ such that any solution of the integral equation

$$y(t) = \lambda^* g\left(t, \frac{1}{\lambda^*} y(t)\right) + \lambda^* TN_{f_1}UN_{f_2}Ay(t), \qquad t \in \Omega,$$

satisfies $\|y\|_{L^1(\Omega, X)} \ne N$.

Remark 2 It is deserved to mention that though the Urysohn operator U maps $L^1(\Omega, X)$ into itself, it does not have to be continuous. Sufficient conditions showing that U maps $L^1(\Omega, X)$ into itself and is continuous can be found in [18].

Before going on, we give crucial Lemma 5.

Lemma 5 *Let X be a finite-dimensional Banach space and let Ω be a compact subset of $\mathcal{R}^n$. If the conditions* (b)-(e) *are satisfied, then the operator $N_{f_1}UN_{f_2}A$ satisfies the condition* H2.

Proof For any nonempty subset D of Ω, let ε be an arbitrary positive real number. We have

$$\begin{aligned}
&\int_D \|N_{f_1}UN_{f_2}Ay(t)\|\,dt \\
&\quad = \int_D \left\| f_1\left(t, \int_\Omega u\big(t,s,f_2\big(s,Ay(s)\big)\big)\,ds\right)\right\|\,dt \\
&\quad \le \int_D \left(\big|a_1(t)\big| + b_1 \int_\Omega \big\|u\big(t,s,f\big(s,Ay(s)\big)\big)\big\|\,ds\right)dt \\
&\quad \le \|a_1\|_{L^1(D)} + b_1 \int_D \left(\int_\Omega \kappa(t,s)\big(\xi(s) + \mu\big\|f_2\big(s,Ay(s)\big)\big\|\big)\,ds\right)dt \\
&\quad \le \|a_1\|_{L^1(D)} + b_1\|K\| \left(\int_D \big(\xi(s) + \mu\big\|f_2\big(s,Ay(s)\big)\big\|\big)\,ds\right) \\
&\quad \le \|a_1\|_{L^1(D)} + b_1\|K\| \left(\|\xi\|_{L^1_+(D)} + \mu \int_D \big(\big|a_2(s)\big| + b_2\big\|Ay(s)\big\|\big)\,ds\right) \\
&\quad \le \|a_1\|_{L^1(D)} + b_1\|K\| \left(\|\xi\|_{L^1_+(D)} + \mu\left(\|a_2\|_{L^1(D)} + b_2\|A\| \int_D \big\|y(s)\big\|\,ds\right)\right) \\
&\quad = \|a_1\|_{L^1(D)} + b_1\|K\|\big(\|\xi\|_{L^1_+(D)} + \mu\|a_2\|_{L^1(D)}\big) + \mu b_1 b_2\|K\|\|A\| \int_D \big\|y(s)\big\|\,ds.
\end{aligned}$$

Now using reference [19, Corollary 11, p.294] together with (3), we have

$$\limsup_{\varepsilon\to 0}\left\{\int_D \big(\big|a_1(t)\big| + \big|\xi(t)\big| + \mu\big|a_2(t)\big|\big)\,dt \;\Big|\; \operatorname{meas}(D) \le \varepsilon\right\} = 0. \tag{5}$$

Accordingly,

$$\omega(N_{f_1}UN_{f_2}AS) \le \mu b_1 b_2\|K\|\|A\|\omega(S) \tag{6}$$

for any bounded subset S of $L^1(\Omega, X)$.

Next, let $(y_n)_{n\in N}$ be a weakly convergent sequence of $L^1(\Omega, X)$. Owing to (6), we infer that $\omega\{N_{f_1}UN_{f_2}A(y_n) : n \in N\} = 0$. This shows that the set $\{N_{f_1}UN_{f_2}A(y_n) : n \in N\}$ is relatively weakly compact in $L^1(\Omega, X)$. This completes the proof. □

Remark 3 Due to the assumption (a), we get

$$\begin{aligned}
\|\mathcal{B}y\| &= \int_\Omega \big\|\mathcal{B}y(t)\big\|\,dt \\
&\le \int_\Omega \big(\big\|g\big(t,y(t)\big) - g(t,0)\big\|\big)\,dt + \int_\Omega \big\|g(t,0)\big\|\,dt
\end{aligned}$$

$$\begin{aligned}
&\leq L \int_{\Omega} \|y(t)\| \, dt + \int_{\Omega} l(t) \, dt \\
&= \|l\|_{L^1(\Omega)} + L \int_{\Omega} \|y(t)\| \, dt,
\end{aligned}$$

where $l(t) = \|g(t,0)\| \in L^1_+(\Omega)$.

This shows that the operator $\mathcal{B}$ is continuous and maps a bounded set of $L^1(\Omega, X)$ into a bounded set of $L^1(\Omega, X)$. According to Lemma 3, we obtain $\mathcal{B}$ satisfies the condition H2.

Now we are in a position to state our main result.

Theorem 2 *Let X be a finite-dimensional Banach space and let Ω be a bounded domain of $\mathcal{R}^n$. Assume that the conditions* (a)-(f) *hold true. Then Eq.* (1) *admits at least one solution in $L^1(\Omega, X)$.*

Proof We apply Theorem 1 with

$$S = \{ y \in L^1(\Omega, X) : \|y\|_{L^1(\Omega,X)} < N \}.$$

Claim 1. Let $x, y \in L^1(\Omega, X)$. It follows from the assumption (a) that

$$\begin{aligned}
\|\mathcal{B}(x) - \mathcal{B}(y)\|_{L^1(\Omega,X)} &= \int_{\Omega} \|g(t, x(t)) - g(t, y(t))\|_X \, dt \\
&\leq L \int_{\Omega} \|x(t) - y(t)\|_X \, dt \\
&= L \|x - y\|_{L^1(\Omega,X)}.
\end{aligned}$$

So, $\mathcal{B}$ is a strict contraction mapping on $L^1(\Omega, X)$, and from Remark 3, $\mathcal{B}$ satisfies the condition H2.

Claim 2. Clearly, by the assumptions (b)-(d), $\mathcal{A} = TN_{f_1}UN_{f_2}A$ is continuous on $L^1(\Omega, X)$. Now we check that $\mathcal{A}$ satisfies the condition H1. To do this, let $(y_n)_{n\in N}$ be a weakly convergent sequence of $L^1(\Omega, X)$. By Lemma 5, $(N_{f_1}UN_{f_2}A(y_n))_{n\in N}$ has a weakly convergent subsequence, say $(N_{f_1}UN_{f_2}A(y_{n_k}))_{k\in N}$. Furthermore, the continuity of the linear operator T implies its weak continuity on $L^1(\Omega, X)$ for almost all $t \in \Omega$. Thus, the sequence $(TN_{f_1}UN_{f_2}A(y_{n_k}))_{k\in N}$, *i.e.*, $(\mathcal{A}(y_{n_k}))_{k\in N}$ converges pointwisely for almost all $t \in \Omega$. By the Vitali convergence theorem [19, p.150], we conclude that $(\mathcal{A}(y_{n_k}))_{k\in N}$ converges strongly in $L^1(\Omega, X)$. Hence, $\mathcal{A}$ satisfies the condition H1.

Claim 3. We show that $\mathcal{A}(\overline{S})$ is relatively weakly compact. For this, we need to prove that

$$\omega(\mathcal{A}(\overline{S})) = \lim_{\varepsilon\to 0} \sup \left\{ \sup_{y\in\overline{S}} \left[\int_D \|\mathcal{A}y(t)\|_X \, dt \,\middle|\, \operatorname{meas}(D) \leq \varepsilon \right] \right\} = 0$$

for all $D \subseteq \Omega$, and $\forall y \in S$. By Lemma 5, we have

$$\begin{aligned}
&\int_D \|\mathcal{A}y(t)\|_X \, dt \\
&\quad = \int_D \|TN_{f_1}UN_{f_2}Ay(t)\| \, dt
\end{aligned}$$

$$\begin{aligned}
&\le \|T\| \int_D \|N_{f_1} U N_{f_2} Ay(t)\| \, dt \\
&\le \|T\| \left(\|a_1\|_{L^1(D)} + b_1\|K\| \left(\|\xi\|_{L^1_+(D)} + \mu \|a_2\|_{L^1(D)} \right) + \mu b_1 b_2 \|K\| \|A\| \int_D \|y(s)\| \, ds \right) \\
&\le \|T\| \left(\|a_1\|_{L^1(D)} + b_1\|K\| \left(\|\xi\|_{L^1_+(D)} + \mu \|a_2\|_{L^1(D)} \right) + \mu b_1 b_2 \|K\| \|A\| N \right).
\end{aligned}$$

Owing to (5), we deduce that $\omega(\mathcal{A}(\overline{S})) = 0$, and hence $\mathcal{A}(\overline{S})$ is relatively weakly compact.

Finally, thanks to the assumption (f), the second situation of Theorem 1 does not occur. Now, applying Theorem 1, we get that $\mathcal{A} + \mathcal{B}$ has a fixed point in $\overline{S}$, that is to say, Eq. (1) has a solution in $\overline{S}$. This completes the proof. □

Remark 4 The requirement that X should be a finite-dimensional Banach space comes from the usage of the relation (3) proved in [15] for bounded subsets in the space of Lebesgue integrable functions with values in a finite-dimensional Banach space.

By Theorem 2, we can get a special existence criterion for Eq. (1).

Corollary 1 *Let X be a finite-dimensional Banach space and let Ω be a bounded domain of $\mathcal{R}^n$. Besides the assumptions* (a)-(e), *we make the following additional assumptions*:

(i) *There exists a continuous function $h : [0,\infty) \to [0,\infty)$ such that $h(u) > 0$ whenever $u > 0$ and*

$$\int_\Omega \|N_{f_1} U N_{f_2} Ay(t)\|_X \, dt \le h\left(\|y\|_{L^1(\Omega,X)}\right) \quad \textit{for every } y \in L^1(\Omega, X).$$

(ii)

$$\sup_{\theta\in[0,\infty)} \left(\frac{(1-L)\theta}{\|l\|_{L^1_+} + \|T\| h(\theta)} \right) > 1,$$

where $l(t) := \|g(t,0)\|$ and L is the constant in the assumption (a). *Then Eq.* (1) *has a solution in $L^1(\Omega, X)$.*

Proof Thanks to Theorem 2, we only need to show that (i) and (ii) imply (g). Let $N > 0$ satisfy

$$\frac{(1-L)N}{\|l\|_{L^1_+} + \|T\| h(N)} > 1. \tag{7}$$

The condition (ii) ensures the existence of such an N. Let $y \in L^1(\Omega, X)$ be any solution of the operator equation

$$y = \lambda^* \mathcal{A} y + \lambda^* \mathcal{B}\left(\frac{y}{\lambda^*}\right), \quad \lambda^* \in (0,1). \tag{8}$$

Then, for $t \in \Omega$, we have the estimate

$$\begin{aligned}
\|y(t)\| &\le \lambda^* \left\| g\left(t, \frac{y(t)}{\lambda^*}\right) - g(t,0) \right\| + \lambda^* \|g(t,0)\| + \lambda^* \|T\| \|N_{f_1} U N_{f_2} Ay(t)\| \\
&\le L\|y(t)\| + l(t) + \|T\| \|N_{f_1} U N_{f_2} Ay(t)\|,
\end{aligned}$$

and so

$$\int_{\Omega} \|y(t)\|\,dt \le L\int_{\Omega}\|y(t)\|\,dt + \int_{\Omega} l(t)\,dt + \|T\|h(\|y\|).$$

Therefore

$$\frac{(1-L)\|y\|_{L^1(\Omega,X)}}{\|l\|_{L^1_+} + \|T\|h(\|y\|_{L^1(\Omega,X)})} \le 1. \tag{9}$$

Assuming that $\|y\|_{L^1(\Omega,X)} = N$. Equation (9) implies $\frac{(1-L)N}{\|l\|_{L^1_+}+\|T\|h(N)} \le 1$ contradicting (7). So, each solution of (8) satisfies $\|y\|_{L^1_+} \neq N$. Accordingly, by Theorem 2, Eq. (1) has a solution $y \in L^1(\Omega, X)$. This completes the proof. □

Corollary 2 *Let X be a finite-dimensional Banach space and let Ω be a bounded domain of $\mathcal{R}^n$. Assume that hypotheses* (a)-(f) *hold true with the additional assumption that*

(iii) $L + \mu b_1 b_2\|K\|\|A\| < 1$, *where the constants b_1, b_2, μ are these in Lemma* 2 *and the hypothesis* (f), *$\|K\|$ denotes the norm of the operator K defined in* (4).

Then Eq. (1) *has a solution in $L^1(\Omega, X)$.*

Proof Let $y \in L^1(\Omega, X)$. By the formula of Lemma 5, we have

$$\begin{aligned}
&\int_{\Omega}\|\mathcal{A}y(t)\|\,dt \\
&\quad\le \|T\|\int_{\Omega}\|N_{f_1}UN_{f_2}Ay(t)\|\,dt \\
&\quad\le \|T\|\bigl(\|a_1\|_{L^1(D)} + b_1\|K\|\bigl(\|\xi\|_{L^1_+(D)} + \mu\|a_2\|_{L^1(D)}\bigr) + \mu b_1 b_2\|K\|\|A\|\|y\|_{L^1(\Omega,X)}\bigr).
\end{aligned}$$

On the other hand, with the same arguments as in the proof of Corollary 1, we have the following estimate:

$$\begin{aligned}
\|y\|_{L^1(\Omega,X)} &\le L\|y\|_{L^1(\Omega,X)} + \|l\|_{L^1_+} + \lambda^*\|\mathcal{A}y\| \\
&\le L\|y\|_{L^1(\Omega,X)} + \|l\|_{L^1_+} + \lambda^*\|T\|\delta\bigl(\|y\|_{L^1(\Omega,X)}\bigr),
\end{aligned}$$

where $\delta(\gamma) = \|a_1\|_{L^1(D)} + b_1\|K\|(\|\xi\|_{L^1_+(D)} + \mu\|a_2\|_{L^1(D)}) + \mu b_1 b_2\|K\|\|A\|\gamma$. For the sake of simplicity, we can set $\|a_1\|_{L^1(D)} + b_1\|K\|(\|\xi\|_{L^1_+(D)} + \mu\|a_2\|_{L^1(D)}) = \nu$. Hence

$$\bigl\{1 - L - \mu b_1 b_2\|T\|\|K\|\|A\|\bigr\}\|y\|_{L^1(\Omega,X)} \le \|l\|_{L^1_+} + \|T\|\nu. \tag{10}$$

Let

$$N > \frac{\|l\|_{L^1_+} + \|T\|\nu}{1 - L - \mu b_1 b_2\|T\|\|K\|\|A\|}.$$

If $\|y\|_{L^1(\Omega,X)} = N$, then (10) implies that

$$N \le \frac{\|l\|_{L^1_+} + \|T\|\nu}{1 - L - \mu b_1 b_2\|T\|\|K\|\|A\|},$$

which is a contradiction. So, the hypothesis (g) is satisfied and the result then follows from Theorem 2. This completes the proof. □

4 Examples

In this section, we provide some examples of a classical integral and functional equation considered in nonlinear analysis which are a particular case of Eq. (1).

Example 1 The existence of solutions of the equation

$$x(t) = g\big(t, x(t)\big) + \lambda \int_\Omega k(t,s) f\big(s, x(s)\big)\,ds$$

has been investigated in [13] by this method under proper assumptions. We denote that it is a special case of Eq. (1) with $T = C$, where C is the Fredholm operator defined as

$$\begin{aligned} &C : L^1(\Omega, Y) \to L^1(\Omega, X), \\ &\psi \mapsto C\psi : \Omega \to X; \qquad C\psi(t) = \int_\Omega k(t,s)\psi(s)\,ds. \end{aligned}$$

Example 2 The following equation proposed in [20]

$$\psi(t) = g\big(t, \psi(t)\big) + (BN_f UA\psi)(t)$$

is also a special case of Eq. (1) with $T = B$ and $f_2(t,y) = y$.

Example 3 The solvability of the nonlinear integral equation of a mixed type

$$x(t) = g(t) + \int_0^1 k_1(t,s) f_1\left(s, \int_0^s k_2(s,\tau) f_2\big(\tau, x(\tau)\big)\,d\tau\right) ds, \quad t \in (0,1)$$

is discussed in the space $L^1(0,1)$ in [21]. If let $X = \mathcal{R}$, $T = K_1$ (defined in [21]) and $u(t,s,x) = k_2(t,s)x$ with $\Omega = [0,1]$, then the above equation is a particular case of Eq. (1) and it is applied to solve fractional order integro-differential equations

$$y(t) = g(t) + \int_0^1 k_1(t,s) f_1\left(s, \int_0^s \frac{(s-\tau)^{-\beta}}{\Gamma(1-\beta)} y(\tau)\,d\tau\right) ds, \quad t \in (0,1).$$

Example 4 Consider the following integral equation of the form

$$x(t) = \frac{1}{6}\big[t\exp(-t) + t^2 x(t)\big] + T\left(\frac{\ln(1+t)}{1+t} + \int_0^t \frac{\exp(-2s)}{\exp(t)+1}\big(\exp(s) + \sin s + 2x(s)\big)\,ds\right)$$

with $0 \le s \le t \le 1$.

Let us take $g : [0,1] \times \mathcal{R} \to \mathcal{R}$, $f_i : [0,1] \times \mathcal{R} \to \mathcal{R}$, $i = 1,2$ and $u : [0,1] \times [0,1] \times \mathcal{R} \to \mathcal{R}$ defined by, respectively,

$$\begin{aligned} g(t,x) &= \frac{1}{6}\big(t\exp(-t) + t^2 x\big), \\ f_1(t,x) &= \frac{\ln(1+t)}{1+t} + x, \end{aligned}$$

$$f_2(t,y) = \sin t + 2y,$$

$$u(t,s,x) = \frac{\exp(-2s)}{\exp(t)+1}\bigl(\exp(s)+x\bigr).$$

We can suppose T to be an arbitrary linear bounded operator on $L[0,1]$. It is easy to see that the function g satisfies the assumption (a) with $L = \frac{1}{6}$, the function $|u(t,s,x)| \leq \frac{\exp(-2s)}{\exp(t)}(\exp(s)+x)$ with $k(t,s) = \frac{\exp(-2s)}{\exp(t)}$ and $\mu = 1$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
LY carried out the main work of this paper; JW and GY participated in work on the part content and modified work of this paper. All authors read and approved the final manuscript.

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